



Existence of multiple weak solutions to elliptic equations with nonstandard growth by the Fountain theorem

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Abstract

We consider an elliptic problem

$$-\operatorname{div}(a(x, \nabla u)) = \eta u |u|^{p(x)-2} + \lambda u |u|^{z(x)-2} + cu |u|^{p^*(x)-2}, \quad u|_{\partial\Omega} = 0$$

in the domain $\Omega \subset \mathbb{R}^l$, $l \geq 3$, under rather general conditions on the elliptic operator involving variable exponent conditions. We show the existence of a weak solution by means of the mountain pass theorem. Our main result is the establishment of the existence of infinitely many nontrivial weak solutions under the assumption of symmetry $A(x, \xi) = A(x, -\xi)$ for all $x \in \Omega$ and all $\xi \in \mathbb{R}^l$.

Keywords: elliptic equation, boundary problem, fountain theorem, divergent form, weak solution, variational method, variable exponent.

MSC: 35J20, 35J60, 58E05.

1 Introduction and preliminary information

In this article, we consider the Dirichlet problem for general elliptic equation involving variable exponent in the form

$$\begin{cases} -\operatorname{div}(a(x, \nabla u)) = \eta u |u|^{p(x)-2} + \lambda u |u|^{z(x)-2} + cu |u|^{p^*(x)-2}, \\ u|_{\partial\Omega} = 0, \end{cases} \quad (1)$$

where Ω is a bounded domain in $\mathbb{R}^l$, $l \geq 3$; $p, z : \Omega \rightarrow \mathbb{R}$ are measurable functions and

$$p^*(x) = \begin{cases} \frac{lp(x)}{l-p(x)} & \text{if } p(x) < l, \\ \infty & \text{if } p(x) \geq l. \end{cases}$$

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Recently, this kind of problem was studied by Y.H. Kim Y. H. Kim (2024); M.C. Chen, Y.F. Xue M. C. Chen and Xue (2024); Y. Cheng, B. Ge and R. P. Agarwal Cheng, B. Ge, and Agarwal (2020); Y. Z. Fang and C. Zhang Fang and C. Zhang (2023); H.H. Ha, K. Ho, I. Sim Ha, Ho, and Sim (2022); B. Ge, L.Y. Ge, and J.F. Lu B. Ge, L. Y. Ge, and Lu (2021); Y. Li, P. Qu, Z. Zeng, and D. Zhang Li et al. (2024); Q. Lou and Z. Zhang Lou and Z. Zhang (2024); W. Zhang, J. Zhang, and V. D. Rădulescu W. Zhang, J. Zhang, and Rădulescu (2023); Y. Zhou, J. Lei, Y. Wang, Z. Xiong Zhou et al. (2022). In particular, Y. Zhou, J. Lei, Y. Wang, and Z. Xiong Zhou et al. (2022) studied a similar quasilinear Schrödinger equation in the case of constant exponent with nonnegative potentials. The multiplicity of solutions of elliptic equations involving the fractional $p(x)$ -Laplacian was investigated in J. M. Kim, Y. H. Kim, and Lee (2020), Y. H. Kim (2024), Lou and Z. Zhang (2024), and J. Zhang, W. Zhang, and Rădulescu (2022). Double phase equations were studied in Fang and C. Zhang (2023), B. Ge, L. Y. Ge, and Lu (2021), Vetro and Winkert (2023), J. Zhang, W. Zhang, and Rădulescu (2022), and W. Zhang, J. Zhang, and Rădulescu (2023). A double-phase problem without any growth and Ambrosetti–Rabinowitz conditions was investigated by B. Ge, L.Y. Ge, and J.F. Lu in B. Ge, L. Y. Ge, and Lu (2021), whose results can be extended to even more general differential equations. The classical results on the Fountain and Intersection theorems can be found in the monograph Willem (1996) by M. Willem, which approaches we are employing. The equations involving terms $a_i(x, \nabla u) = |\nabla u|^{p(x)-2} \nabla_i u$ are widely applied in physical models describing electrorheological fluids, image processing, and others Meng, Huang, and J. Chen (2021) and Zhou et al. (2022).

A typical example of the problem (1) can be thought as

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p(x)-2} \nabla_i u) = \lambda u |u|^{z(x)-2} + cu |u|^{p^*(x)-2} \\ u|_{\partial\Omega} = 0 \end{cases}$$

in a bounded domain $\Omega \subset \mathbb{R}^l$, $l \geq 3$. A more example with subcritical coefficients of this kind was studied by Xianling Fan Fan (2005), where the author dealt with a compact embedding theorem of the subcritical Sobolev-Hardy type, which was later used to formulate the variational principle and to prove the existence and multiplicity results by the critical point theory.

The remainder of the paper is organized as follows. In Section 2, we present the conditions and necessary preliminaries. In Section 3, we prove the properties of the elliptic operator comprising the given elliptic equation under the conditions of Section 2, and the mountain-pass theorem for problem (1) is studied. Section 4 is dedicated to the existence of infinitely many nontrivial solutions to the problem (1) under the conditions of Section 2.

2 The assumptions and frameworks

Definition 1 – Let a function $p : \Omega \rightarrow \mathbb{R}$.

- p is called log-Hölder continuous if for some $c > 0$,

$$\forall x, y \in \Omega \quad |p(x) - p(y)| \leq \frac{c}{\ln(e + |x - y|^{-1})}.$$

- p is said to satisfy the log-Hölder decay condition if for some $c_1 > 0$ and some real constant p_∞ ,

$$\forall x \in \Omega \quad |p(x) - p_\infty| \leq \frac{c_1}{\ln(e + |x|)}.$$

- The class $P^{\ln}(\Omega)$ consists of all functions $p(\cdot)$ such that $p(\cdot)^{-1}$ is log-Hölder continuous and satisfies the log-Hölder decay condition.

2.1 Assumptions for the Dirichlet problem (1)

We assume that the exponents p and z in problem (1) belong to $P^{\ln}(\Omega)$. Let

$$p_m = \inf_{x \in \Omega} p(x), \quad p_S = \sup_{x \in \Omega} p(x), \quad z_m = \inf_{x \in \Omega} z(x), \quad z_S = \sup_{x \in \Omega} z(x), \quad \text{and} \quad p_m^* = \inf_{x \in \Omega} p^*(x).$$

We assume that

$$1 < z_m \leq z_S < p_m \leq p_S < p_m^*.$$

Functions $a_i(x, \xi)$ are defined for all $x \in \text{clos}(\Omega)$, $\xi \in \mathbb{R}^l$; $a_i(x, \xi)$ are continuous at $\xi \in \mathbb{R}^l$. There exists A a measurable symmetric function on $\Omega \times \mathbb{R}^l$ such that $A(x, 0) = 0$, $a(x, 0) = 0$ for all $x \in \Omega$, $A(x, \xi) = A(x, -\xi)$ for all $x \in \Omega$ and all $\xi \in \mathbb{R}^l$, and $a_i(x, \xi) = \frac{\partial A(x, \xi)}{\partial \xi_i}$ is a Carathéodory function on $\Omega \times \mathbb{R}^l$. We assume that:

1. $|a_i(x, \xi)| \leq \gamma_1(x) + \mu_1(x) |\xi|^{p(x)-1}$ for all $\xi \in \mathbb{R}^l$, $l \geq 3$ and almost all $x \in \text{clos}(\Omega)$, where $p \in P^{\ln}(\Omega)$, $\gamma_1 \in L^{q(\cdot)}(\Omega)$ and $\mu_1 \in L^1_{\text{loc}}(\Omega)$;
2. $\tau A(x, \xi_1) + (1 - \tau) A(x, \xi_2) \geq A(x, \tau \xi_1 + (1 - \tau) \xi_2) + \tilde{\mu} \mu_1(x) |\xi_1 - \xi_2|^{p(x)}$ for almost all $x \in \text{clos}(\Omega)$ and all $\xi_1, \xi_2 \in \mathbb{R}^l$ and for all $\tau \in (0, 1)$;
3. $A(x, \xi) \geq \nu \mu_1(x) |\xi|^{p(x)}$ for almost all $x \in \text{clos}(\Omega)$ and $\xi \in \mathbb{R}^l$;
4. $0 \leq a_i(x, \xi) \xi_i \leq p(x) A(x, \xi)$ for almost all $x \in \text{clos}(\Omega)$ and all $\xi \in \mathbb{R}^l$.

To ensure the validity of the weighted Poincaré inequality and the compactness properties required in our analysis, we further assume that the weight function μ_1 satisfies the following condition:

5. $\mu_1(x) \geq \mu_0 > 0$ for almost every $x \in \Omega$ and $\mu_1 \in A_{p(\cdot)}$, i.e., μ_1 belongs to the Muckenhoupt class for variable exponents. This condition guarantees the equivalence of the norm defined in (2) with the standard norm in $W_0^{1,p(\cdot)}(\Omega)$ and ensures the compact embedding results needed later. For details on weighted variable exponent spaces and Poincaré inequalities, we refer to Diening et al. (2011).

2.2 Frameworks

We introduce a Banach space

$$W = \left\{ u \in W_{1,0}^{p(\cdot)}(\Omega) : \rho_{p(\cdot)}^{\mu_1}(\nabla u) = \int_{\Omega} \mu_1(x) |\nabla u|^{p(x)} dx < \infty \right\}$$

with the norm defined by

$$\|u\|_W = \inf \left\{ \zeta > 0 : \int_{\Omega} \mu_1(x) \left| \frac{\nabla u}{\zeta} \right|^{p(x)} dx \leq 1 \right\}, \quad (2)$$

and a number

$$\sigma_1 = \inf \left\{ \frac{\rho_{p(\cdot)}^{\mu_1}(\nabla u)}{\rho_{p(\cdot)}(u)} : u \in W \setminus \{0\} \right\} \quad \text{where} \quad \rho_{p(\cdot)}(u) = \int_{\Omega} |u|^{p(x)} dx,$$

and

$$\|u\|_{W(p^*)} = \inf \left\{ \lambda > 0 : \int_{\Omega} \left| \frac{u}{\lambda} \right|^{p^*(x)} dx \leq 1 \right\}.$$

We define a reflexive separable Banach space

$$E = \left\{ u \in W_{1,0}^{p(\cdot)}(\Omega) : \begin{aligned} &\rho_{p(\cdot)}^{\mu_1}(u) = \int_{\Omega} \mu_1(x) |\nabla u|^{p(x)} dx < \infty, \\ &\rho_{p^*(\cdot)}(u) = \int_{\Omega} |u|^{p^*(x)} dx < \infty \end{aligned} \right\}$$

endowed with the norm

$$\|u\| = \|u\|_E = \inf \left\{ \lambda > 0 : \int_{\Omega} \mu_1(x) \left| \frac{\nabla u}{\lambda} \right|^{p(x)} + \left| \frac{u}{\lambda} \right|^{p^*(x)} dx \leq 1 \right\}.$$

We assume that E is a reflexive separable Banach space. Then, there exist two sequences $\{e_k\} \subset E$ and $\{e_k^*\} \subset E^*$ such that

$$\langle e_k^*, e_m \rangle = \delta_{k,m}, \quad E = \text{clos}(\text{span}(\{e_k\}_{k \geq 1})), \quad E^* = \text{clos}(\text{span}(\{e_k^*\}_{k \geq 1})).$$

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For each integer $k \geq 1$, we define

$$X_k = \text{clos}(\text{span}(e_k)), \quad Y_k = \bigoplus_{1 \leq j \leq k} X_j, \quad Z_k = \text{clos}\left(\bigoplus_{j \geq k+1} X_j\right).$$

We recall the Poincaré inequality, which ensures the existence of a positive constant C_P such that, for all $u \in W$,

$$\int_{\Omega} |u|^{p(x)} dx \leq C_P \rho_{p(\cdot)}^{\mu_1}(u).$$

We define a functional $J \in C^1(E, \mathbb{R})$ by $J(u) = J_1(u) - J_2(u)$, where

$$J_1(u) = \int_{\Omega} A(x, \nabla u) dx,$$

$$J_2(u) = \int_{\Omega} \left(\eta \frac{|u|^{p(x)}}{p(x)} + \lambda \frac{|u|^{z(x)}}{z(x)} \right) dx + \int_{\Omega} c \frac{|u|^{p^*(x)}}{p^*(x)} dx.$$

Then J' is given by

$$\begin{aligned} \langle J'(u), v \rangle &= \int_{\Omega} a_i(x, \nabla u) \nabla_i v dx \\ &\quad - \int_{\Omega} \left(\eta |u|^{p(x)-2} uv + \lambda |u|^{z(x)-2} uv \right) dx - \int_{\Omega} c |u|^{p^*(x)-2} uv dx \end{aligned}$$

for all $u, v \in W_{1,0}^{p(\cdot)}(\Omega)$.

3 The mountain-pass theorem for an elliptic problem with nonstandard growth

In this Section, we establish some properties of the terms of the elliptic equation under our conditions of previous Section.

Lemma 1 – *We have the following properties:*

1. $|A(x, \xi)| \leq \gamma_1(x)|\xi| + \mu_1(x)|\xi|^{p(x)}$ for all $\xi \in \mathbb{R}^l$ and almost all $x \in \Omega$;
2. for $J_1(u) = \int_{\Omega} A(x, \nabla u) dx$ the inequality

$$\tau J_1(u) + (1 - \tau) J_1(v) \geq J_1(\tau u + (1 - \tau)v) + \tilde{\mu} \rho_{p(\cdot)}^{\mu_1}(u - v)$$

holds for all $u, v \in W$ and all $\tau \in [0, 1]$.

Proof. By assumption 1., we have

$$\begin{aligned}
 |A(x, \xi)| &= \left| \int_0^1 \frac{dA(x, \xi \tau)}{d\tau} d\tau \right| = \left| \int_0^1 a_i(x, \xi \tau) \cdot \xi d\tau \right| \\
 &\leq \int_0^1 |a_i(x, \xi \tau)| |\xi| d\tau \\
 &\leq \int_0^1 (\gamma_1(x) + \mu_1(x) |\xi|^{p(x)-1} \tau^{p(x)-1}) |\xi| d\tau \\
 &\leq \gamma_1(x) |\xi| + \mu_1(x) |\xi|^{p(x)}
 \end{aligned}$$

for all $\xi \in \mathbb{R}^l$. The second statement can be proven by similar arguments. $\square$

Lemma 2 – *The functional J_1 is continuous on W and J_1 is weakly continuously differentiable on W and such that, for all $u, v \in W$,*

$$\langle J'_1(u), v \rangle = \int_{\Omega} a_i(x, \nabla u) \nabla_i v dx \quad \text{and} \quad J_1(u) - J_1(v) \geq J'_1(v)(u - v).$$

Proof. For all $u, v \in W$ and all $\tau \in [0, 1]$, we have

$$\begin{aligned}
 &\left| \frac{A(x, \nabla u(x) + \tau \nabla v(x)) - A(x, \nabla u(x))}{\tau} \right| \\
 &= \left| \int_0^1 a_i(x, \nabla u(x) + \tau \nabla v(x)) \nabla_i v(x) d\tau \right| \\
 &\leq \int_0^1 (\gamma_1(x) + \mu_1(x) |\nabla u(x) + \tau \nabla v(x)|^{p(x)-1}) |\nabla v(x)| d\tau \\
 &\leq \gamma_1(x) |\nabla v(x)| \\
 &\quad + (\mu_1(x))^{\frac{1}{p(x)}} |\nabla v(x)| \left((\mu_1(x))^{\frac{1}{p(x)}} |\nabla u(x)| + (\mu_1(x))^{\frac{1}{p(x)}} |\nabla v(x)| \right)^{p(x)-1}.
 \end{aligned}$$

By the Lebesgue dominated convergence theorem, we obtain

$$\begin{aligned}
 \langle J'_1(u), v \rangle &= \lim_{\tau \rightarrow 0} \int_{\Omega} \frac{A(x, \nabla u(x) + \tau \nabla v(x)) - A(x, \nabla u(x))}{\tau} dx \\
 &= \int_{\Omega} a_i(x, \nabla u) \nabla_i v dx.
 \end{aligned}$$

Let $w \in W$. Arguing by contradiction, we assume that mapping $u \mapsto \langle J'_1(u), v \rangle$ is not continuous at w for each fixed $v \in W$. So, we obtain that there is a number $\varepsilon > 0$ and a sequence $\{w_n\} \subset W$ such that

$$\lim_{n \rightarrow \infty} \|w - w_n\|_W = 0 \quad \text{and} \quad |\langle J'_1(w), v \rangle - \langle J'_1(w_n), v \rangle| > \varepsilon \quad (n \in \mathbb{N}).$$

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We can choose a subsequence of $\{w_n\} \subset W$, which is still indexed by n , and a function $f \in L^1(\Omega)$ such that the following estimate holds almost everywhere in Ω for all $n \in \mathbb{N}$:

$$|\nabla w_n|^{p(x)} \leq \mu_1 |\nabla w_n|^{p(x)} \leq f.$$

Thus, for all $n \in \mathbb{N}$, we deduce

$$\begin{aligned} & |a_i(x, \nabla w_n(x))| |\nabla v(x)| \\ & \leq \left(\gamma_1(x) + \mu_1(x) |\nabla w_n|^{p(x)-1}(x) \right) |\nabla v(x)| \\ & = \gamma_1(x) |\nabla v(x)| + \left(\mu_1(x) |\nabla w_n(x)|^{p(x)} \right)^{\frac{p(x)-1}{p(x)}} (\mu_1(x))^{\frac{1}{p(x)}} |\nabla v(x)| \\ & \leq \gamma_1(x) |\nabla v(x)| + \mu_1(x) |f(x)|^{\frac{p(x)-1}{p(x)}} (\mu_1(x))^{\frac{1}{p(x)}} |\nabla v(x)|. \end{aligned}$$

Since $\gamma_1, f^{\frac{p(\cdot)-1}{p(\cdot)}} \in L^{\frac{p(\cdot)}{p(\cdot)-1}}(\Omega)$ and $\mu_1^{\frac{1}{p(\cdot)}}, |\nabla v| \in L^{p(\cdot)}(\Omega)$ we obtain the integrability of the right hand terms. Also, we have

$$\nabla w_n \xrightarrow[n \rightarrow \infty]{\text{a.e.}} \nabla w \quad \text{and} \quad a_i(x, \nabla w_n) \nabla_i v \xrightarrow[n \rightarrow \infty]{\text{a.e.}} a_i(x, \nabla w) \nabla_i v.$$

Applying the Lebesgue dominated convergence theorem, we conclude

$$\int_{\Omega} a_i(x, \nabla w_n) \nabla_i v dx \xrightarrow[n \rightarrow \infty]{} \int_{\Omega} a_i(x, \nabla w) \nabla_i v dx$$

or $\langle J'_1(w_n), v \rangle \xrightarrow[n \rightarrow \infty]{} \langle J'_1(w), v \rangle$. This contradiction shows the continuity of the mapping $u \mapsto \langle J'_1(u), v \rangle$ on W and this weakly differentiability at u .

By convexity, we estimate for all $\tau \in (0, 1)$

$$\begin{aligned} \frac{J_1(u_n + \tau(u - u_n)) - J_1(u_n)}{\tau} &= \frac{J_1((1 - \tau)u_n + \tau u) - J_1(u_n)}{\tau} \\ &\leq \frac{(1 - \tau)J_1(u_n) + \tau J_1(u) - J_1(u_n)}{\tau} \\ &= J_1(u) - J_1(u_n). \end{aligned}$$

By passing to the limit as $\tau \rightarrow 0$, we deduce

$$J'_1(u_n)(u - u_n) \leq J_1(u) - J_1(u_n).$$

The lemma is proven. $\square$

Definition 2 (Palais-Smale condition) – The functional $J \in C^1(E, \mathbb{R})$ is said to satisfy the Palais-Smale condition if each sequence $\{u_n\} \subset E$ such that sequence $\{J(u_n)\}$ is bounded ($|J(u_n)| \leq M$), $J(u_n) \xrightarrow[n \rightarrow \infty]{} c$ and $J'(u_n) \xrightarrow[n \rightarrow \infty]{} 0$ then the sequence $\{u_n\}$ contains a convergent subsequence.

We denote $\Gamma = \{g \in C([0, 1], E) : g(0) = 0, g(1) = w\}$, where the set Γ is non-empty since assuming $g \in C([0, 1], E)$ and taking $g(s) = sw$ for all $s \in [0, 1]$.

Lemma 3 – Let $\{u_n\} \subset E$ satisfy the following conditions $|J(u_n)| \leq M$, $J(u_n) \xrightarrow{n \rightarrow \infty} c$ and $J'(u_n) \xrightarrow{n \rightarrow \infty} 0$ then the sequence $\{u_n\}$ has a convergent subsequence.

Proof. We assume $\{u_n\} \subset E$ and $\|u_n\| > 1$ for all $n \in \mathbb{N}$. We estimate

$$\begin{aligned}
M &> J(u_n) - \frac{1}{p_m^*} \langle J'(u_n), u_n \rangle \\
&= \int_{\Omega} A(x, \nabla u_n) dx - \int_{\Omega} \eta \frac{|u_n|^{p(x)}}{p(x)} dx - \int_{\Omega} \lambda \frac{|u_n|^{z(x)}}{z(x)} dx - \int_{\Omega} c \frac{|u_n|^{p^*(x)}}{p^*(x)} dx \\
&\quad + \frac{1}{p_m^*} - \frac{1}{p_m^*} \int_{\Omega} a_i(x, \nabla u_n) \nabla_i u_n dx + \frac{1}{p_m^*} \int_{\Omega} \eta |u_n|^{p(x)} dx \\
&\quad + \frac{1}{p_m^*} \int_{\Omega} \lambda |u_n|^{z(x)} dx + \frac{1}{p_m^*} \int_{\Omega} c |u_n|^{p^*(x)} dx \\
&\geq \int_{\Omega} A(x, \nabla u_n) dx - \frac{\eta}{p_m} \int_{\Omega} |u_n|^{p(x)} dx - \frac{\lambda}{z_m} \int_{\Omega} |u_n|^{z(x)} dx - \frac{c}{p_m^*} \int_{\Omega} |u_n|^{p^*(x)} dx \\
&\quad - \frac{1}{p_m^*} \int_{\Omega} p(x) A(x, \nabla u_n) dx + \frac{\eta}{p_m^*} \int_{\Omega} |u|^{p(x)} dx \\
&\quad + \frac{\lambda}{p_m^*} \int_{\Omega} |u|^{z(x)} dx + \frac{c}{p_m^*} \int_{\Omega} |u_n|^{p^*(x)} dx \\
&\geq \left(1 - \frac{p_S}{p_m^*}\right) \nu \int_{\Omega} \mu_1(x) |\nabla u_n|^{p(x)} dx \\
&\quad + \eta \left(\frac{1}{p_m^*} - \frac{1}{p_m}\right) \int_{\Omega} |u_n|^{p(x)} dx + \lambda \left(\frac{1}{p_m^*} - \frac{1}{z_m}\right) \int_{\Omega} |u_n|^{z(x)} dx.
\end{aligned}$$

Since $\frac{1}{p_m^*} - \frac{1}{p_m} < 0$, applying the Poincaré inequality in the form

$$\int_{\Omega} |u|^{p(x)} dx \leq C_P \rho_{p(\cdot)}^{\mu_1}(u),$$

we obtain

$$\begin{aligned}
M &> J(u_n) - \frac{1}{p_m^*} \langle J'(u_n), u_n \rangle \\
&\geq \left(\left(1 - \frac{p_S}{p_m^*}\right) \nu + C_P \eta \left(\frac{1}{p_m^*} - \frac{1}{p_m}\right) \right) \int_{\Omega} \mu_1(x) |\nabla u_n|^{p(x)} dx \\
&\quad + \lambda \left(\frac{1}{p_m^*} - \frac{1}{z_m}\right) \int_{\Omega} |u_n|^{z(x)} dx
\end{aligned}$$

(Cont. next page)

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$$\geq \left(\left(1 - \frac{p_S}{p_m^*} \right) \nu + C_P \eta \left(\frac{1}{p_m^*} - \frac{1}{p_m} \right) \right) \|u_n\|_W^{p_m} - \lambda \left(\frac{1}{z_m} - \frac{1}{p_m^*} \right) c_4 \|u_n\|_W^{z_S},$$

where in the last term we use the Poincaré inequality. We assume

$$\frac{1}{C_P} \left(1 - \frac{p_S}{p_m^*} \right) \left(\frac{1}{p_m} - \frac{1}{p_m^*} \right)^{-1} \nu > \eta.$$

Dividing by $\|u_n\|_W^{p_m}$ and passing to the limit as $n \rightarrow \infty$, we obtain that

$$\left(1 - \frac{p_S}{p_m^*} \right) \nu + C_P \eta \left(\frac{1}{p_m^*} - \frac{1}{p_m} \right) < 0.$$

This contradiction proves that the sequence $\{u_n\}$ must be bounded since $z_S \in [z_m, p_m^*)$.

Now, we have to establish the strong convergence of $\{u_n\}$ to $u \in E$. Hence the sequence $\{u_n\}$ is bounded, there is a $u \in E$ such that

$$u_m \xrightarrow[m \rightarrow \infty]{\text{weakly in } W_{1,0}^{p(\cdot)}(\Omega)} u.$$

By the compact embedding theorem for variable exponent Sobolev spaces (see Diening et al. (2011) and Fan (2005)), we have that

$$u_m \xrightarrow[m \rightarrow \infty]{L^{p(\cdot)}(\Omega)} u$$

and, due to the local nature of the critical exponent and the boundedness of Ω ,

$$u_m \xrightarrow[m \rightarrow \infty]{L^{p^*(\cdot)}(K)} u$$

for each compact set $K \subset \Omega$. Therefore, we obtain

$$\langle J'(u) - J'(u_n), u - u_n \rangle \xrightarrow[n \rightarrow \infty]{} 0.$$

Next, we can write

$$\begin{aligned} & \int_{\Omega} (a_i(x, \nabla u) - a_i(x, \nabla u_n)) (\nabla_i u - \nabla_i u_n) dx \\ &= \langle J'(u) - J'(u_n), u - u_n \rangle - \int_{\Omega} \eta (|u|^{p(x)-2} u - |u_n|^{p(x)-2} u_n) (u - u_n) dx \\ & \quad - \int_{\Omega} \lambda (|u|^{z(x)-2} u - |u_n|^{z(x)-2} u_n) (u - u_n) dx \\ & \quad - \int_{\Omega} c (|u|^{p^*(x)-2} u - |u_n|^{p^*(x)-2} u_n) (u - u_n) dx. \end{aligned}$$

By general Hölder inequality, we estimate

$$\begin{aligned}
& \left| \int_{\Omega} \eta \left(|u|^{p(x)-2} u - |u_n|^{p(x)-2} u_n \right) (u - u_n) dx \right| \\
& \leq \eta k_{\text{Hol}} \left\| u^{p(\cdot)-1} \right\|_{L^{q(\cdot)}(\Omega)} \|u - u_n\|_{L^{p(\cdot)}(\Omega)} + \eta k_{\text{Hol}} \left\| u_n^{p(\cdot)-1} \right\|_{L^{q(\cdot)}(\Omega)} \|u - u_n\|_{L^{p(\cdot)}(\Omega)} \\
& \leq \eta k_{\text{Hol}} \|u\|_{L^{p(\cdot)}(\Omega)}^{p_S-1} \|u - u_n\|_{L^{p(\cdot)}(\Omega)} + \eta k_{\text{Hol}} \|u_n\|_{L^{p(\cdot)}(\Omega)}^{p_S-1} \|u - u_n\|_{L^{p(\cdot)}(\Omega)}, \\
& \left| \int_{\Omega} \lambda \left(|u|^{z(x)-2} u - |u_n|^{z(x)-2} u_n \right) (u - u_n) dx \right| \\
& \leq k_{\text{Hol}} \|\lambda\|_{L^{s(\cdot)}(\Omega)} \left\| u^{z(\cdot)-1} \right\|_{L^{z(\cdot)}(\Omega)} \|u - u_n\|_{L^{p(\cdot)}(\Omega)} \\
& \quad + k_{\text{Hol}} \|\lambda\|_{L^{s(\cdot)}(\Omega)} \left\| u_n^{z(\cdot)-1} \right\|_{L^{z(\cdot)}(\Omega)} \|u - u_n\|_{L^{p(\cdot)}(\Omega)} \\
& \leq k_{\text{Hol}} \lambda \|1\|_{L^{s(\cdot)}(\Omega)} \|u\|_{L^{p(\cdot)}(\Omega)}^{z_S-1} \|u - u_n\|_{L^{p(\cdot)}(\Omega)} \\
& \quad + k_{\text{Hol}} \lambda \|1\|_{L^{s(\cdot)}(\Omega)} \|u_n\|_{L^{p(\cdot)}(\Omega)}^{z_S-1} \|u - u_n\|_{L^{p(\cdot)}(\Omega)},
\end{aligned}$$

where $\frac{1}{s(x)} + \frac{1}{z(x)} + \frac{1}{p(x)} = 1$, and similarly, we have

$$\begin{aligned}
& \left| \int_{\Omega} c \left(|u|^{p^*(x)-2} u - |u_n|^{p^*(x)-2} u_n \right) (u - u_n) dx \right| \\
& \leq k_{\text{Hol}} c \left\| u^{p^*(\cdot)-1} \right\|_{L^{\frac{p^*(\cdot)}{p^*(\cdot)-1}}(\Omega)} \|u - u_n\|_{L^{p^*(\cdot)}(\Omega)} \\
& \quad + k_{\text{Hol}} c \left\| u_n^{p^*(\cdot)-1} \right\|_{L^{\frac{p^*(\cdot)}{p^*(\cdot)-1}}(\Omega)} \|u - u_n\|_{L^{p^*(\cdot)}(\Omega)} \\
& \leq k_{\text{Hol}} c \|u\|_{L^{p^*(\cdot)}(\Omega)}^{p_S^*-1} \|u - u_n\|_{L^{p^*(\cdot)}(\Omega)} \\
& \quad + k_{\text{Hol}} c \|u_n\|_{L^{p^*(\cdot)}(\Omega)}^{p_S^*-1} \|u - u_n\|_{L^{p^*(\cdot)}(\Omega)}.
\end{aligned}$$

Thus, we conclude

$$\begin{aligned}
& \lim_{n \rightarrow \infty} \int_{\Omega} \eta \left(|u|^{p(x)-2} u - |u_n|^{p(x)-2} u_n \right) (u - u_n) dx = 0 \\
& \lim_{n \rightarrow \infty} \int_{\Omega} \lambda \left(|u|^{z(x)-2} u - |u_n|^{z(x)-2} u_n \right) (u - u_n) dx = 0 \\
& \lim_{n \rightarrow \infty} \int_{\Omega} c \left(|u|^{p^*(x)-2} u - |u_n|^{p^*(x)-2} u_n \right) (u - u_n) dx = 0.
\end{aligned}$$

So, we have

$$\lim_{n \rightarrow \infty} \int_{\Omega} (a_i(x, \nabla u) - a_i(x, \nabla u_n)) (\nabla_i u - \nabla_i u_n) dx = 0$$

3. The mountain-pass theorem for an elliptic problem with nonstandard growth

and

$$\lim_{n \rightarrow \infty} \int_{\Omega} |\nabla u - \nabla u_n|^{p(x)} dx = 0,$$

this yields $\lim_{n \rightarrow \infty} \|\nabla u - \nabla u_n\|_{L^{p(\cdot)}(\Omega)} = 0$. The lemma is proven. $\square$

Now, we will prove the following lemma.

Lemma 4 (mountain-pass) – *Let $J \in C^1(E, \mathbb{R})$. Then, we have that:*

1. *there is Λ and $r \in (0, 1)$ such that $J(u) \geq \Lambda$ for all $u \in W$ such that $\|u\|_W = r$;*
2. *there is $w \in W_{1,0}^{p(\cdot)}(\Omega)$ such that $\lim_{\tau \rightarrow \infty} J(\tau w) = -\infty$.*

Proof. 1. By the Poincaré inequality, for each $u \in W_{1,0}^{p(\cdot)}(\Omega)$ such that $\|u\|_W = r$ with $r \in (0, 1)$, we estimate

$$\begin{aligned} J(u) &\geq \nu \int_{\Omega} \mu_1(x) |\nabla u|^{p(x)} dx \\ &\quad - \frac{\eta}{p_m} \int_{\Omega} |u|^{p(x)} dx - \frac{1}{z_m} \int_{\Omega} \lambda |u|^{z(x)} dx - \frac{1}{p_m^*} \int_{\Omega} c |u|^{p^*(x)} dx \\ &\geq \nu \rho_{p(\cdot)}^{\mu_1}(u) - \frac{\eta}{p_m} C_P \rho_{p(\cdot)}^{\mu_1}(u) - \frac{1}{z_m} k_{\text{Hol}} \lambda \|1\|_{L^{s(\cdot)}(\Omega)} \|u\|_{L^{p(\cdot)}(\Omega)}^{z_m} - \frac{1}{p_m^*} c \|u_n\|_{L^{p^*(\cdot)}(\Omega)}^{p_m^*} \\ &\geq \left(\nu - \frac{\eta}{p_m} C_P \right) \|u\|_W^{p_S} - \frac{1}{z_m} C_P^{z_m} k_{\text{Hol}} \lambda \|1\|_{L^{s(\cdot)}(\Omega)} \|u\|_W^{z_m} - \frac{1}{p_m^*} C_P^{p_m^*} c \|u_n\|_W^{p_m^*}. \end{aligned}$$

Assume $\lambda = \kappa c$ with a given positive coefficient $c > 0$. Then, for sufficiently small $r > 0$, we can choose c and κ such that the expression in brackets is positive, so that $J(u) \geq c_{\kappa} r^{p_S}$ for some positive constant c_{κ} .

2. We assume $w \in C_0^{\infty}(\Omega)$, $w \geq 0$, $w \neq 0$ and $\tau > 0$, and we have

$$\begin{aligned} J(\tau w) &= \int_{\Omega} A(x, \tau \nabla w) dx \\ &\quad - \int_{\Omega} \left(\eta \tau^{p(x)} \frac{|w|^{p(x)}}{p(x)} + \lambda \tau^{z(x)} \frac{|w|^{z(x)}}{z(x)} \right) dx - \int_{\Omega} c \tau^{p^*(x)} \frac{|w|^{p^*(x)}}{p^*(x)} dx \\ &\leq \int_{\Omega} (\gamma_1 \tau |\nabla w| + \mu_1 \tau^{p(x)} |\nabla w|^{p(x)}) dx \\ &\quad - \eta \frac{\tau^{p_m}}{p_S} \int_{\Omega} |w|^{p(x)} dx - \lambda \frac{\tau^{z_m}}{z_S} \int_{\Omega} |w|^{z(x)} dx - c \frac{\tau^{p_m^*}}{p_S^*} \int_{\Omega} |w|^{p^*(x)} dx \\ &\leq \tau \int_{\Omega} \gamma_1 |\nabla w| dx + \tau^{p_S} \int_{\Omega} \mu_1 |\nabla w|^{p(x)} dx - \eta \frac{\tau^{p_m}}{p_S} \int_{\Omega} |w|^{p(x)} dx \end{aligned}$$

(Cont. next page)

$$-\lambda \frac{\tau^{z_m}}{z_S} \int_{\Omega} |w|^{z(x)} dx - c \frac{\tau^{p_m^*}}{p_S^*} \int_{\Omega} |w|^{p^*(x)} dx \xrightarrow{\tau \rightarrow \infty} -\infty$$

since the term $\tau^{p_m^*}$ becomes dominant. $\square$

Lemma 5 – Let $J \in C^1(E, \mathbb{R})$ and satisfy Palais-Smale condition. Let $J(0) = 0$. Then, there is $r > 0$ and $\phi \in W$ such that

$$\|\phi\|_W > r, \quad J(\phi) \leq J(0), \quad \text{and} \quad \Lambda = \inf \{J(u) : u \in W, \|u\| = r\} > 0.$$

Proof. By the previous lemma, there exists $w \in W_{1,0}^{p(\cdot)}(\Omega)$ such that

$$\lim_{\tau \rightarrow \infty} J(\tau w) = -\infty.$$

Thus, we can choose $\tilde{\tau} > 0$ such that $\|\tilde{\tau} w\|_W > r$ and $J(\tilde{\tau} w) \leq 0$. Now, we take $\phi = \tilde{\tau} w$ and we obtain $\|\phi\|_W > r$ and $J(\phi) \leq 0$. The lemma is proven. $\square$

Theorem 1 (Mountain-Pass theorem) – Let $J \in C^1(E, \mathbb{R})$ satisfy the Palais-Smale condition. Let there are $r > 0$ and $u \in E$ such that

1. $\|u\| > r$ and $J(u) \leq J(0) = 0$;
2. $\Lambda = \inf \{J(u) : u \in W, \|u\| = r\} > 0$.

Then, we obtain $\Upsilon = \inf_{g \in \Gamma} \sup_{\tau \in [0, 1]} \{J(g(\tau))\} \geq \Lambda$ and Υ is a critical value of the functional J .

Proof. By previous lemmas, we can consider $g \in C([0, 1], E)$ given by $g \stackrel{\text{def}}{=} \tau \phi$ for all $\tau \in [0, 1]$ then $g \in \Gamma$ and $g \neq 0$. The theorem has been proven. $\square$

Theorem 2 (existence of solution) – Let the assumptions of Section 2 be satisfied. Then, there exists a nontrivial solution to the problem (1).

Proof. Applying lemmas, we can choose $u \in E$ such that $\|u\| > r$ thus every path $g \in \Gamma$ intersects the sphere $\|u\| = r$, then

$$0 < \Lambda \leq \inf_{\|u\|=r} \{J(u)\} \leq \Upsilon,$$

where $\Upsilon = \inf_{g \in \Gamma} \sup_{\tau \in [0, 1]} \{J(g(\tau))\}$. Then, there exists a sequence $\{u_n\} \subset E$ such that $J'(u_n) \xrightarrow{n \rightarrow \infty} J'(u)$ and $J'(u) = 0$. By lemmas, we deduce that $J(u) > 0 = J(0)$ so the function u is a nontrivial weak solution to (1). The theorem is proven. $\square$

4. The main result of infinite numbers of solutions

4 The main result of infinite numbers of solutions

Since we will use the Fountain theorem (see M. Willem, Theorem 3.6) we recall the intersection theorem, on which it is based. We will use notation $\|\cdot\|_E \equiv \|\cdot\|$.

Theorem 3 (Intersection theorem) – We denote for all $r_k > \gamma_k > 0$

$$B_k = \{u \in Y_k : \|u\| \leq r_k\} \quad \text{and} \quad N_k = \{u \in Z_k : \|u\| = r_k\}.$$

Let $\psi \in C(B_k, E)$ be equivariant and $\psi|_{\partial B_k} = id$. Then the intersection $\psi(B_k) \cap N_k$ is not empty.

Proof. The continuous equivariant mapping

$$\partial\{u \in B_k : \|\psi(u)\| < r_k\} \rightarrow Y_{k-1} \approx V^{k-1} : u \mapsto P_k(\psi(u))$$

possesses zero, where the operator P_k is the projector on Y_{k-1} such that $P_k Z_k = \{0\}$. Thus, we obtain $\psi(B_k) \cap N_k \neq \emptyset$. $\square$

Theorem 4 (Fountain theorem) – Let $J \in C^1(E, \mathbb{R})$ satisfy the Palais-Smale condition, and spaces X_k, Y_k, Z_k be defined as above. If there are numbers $r_k > \gamma_k > 0$ such that

1. $\inf_{u \in Z_k, \|u\|=\gamma_k} \{J(u)\} \xrightarrow{k \rightarrow \infty} \infty;$
2. $\max_{u \in Y_k, \|u\|=r_k} \{J(u)\} \leq 0.$

Then, there exists an unbounded sequence $\{u_k\}$ of critical values of functional J .

Now, we are ready to present the main result of the paper.

Theorem 5 (existence of infinity of solutions) – Let the assumptions of Section 2 be satisfied. Then, the Dirichlet problem (1) has infinitely many nontrivial solutions, more precisely, there exists a sequence $\{u_k\}$ of solutions such that $J(u_k) \xrightarrow{k \rightarrow \infty} \infty$.

Proof. We define $\beta_k = \sup_{u \in Z_k, \|u\|=1} \|u\|_W$. By the compact embedding of W into $L^{p(\cdot)}(\Omega)$ and the fact that Z_k is a subspace of finite codimension, we have $\beta_k \rightarrow 0$ as $k \rightarrow \infty$ (see Willem (1996) for details). Now, for $u \in Z_k$ with $\|u\| = \gamma_k$, we estimate using the Poincaré inequality and the definition of β_k :

$$J(u) \geq \nu \|u\|_W^{p_S} - C_1 \|u\|_W^{z_m} - C_2 \|u\|_W^{p_m^*} \geq \nu \beta_k^{p_S} \gamma_k^{p_S} - C_1 \gamma_k^{z_m} - C_2 \gamma_k^{p_m^*}.$$

Choosing $\gamma_k = \beta_k^{-1}$, we obtain

$$J(u) \geq \nu \beta_k^{p_S-1} - C_1 \beta_k^{-z_m} - C_2 \beta_k^{-p_m^*} \rightarrow \infty \quad \text{as } k \rightarrow \infty,$$

since $p_S > 1$ and $\beta_k \rightarrow 0$. This verifies condition (1) of the Fountain theorem.

For condition (2), on Y_k (which is finite-dimensional), all norms are equivalent. We can choose r_k sufficiently large such that for $u \in Y_k$ with $\|u\| = r_k$, the term involving c (the critical exponent term) dominates and forces $J(u) \leq 0$. Specifically, using the equivalence of norms and the growth condition on A , we have

$$J(u) \leq C_3 \|u\|^{p_S} - c \|u\|^{p_m^*} \leq 0$$

for $\|u\| = r_k$ large enough, since $p_m^* > p_S$.

Thus, by the Fountain theorem, there exists an unbounded sequence of critical values $J(u_k) \rightarrow \infty$, and consequently infinitely many nontrivial solutions to problem (1). $\square$

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