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Theoretical and numerical analysis of the Minimal Residual Method for solving the neutron transport equation in spherical geometry

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Abstract

This paper introduces an infinite-dimensional adaptation of the Minimal Residual method for solving the neutron transport equation in spherical geometry. The method is based on a novel splitting strategy of the collision operator, designed to account for the distinct characteristics of the transport operator. We provide both theoretical and numerical analyses of the algorithm, demonstrating its convergence and computational efficiency. Compared to previous approaches Tizaoui 2007b, 2009, our method offers improved accuracy and a significant reduction in computational cost, particularly for large-scale systems. These results underline the potential of the MR method in solving complex transport problems, with applications in nuclear physics and engineering.

Keywords: Neutron transport equation, Integro-differential operators, Splitting, Minimal Residual method.

MSC: 82D75, 65-XX, 65Yxx, 65Bxx.

1 Introduction

Neutron transport problems are crucial in many industrial and scientific applications, such as nuclear reactor simulations, where they are traditionally solved using diffusion equations Brisbois et al. 1974; Cox, Harris, and E. L. Horton 2019; E. Horton, Kyprianou, and Villemonais 2020; Larsen and Morel 1978; Mohanakrishnan, Singh, and Umasankari 2021. However, these approaches, while computationally efficient, do not capture the directional behavior of neutrons accurately Akesbi and Lesaint 1995; Duderstadt and Hamilton 1976; Tizaoui 2007b, 2009. To model

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neutron populations in complex geometries, particularly spherical geometry, the transport equation — an integro-differential equation — must be solved. The numerical resolution of this equation has been a subject of extensive study Anli et al. 2017; Kophazi and Turzo 2018; Lewis and Miller 1993; Mokhtar-Kharroubi 1993; Siewert and Grandjean 1979, with iterative algorithms playing a central role in improving computational efficiency.

Direct discretization of the transport equation leads to large, nearly full linear systems due to the integral operator's contribution, making it computationally expensive. To overcome this challenge, iterative methods aim to separate the integral term and solve a hyperbolic partial differential equation at each iteration. Despite their advantages, these methods face two significant difficulties:

- Slow convergence of the iterative algorithms, especially when solving large-scale problems Akesbi and Lesaint 1995; Manteuffel 1980.

- A large number of unknowns in the discretized system, which can be overwhelming even with fine angular resolution.

To address these issues, a splitting strategy for the collision operator Tizaoui 2007b has been proposed, leading to algorithms such as the Jacobi and Gauss-Seidel methods in infinite dimensions Tizaoui 2009. Studies have shown that the Gauss-Seidel method converges faster than the Jacobi method and performs at least as well as traditional methods like the diffusion synthetic acceleration (DSA) method Akesbi and Lesaint 1995; Anli et al. 2017; Kophazi and Turzo 2018; Larsen and Morel 1978.

However, due to the non-self-adjoint nature of the operators involved in the transport equation, conventional methods like the conjugate gradient algorithm cannot be directly applied. In response, we explore the adaptation of the Minimal Residual method, which is particularly suited to non-self-adjoint systems. This approach aims to accelerate convergence while remaining independent of the discretization method used Bell and Glasstone 1970; Duderstadt and Hamilton 1976; Siewert and Grandjean 1979.

This paper is structured as follows: Section 2 introduces the mathematical model of the neutron population and defines the functional spaces required to ensure well-posed boundary conditions and the existence of unique solutions. Section 3 revisits the splitting method for the collision operator Tizaoui 2007b, 2017. Section 4 presents the infinite-dimensional adaptation of the Minimal Residual method based on the splitting strategy and provides a theoretical analysis of its convergence and efficiency. Section 5 presents the numerical results, comparing the proposed method with existing approaches Tizaoui 2007b, 2009.

2 Mathematical Model

The behavior of neutrons within a nuclear reactor core is determined by their transport and interactions with atomic nuclei. These interactions—comprising scattering, absorption, and fission—are fundamental to sustaining a controlled chain reaction. Consequently, solving the neutron transport equation in spherical geometry is critical for the design, analysis, and safety assessment of nuclear reactors Ahmet and Fikret 2015; Bussac and Reuss 1978; Khattab and Larsen 1991; Tizaoui 2007b, 2009, 2017. The neutron transport equation in spherical coordinates is given by

$$\frac{\mu}{r^2}\frac{\partial}{\partial r}\left(r^2 u(r,\mu)\right) + \frac{1}{r}\frac{\partial}{\partial \mu}\left[\left(1-\mu^2\right)u(r,\mu)\right] + \sigma u(r,\mu) = \int_{-1}^{1} k(\mu,\mu')u(r,\mu')d\mu' + S(r,\mu), \tag{1}$$

for all $(r,\mu) \in (0,R)\times(-1,1)$, with the boundary condition

$$u(R,\mu) = 0 \quad \forall \mu < 0. \tag{2}$$

Here, r is the radial distance from the center of the spherical domain, μ is the cosine of the angle between the neutron velocity and the radial vector, σ is the scattering cross-section (assumed to be constant), $k(\mu,\mu')$ is the scattering kernel, and $S(r,\mu)$ is a given non-negative source term in $L^2(\Omega)$. The equation can also be written in a non-conservative form

$$\mu\frac{\partial u}{\partial r}(r,\mu) + \frac{1-\mu^2}{r}\frac{\partial u}{\partial \mu}(r,\mu) + \sigma u(r,\mu) = \int_{-1}^{1} k(\mu,\mu')u(r,\mu')d\mu' + S(r,\mu). \tag{3}$$

To facilitate analysis and numerical approximation, the problem (1)–(2) is reformulated as an operator equation. Let $\Omega = (0,R)\times(-1,1)$. The goal is to find the neutron flux $u : \Omega \to \mathbb{R}^+$ satisfying

$$\begin{cases} Tu(r,\mu) = Ku(r,\mu) + S(r,\mu), & \text{for all } (r,\mu) \in \Omega, \\ u \in \mathcal{W} \end{cases} \tag{4}$$

where

- T is the transport operator, defined by

$$Tu(r,\mu) = \frac{\mu}{r^2}\frac{\partial}{\partial r}\left(r^2 u(r,\mu)\right) + \frac{1}{r}\frac{\partial}{\partial \mu}\left[\left(1-\mu^2\right)u(r,\mu)\right] + \sigma u(r,\mu),$$

- K is the integral collision operator with a positive kernel k, given by

$$Ku(r,\mu) = \int_{-1}^{1} k(\mu,\mu')u(r,\mu')d\mu',$$

- $S(r,\mu)$ is the source term.
- The solution u belongs to the function space

$$\mathcal{W} := \Big\{ u \in L^2(\Omega) \;\Big|\; \frac{\mu}{r^2}\frac{\partial}{\partial r}(r^2 u(r,\mu)) + \frac{1}{r}\frac{\partial}{\partial \mu}[(1-\mu^2)u(r,\mu)] \in L^2(\Omega), \\ \text{and } u(R,\mu) = 0 \;\forall \mu < 0 \Big\}. \tag{5}$$

The transport operator T represents the movement and interaction of neutrons within the domain, while the collision operator K accounts for scattering events. The boundary condition $u(R,\mu) = 0$ ensures that no neutrons enter the domain from the outer boundary for directions where $\mu < 0$. This mathematical framework allows for a detailed analysis of neutron behavior in nuclear reactors and serves as the foundation for further numerical and theoretical studies.

We make the following hypothesis:

(H_1) $\rho(\Theta) < \dfrac{c}{2}$, with $\Theta = T^{-1}K$ and $0 < c \leq 1$, where ρ designates the spectral radius.
(H_2) k is nonnegative and bounded function.

Remark 1 – The operator $\mathcal{L}$ defined by

$$\mathcal{L} := \frac{\mu}{r^2}\frac{\partial}{\partial r}\left(r^2 u(r,\mu)\right) + \frac{1}{r}\frac{\partial}{\partial \mu}\left[\left(1-\mu^2\right)u(r,\mu)\right]$$

is defined, then T^{-1} exists and the operator $T^{-1}K$ is compact Mokhtar-Kharroubi 1993; Tizaoui 2017.

3 Splitting Method

In the previous section, we laid the foundation for the neutron transport problem by defining the transport operators on the domains Ω_1 and Ω_2. To solve this problem efficiently, we introduce a splitting method, which involves decomposing the transport operator into several simpler operators. This approach allows us to treat the problem more effectively by separating it into coupled subproblems, thus facilitating its numerical resolution. Below, we provide a detailed explanation of this splitting method and its application to our problem. We define the neutron transport operator T_1 (respectively T_2) on the domains $\Omega_1 = (0,R)\times(0,1)$ (respectively $\Omega_2 = (0,R)\times(-1,0)$). Let K_{ij}, for $i,j \in \{1,2\}$, be the integral operators whose kernels are defined as:

$$k_{ij}(r,\mu,\mu') = k(\mu,\mu') \times \mathbb{1}_{\Omega_i}(r,\mu) \times \mathbb{1}_{\Omega_j}(r,\mu'),$$

where $\mathbb{1}_{\Omega_i}$ is the indicator function of Ω_i, $i \in \{1,2\}$. From this, we deduce that

$$K_{ij}(u) = K\left(u \times \mathbb{1}_{\Omega_j}\right) \times \mathbb{1}_{\Omega_i}, \quad i \in \{1,2\}.$$

Thus, we obtain a splitting of the integral operator K in the form:

$$K = \sum_{i=1}^{2} \sum_{j=1}^{2} K_{ij}.$$

Note that K_{ij} is an operator acting on $L^2(\Omega)$, using only the values of u on Ω_j, such that $K_{ij}u$ has its support in Ω_i. The problem (4) is then split into two coupled problems, defined respectively on Ω_1 and Ω_2, with solutions u_1 and u_2. This method involves adjusting boundary conditions on $\Gamma = (0,R) \times \{0\}$ (see Tizaoui 2007b). The solution u of (4) is then given by the form $u = (u_1, u_2)$, where u_1 and u_2 are the solutions of the coupled system

$$\begin{pmatrix} T_1 - K_{11} & -K_{12} \\ -K_{21} & T_2 - K_{22} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} S_1 \\ S_2 \end{pmatrix}, \tag{6}$$

with $S_i = S \times \mathbb{1}_{\Omega_i}$ for $i \in \{1,2\}$.
The idea in Tizaoui 2009 is to introduce and study various algorithms, based on a splitting of the collision operator, adapted from the Jacobi and Gauss-Seidel methods. We aim to develop a method that ensures a good convergence rate without requiring any additional parameter calculation.
Let $\theta_{ij} = T_i^{-1} K_{ij}$ and $\widetilde{S}_i = T_i^{-1} S_i$, where $i,j \in \{1,2\}$. The system (6) can then be written as

$$\begin{pmatrix} I - \theta_{11} & \theta_{12} \\ \theta_{21} & I - \theta_{22} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \widetilde{S}_1 \\ \widetilde{S}_2 \end{pmatrix}. \tag{7}$$

Applying the diagonal preconditioning method to this system, we obtain

$$\underbrace{\begin{pmatrix} I & -(I-\theta_{11})^{-1}\theta_{12} \\ -(I-\theta_{22})^{-1}\theta_{21} & I \end{pmatrix}}_{\mathcal{A}} \underbrace{\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}}_{X} = \underbrace{\begin{pmatrix} -(I-\theta_{11})^{-1}\widetilde{S}_1 \\ -(I-\theta_{22})^{-1}\widetilde{S}_2 \end{pmatrix}}_{\mathcal{B}}. \tag{8}$$

Remark 2 – For all $i \in \{1,2\}$, we have $\|\theta_{ij}\|_2 = \|T_i^{-1}K_{ij}\|_2 \leq \rho\left(\theta_{ij} = T_i^{-1}K_{ij}\right)$. Then

$$\| \theta_{ij} \|_2 \leq \frac{c}{2} \text{ for } (i,j) \in \{1,2\} \times \{1,2\}. \tag{9}$$

Remark 3 – From (9), we deduce

$$\|(I-\theta_{ii})^{-1}\theta_{ij}\| \leq \|(I-\theta_{ii})^{-1}\| \times \|\theta_{ij}\|_2,$$

which leads to

$$\|(I-\theta_{ii})^{-1}\,\theta_{ij}\| \leq \frac{c}{2}\sum_{p=0}^{+\infty}\|\theta_{ii}^{p}\|_2 \leq \frac{c}{2}\frac{1}{1-\frac{c}{2}} = \frac{c}{2-c} =: d. \tag{10}$$

The implicit definition of θ_{ij} through the operator equation is consistent with the explicit form used for the majoration. This explicit form is derived from the underlying system and is crucial for bounding the terms as shown in this remark. Both the implicit and explicit definitions of θ_{ij} are fully compatible and provide a consistent framework for the analysis presented.

Remark 4 – The operator matrix $\mathcal{A}$ is invertible, as the operators $I-\theta_{11}$ and $I-\theta_{22}$ are nonsingular, as shown in the previous remarks. This guarantees the invertibility of $\mathcal{A}$.

4 Minimal Residual Algorithm

The Minimal Residual algorithm is a widely used iterative method for solving large, sparse systems of equations, which are common in many scientific and engineering applications. It is particularly effective for symmetric indefinite systems, where traditional direct methods may be computationally expensive. By minimizing the residual vector at each step, this algorithm progressively refines the solution, making it an efficient tool for large-scale numerical simulations.

4.1 Benchmark or Algorithm Overview

In this subsection, we present an overview of the Minimal Residual algorithm, highlighting its key features, such as its ability to solve large, sparse systems efficiently and its applicability in fields such as computational fluid dynamics and structural simulations. The algorithm's main advantage is its iterative nature, which allows for the progressive improvement of the solution while keeping computational costs low.

The Minimal Residual method, introduced by Axelsson 1996, is an iterative technique designed to solve symmetric indefinite systems by minimizing the residual vector at each iteration. This method is especially useful for large-scale problems where direct methods are too costly or impractical. The algorithm is defined as follows to solve the system (8)

1. Start by selecting X^0, calculating the initial residual $r^0 = \mathcal{B} - \mathcal{A}X^0$, and setting $p^0 = r^0$, $q^0 = \mathcal{A}p^0$.

2. For $k = 0, 1, \ldots$ until $\|r^k\|_2 < \varepsilon$, perform the following steps:

$$\begin{aligned}
&\alpha^k = \frac{\langle r^k, q^k \rangle}{\langle q^k, q^k \rangle}, \\
&X^{k+1} = X^k + \alpha^k p^k, \\
&r^{k+1} = r^k - \alpha^k q^k, \\
&\beta^{k+1} = -\frac{\langle \mathcal{A} r^{k+1}, q^k \rangle}{\langle q^k, q^k \rangle}, \\
&p^{k+1} = r^{k+1} + \beta^{k+1} p^k, \\
&q^{k+1} = \mathcal{A} r^{k+1} + \beta^{k+1} q^k.
\end{aligned}$$

This algorithm aims to minimize the function

$$\mathcal{E}(X) = \|\mathcal{B} - \mathcal{A}X\|_2^2,$$

which represents the squared error between the current approximation of the solution and the exact solution. By iteratively improving the approximation of X^k, the algorithm seeks to reduce this error and converge to the true solution.

4.2 Convergence of the Minimal Residual Algorithm

In this section, we examine the convergence properties of the Minimal Residual algorithm. To this end, we first establish some essential notation and preliminary results. Let $\langle \cdot, \cdot \rangle$ denote the standard inner product in $L^2(\Omega) \times L^2(\Omega)$, and let $\| \cdot \|_2$ represent the corresponding $L^2(\Omega)$-norm. With this notation in place, we focus on the iterative solution of the system of equations. The Minimal Residual method is widely regarded as one of the most efficient iterative techniques for solving large-scale systems, particularly those involving symmetric indefinite matrices. By minimizing the norm of the residual at each iteration, the algorithm achieves progressive refinement of the approximate solution, making it highly effective for tackling computationally challenging problems.

Proposition 1 – *Let X^k be constructed by the algorithm described above, starting from X^0. For $k \geq 0$, the following estimate for the residual holds:*

$$\mathcal{E}(X^{k+1}) \leq \mathcal{E}(X^k) \left(1 - \frac{\langle r^k, \mathcal{A} r^k \rangle}{\langle r^k, r^k \rangle} \frac{\langle r^k, \mathcal{A} r^k \rangle}{\langle \mathcal{A} r^k, \mathcal{A} r^k \rangle} \right). \tag{11}$$

Proposition 2 – *Under the assumptions (H_1) and (H_2), for all X, the operator $\mathcal{A}$ satisfies*

$$\langle \mathcal{A}X, X \rangle \geq (1 - d) \langle X, X \rangle \quad \text{and} \quad \langle \mathcal{A}X, X \rangle \geq \frac{1}{1 + d} \langle \mathcal{A}X, \mathcal{A}X \rangle, \tag{12}$$

where $d = \dfrac{c}{2 - c}$.

Proof. Let $X = (u_1, u_2)$. We have

$$\langle \mathcal{A}X, X\rangle = \|X\|_2^2 - \langle (I-\theta_{11})^{-1}\theta_{12}u_2, u_1\rangle - \langle (I-\theta_{22})^{-1}\theta_{21}u_1, u_2\rangle.$$

Using (10), we obtain

$$\langle \mathcal{A}X, X\rangle \geq \left(1 - \frac{c}{2-c}\right)\|X\|_2^2,$$

thus proving the left-hand inequality in (12). To prove the right-hand inequality of (12), assume $\xi > 0$, and we have

$$\begin{aligned}\langle \mathcal{A}X, X\rangle - \xi\langle \mathcal{A}X, \mathcal{A}X\rangle &= (1-\xi)\|X\|_2^2 \\ &\quad - (1-2\xi)\Big[\langle (I-\theta_{11})^{-1}\theta_{12}u_2, u_1\rangle + \langle (I-\theta_{22})^{-1}\theta_{21}u_1, u_2\rangle\Big] \\ &\quad - \xi\Big[\langle (I-\theta_{22})^{-1}\theta_{21}u_1, (I-\theta_{11})^{-1}\theta_{21}u_1\rangle \\ &\qquad + \langle (I-\theta_{11})^{-1}\theta_{12}u_2, (I-\theta_{11})^{-1}\theta_{12}u_2\rangle\Big]. \end{aligned} \tag{13}$$

Choosing $\xi > \frac{1}{2}$, and using (10), we derive the inequality

$$\langle \mathcal{A}X, X\rangle - \xi\langle \mathcal{A}X, \mathcal{A}X\rangle \geq \left(1 + d - \xi(1+d)^2\right)\|X\|_2^2.$$

By selecting ξ such that $1 + d - \xi(1+d)^2 = 0$, we obtain $\xi = \frac{1}{1+d}$. Therefore, the right-hand inequality of (12) is also proven. □

The next theorem illustrates the convergence of the Minimal Residual algorithm and how $\mathcal{A}$ can occur.

Theorem 1 – *Under the assumptions* (H_1) *and* (H_2)*, the Minimal Residual method converges, and the residual decays at least at the following exponential rate*

$$\mathcal{E}(X^k) \leq c^k \mathcal{E}(X^0). \tag{14}$$

Proof. By inserting the inequalities of Proposition 2 into (11), we get

$$\mathcal{E}(X^{k+1}) \leq c\,\mathcal{E}(X^k), \tag{15}$$

and consequently (14). Since $c \in (0,1)$, then $\mathcal{E}(X^k)$ converges towards 0 when k goes to infinity. Using the left-hand inequality of (12), we get

$$\|X^{k+1} - \mathcal{A}^{-1}\mathcal{B}\|_2^2 \leq \frac{1}{1-d}\langle \mathcal{A}X^{k+1} - \mathcal{B}, X^{k+1} - \mathcal{A}^{-1}\mathcal{B}\rangle; \tag{16}$$

so that

$$\|X^{k+1} - \mathcal{A}^{-1}\mathcal{B}\|_2 \leq \frac{1}{\sqrt{1-d}}\sqrt{\mathcal{E}(X^{k+1})}, \tag{17}$$

which means $X^{k+1} \to X$ with $\mathcal{A}X = \mathcal{B}$. Hence, the Minimal Residual method converges. □

5 Numerical results

In this section, we are mainly interested in solving the following neutron transport problem: Find $u : (0,R)\times(-1,1) \to \mathbb{R}^+$ such that

$$\begin{cases} \mu\frac{\partial u}{\partial r}(r,\mu) + \frac{1-\mu^2}{r}\frac{\partial u}{\partial \mu}(r,\mu) + \sigma u(r,\mu) = \dfrac{\sigma c}{2}\displaystyle\int_{-1}^{1} u(r,\mu')d\mu' + S(r,\mu), \\ u(R,\mu) = 0,\ \forall\ \mu < 0. \end{cases} \tag{18}$$

5.1 Discretization

The presence of spatial and angular derivatives in the studied equation imposes significant constraints on the choice of the numerical method. To ensure a consistent and efficient discretization while limiting the number of degrees of freedom for computational feasibility, we have adopted a specific scheme for each mesh element. This scheme is designed to preserve the fundamental properties of the continuous equation while meeting the requirements of simplicity and robustness in numerical processing. To this end, we define a suitable discrete space, denoted by V_h, which consists of functions whose restriction to each rectangle Ω_{ij} is expressed in the form $a + br + br\mu$, where a and b are real numbers. This structure effectively captures spatial and angular variations while maintaining computational complexity at a reasonable level. Let the integers $N \geq 1$ and $M \geq 1$. We discretize the intervals $(-1,1)$ and $(0,R)$ as follows

$$-1 = \mu_{-M} < \mu_{-M+1} < \cdots < \mu_0 = 0 < \mu_1 < \cdots < \mu_M = 1,$$

$$0 = r_0 < r_1 < r_2 < \cdots < r_{N-1} < r_N = R.$$

We adopt the following notations:

$$\overline{\Omega} = \bigcup_{i,j}\Omega_{ij}, \quad \text{where } \Omega_{ij} = [r_i, r_{i+1}]\times[\mu_j, \mu_{j+1}],$$

$$\tau = \mu_{j+1} - \mu_j, \quad \mu_{j+\frac{1}{2}} = \frac{\mu_j + \mu_{j+1}}{2},$$

$$h = r_{i+1} - r_i, \quad r_{i+\frac{1}{2}} = \frac{r_i + r_{i+1}}{2}.$$

The average values m_{ij} and $\gamma_{i,j}$ are defined as:

$$m_{ij} = \frac{1}{h\tau} \int_{r_i}^{r_{i+1}} \int_{\mu_j}^{\mu_{j+1}} u(r,\mu)\, d\mu\, dr,$$
$$\gamma_{i,j} = \frac{1}{\tau} \int_{\mu_j}^{\mu_{j+1}} u(r_i,\mu)\, d\mu,$$

for all $(i,j) \in [0, N-1] \times [0, M-1]$. It is straightforward to prove that:

$$m_{i,j} = \frac{1}{2}\left(\gamma_{i+1,j} + \gamma_{i,j}\right), \quad \forall (i,j) \in [0, N-1] \times [0, M-1]. \tag{19}$$

On the space V_h, the function $u(r,\mu)$ can be written as:

$$u(r,\mu) = \frac{2}{h}\left(r_{i+1} - \frac{r\mu}{\mu_{j+\frac{1}{2}}}\right) m_{i,j} - \frac{2}{h}\left(r_{i+1} - \frac{r}{\mu_{j+\frac{1}{2}}}\right) \gamma_{i+1,j}. \tag{20}$$

5.2 Resolution on Ω_2

In the processing on Ω_2, the values of $\gamma_{i+1,j}$ are known, so we will look for the values of $\gamma_{i,j}$ and m_{ij}. We consider the following algorithm: Find $u^{k+1} \in V$ such that

$$\mu \frac{\partial u^{k+1}}{\partial r}(r,\mu) + \frac{1-\mu^2}{r} \frac{\partial u^{k+1}}{\partial \mu}(r,\mu) + \sigma u^{k+1}(r,\mu) = \frac{\sigma c}{2} \int_{-1}^{0} u^{k+1}(r,\mu')\, d\mu'$$
$$+ \frac{\sigma c}{2} \int_{0}^{1} u^{k}(r,\mu')\, d\mu' + S(r,\mu). \tag{21}$$

This is equivalent to

$$\mu \frac{\partial u^{k+1}}{\partial r}(r,\mu) + \frac{1-\mu^2}{r} \frac{\partial u^{k+1}}{\partial \mu}(r,\mu) + \sigma u^{k+1}(r,\mu) = \frac{\sigma c}{2} \sum_{p=-M}^{p=-1} \int_{\mu_p}^{\mu_{p+1}} u^{k+1}(r,\mu')\, d\mu'$$
$$+ \frac{\sigma c}{2} \sum_{p=0}^{p=M-1} \int_{\mu_p}^{\mu_{p+1}} u^{k}(r,\mu')\, d\mu' + S(r,\mu). \tag{22}$$

By integrating (22) over Ω_{ij}, and dividing by $h\tau$, we get:

(Cont. next page)

$$\frac{1}{h\tau}\int_{r_i}^{r_{i+1}}\int_{\mu_j}^{\mu_{j+1}}\mu\frac{\partial u^{k+1}}{\partial r}(r,\mu)drd\mu+\frac{1}{h\tau}\int_{r_i}^{r_{i+1}}\int_{\mu_j}^{\mu_{j+1}}\frac{1-\mu^2}{r}\frac{\partial u^{k+1}}{\partial \mu}(r,\mu)drd\mu$$

$$+\sigma m_{ij}^{k+1}=\frac{\sigma c}{2}\sum_{p=-M}^{p=-1}m_{ip}^{k+1}+S_{ij}^{k}, \tag{23}$$

where

$$S_{ij}^{k}=\frac{\sigma c}{2}\sum_{p=0}^{p=M-1}m_{ip}^{k}+\frac{1}{h\tau}\int_{r_i}^{r_{i+1}}\int_{\mu_j}^{\mu_{j+1}}S(r,\mu)drd\mu.$$

Using the equations (19) and (20), the equality (23) becomes

$$\underbrace{\left(\sigma-\frac{2}{h\mu_{j+\frac{1}{2}}}\right)}_{C_j^{-1}>0}m_{ij}^{k+1}+\frac{2}{h\mu_{j+\frac{1}{2}}}\gamma_{i+1,j}^{k+1}=\frac{\sigma c}{2}\sum_{p=-M}^{p=-1}m_{ip}^{k+1}+S_{ij}^{k}.$$

Hence, we have for the expression m_{ij}^{k+1}

$$m_{ij}^{k+1}=\frac{\sigma c}{2}C_j\sum_{p=-M}^{p=-1}m_{ip}^{k+1}+C_j\left(-\frac{2}{h\mu_{j+\frac{1}{2}}}\gamma_{i+1,j}^{k+1}+S_{ij}^{k}\right). \tag{24}$$

We put

$$X_i^k=\sum_{p=-M}^{p=-1}m_{ip}^{k}$$
$$\widetilde{S}_{ij}^{k}=C_j\left(-\tfrac{2}{h\mu_{j+\frac{1}{2}}}\gamma_{i+1,j}^{k+1}+S_{ij}^{k}\right).$$

Hence, we have

$$m_{ij}^{k+1}=C_j\frac{\sigma c}{2}X_i^{k+1}+\tilde{S}_{i,j}^{k}. \tag{25}$$

Remember that all the terms $\tilde{S}_{i,j}^{k+1}$ are known.
Before determining the unknowns m_{ij}^{k+1}, we first propose the determination of the

auxiliary unknowns X_i^{k+1} by summing over the index j ($-M \leq j \leq -1$). We then deduce

$$\left(\underbrace{1 - \frac{\sigma c}{2} \sum_{j=-M}^{j=-1} C_j}_{\omega^{-1}} \right) X_i^{k+1} = \sum_{j=-M}^{j=-1} \widetilde{S}_{i,j}^k. \tag{26}$$

Consequently

$$X_i^{k+1} = \omega \sum_{j=-M}^{j=-1} \widetilde{S}_{i,j}^k. \tag{27}$$

For i fixed in $[0, N-1]$, equation (27) gives the values of X_i^{k+1}, then using (25), we determine the terms m_{ij}^{k+1}. Finally, we determine the values of $\gamma_{i,j}^{k+1}$ by the equality (19). The numerical resolution on the sub-domain Ω_2 is then completed.

5.3 Resolution on Ω_1

Similarly, for the resolution on Ω_1, we will first use the symmetry equality $u(0,\mu) = u(0,-\mu)$, which will allow us to know $\gamma_{0,j} = \gamma_{0,-j}$ for $j \geq 0$. Then, we proceed as on Ω_2 except that the direction of travel on the characteristics is opposite. This statement therefore means that the terms $\gamma_{i,j}$ are known and the terms $\gamma_{i+1,j}$ and m_{ij} are unknown. In this case, we take the expression of u in the following form

$$u(r,\mu) = \frac{2}{h}\left(-r_i + \frac{1}{\mu_{j+\frac{1}{2}}} r\mu\right) m_{ij} + \frac{2}{h}\left(r_{i+\frac{1}{2}} - \frac{1}{\mu_{j+\frac{1}{2}}} r\mu\right) \gamma_{i,j}.$$

For more details, the reader is referred to, for example, Tizaoui 2007b, 2017.

Remark 5 – The above numerical scheme can be generated for collision kernels of the form:

$$k(r,\mu,\mu') = C(r) \sum_{l=1}^{l=N_k} \alpha_l(\mu)\alpha_l(\mu')$$

with positive measurable and bounded function C and $N_k \in \mathbb{N}^\star$. One of the kernels taking this form is Thomson's kernel:

$$k(\mu,\mu') = \frac{9\sigma}{16}\left[\left(1 - \frac{\mu^2}{3}\right)\left(1 - \frac{\mu'^2}{3}\right) + \frac{8}{9}\mu^2\mu'^2\right].$$

5.4 Numerical Tests

For the numerical experiments presented, we consider the domain $\Omega = (0,1)\times(-1,1)$, with a spatial step size $h = \frac{1}{500}$ and a time step size $\tau = \frac{1}{10}$. Due to the computational cost and space limitations of this study, we have chosen not to perform excessive refinement of the angular variable in our discretization. We have ensured that the current level of refinement captures the essential features of the solution while maintaining the accuracy of the method. The source term is chosen in the following form.

$$S(r,\mu) = 1 + \mu + \sigma\left[\mu\left(r - e^{-\sigma r}\right) + (1-c)\left(1 + r + e^{-\sigma r}\right)\right].$$

For this choice of S, the exact analytical solution of the test is given by:

$$u(r,\mu) = 1 + r + r\mu + e^{-\sigma r}.$$

The boundary condition is specified as:

$$\varphi(1,\mu) := 2 + \mu + e^{-\sigma}, \quad \text{for all } \mu < 0.$$

For each iterative method tested, iterations are stopped when the following convergence criterion is met:

$$\frac{\|\varphi^{k+1} - \varphi^{k}\|_2}{\|\varphi^{k}\|_2} < 10^{-9}.$$

In the left-hand panel of Figure 1, we plot the exact solution, and in the right-hand panel, we plot the corresponding source term for $\sigma = 10$ and $c = 0.9$.

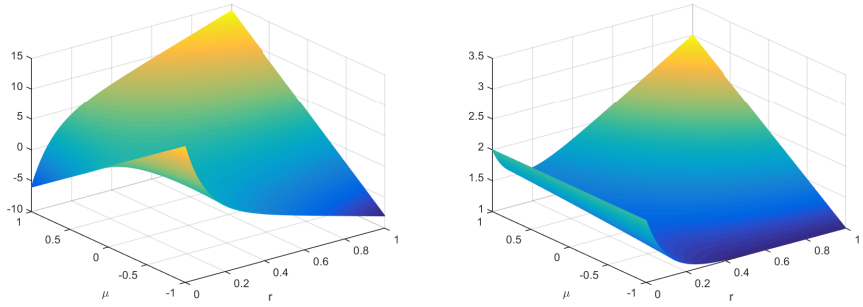

Figure 1 – On the left, the exact solution $u(r,\mu)$, and on the right, the source term for $\sigma = 10$ and $c = 0.9$.

There are two sets of tests: one for a fixed value of σ, and another for a fixed value of c. In each case, we compare the number of iterations required for convergence using the Minimal Residual algorithm and the Gauss-Seidel method Tizaoui 2009.

Figure 2 shows, as a function of c ($0 < c \leq 1$), the number of iterations required for convergence using the Gauss-Seidel method and the Minimal Residual Method, with σ fixed at 50. Figure 3 illustrates the number of iterations for convergence as a function of c near 1, with σ fixed at 50, for both the Gauss-Seidel and the Minimal Residual Methods. Figure 4 presents the number of iterations with respect to σ at a fixed value of $c = 0.98$, using the same example as in the previous figures.

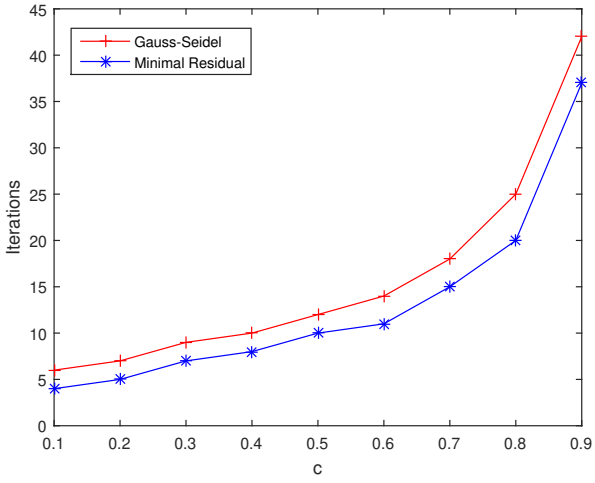

Figure 2 – Comparison of the iterations number at $\sigma = 50$ ($0 < c < 1$).

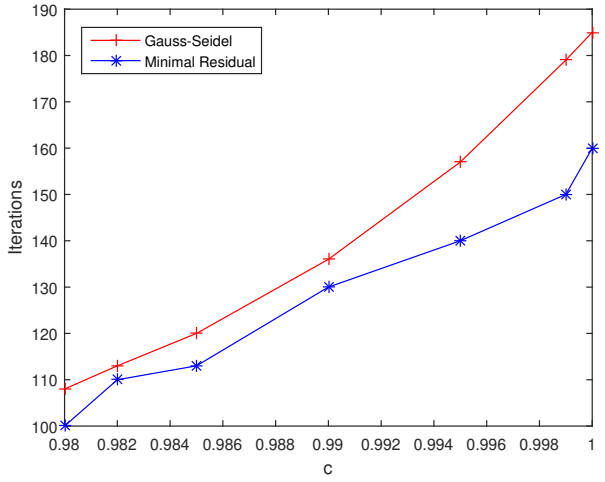

Figure 3 – Comparison of the number of iterations for $\sigma = 50$ and $c \approx 1$.

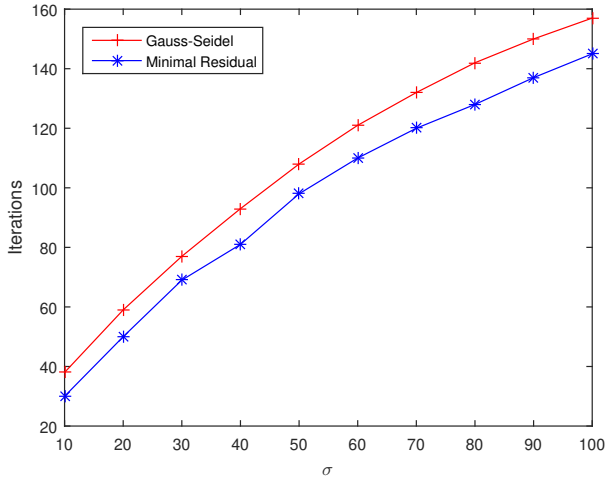

Figure 4 – Comparison of the iterations number $c = 0.98$.

Observing the numerical results, we deduce that the Minimal Residual algorithm is efficient compared to the Gauss-Seidel method, and consequently, it outperforms

the methods presented in Akesbi and Lesaint 1995; Tizaoui 2007b, 2009.

6 Conclusion

In the present work, an infinite-dimensional adaptation of the Minimal Residual method for solving the stationary neutron transport equation in spherical geometry is presented. This method offers a robust and efficient solution framework, with a theoretical proof of its convergence rate and numerical results illustrating its advantages over existing methods Akesbi and Lesaint 1995; Tizaoui 2007b, 2009. Notably, the convergence of the algorithm is shown to be independent of the discretization choice, highlighting its flexibility and robustness. Building on these results, ongoing efforts aim to accelerate the algorithm through preconditioning techniques Tizaoui 2007a, 2021 and to further validate its performance through extensive numerical experiments, paving the way for significant advancements in computational methods applied to neutron transport and their practical applications.

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