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Global well-posedness of homogeneous Boltzmann equation in modulation spaces

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Abstract

In this paper, we study the Cauchy problem of Boltzmann equation with soft potential in modulation spaces. Our aim is to obtain the global existence of solution to the space-homogeneous Boltzmann equation. To realize this goal, the boundedness of Boltzmann operator in modulation space is established. In addition, Banach fixed-point theorem is applied with careful estimate of time integral for the contraction mapping.

Keywords: Global well-posedness, Boltzmann equation, Modulation space.

MSC: 35Q20, 35A01, 42B35.

1 Introduction

In last decades, the study of kinetic models for granular flow[3] attracted many mathematicians. Depending on the external conditions (geometry, gravity, interactions with surface of a vessel), granular system may be in a variety of regimes, displaying typical features of solids, liquids or gases and also producing quite surprising effects[4]. In the case of rapid, dilute flows, the binary collisions between particles may be considered the main mechanism of inter-practice interactions in the system. In this paper, we study a model in the space homogeneous regime, described by the following equation:

$$\begin{cases} \partial_t f - \Delta_v f = Q(f,f)(v,t), \quad v \in \mathbb{R}^n, \quad t > 0, \\ f(v,0) = f_0(v), \end{cases} \tag{1}$$

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[3] Cercignani, 1995, "Recent developments in the mechanics of granular materials"; Jenkins, 1998, "Kinetic theory for nearly elastic spheres".

[4] Umbanhowar, Melo, and Swinney, 1996, "Localized excitations in a vertically vibrated granular layer".

where f is the distribution function of particle, depending only on the microscopic velocity v and time t, and $Q(f,f)$ the Boltzmann operator is defined in section 1.2.

Gamba, Panferov, and Villani[5] studied Boltzmann equation with hard sphere case for (1) and obtained solution in $L_2^1(\mathbb{R}^n) \cap L\ LogL(\mathbb{R}^n)$ space, where the spaces are defined as below:

$$L_2^1(\mathbb{R}^n) = \left\{ f:\ \int_{\mathbb{R}^n} f(v)(1+|v|^2)dv < \infty \right\},$$

and

$$L\ LogL(\mathbb{R}^n) = \left\{ f:\ \int_{\mathbb{R}^n} f\ logf\,dv < \infty \right\}.$$

For the solutions near Maxwellian in the L^2-framework, Caflisch[6] studied the spatially-dependent nonlinear Boltzmann equation for soft potential case. More precisely, Caflisch obtained solution in $\mathscr{H}_\alpha$, which is defined by

$$\mathscr{H}_\alpha = \left\{ f:\ \|f\|_\alpha =: \sup_\xi e^{\alpha|\xi|^2} \sum_{s=1}^{4} \Big(\int_{\mathbb{T}^3} |\nabla^s f(x,\xi)|^2 dx \Big)^{\frac{1}{2}} < \infty \right\},$$

where f is periodic in spatial variable $x \in \mathbb{T}^3 = (\mathbb{R}/2\pi\mathbb{Z})^3, \xi \in \mathbb{R}^3$ is microscopic velocity.

Duan and Yu[7] studied the relativistic Boltzmann equation for soft potential in weighted H^N Sobolev spaces. One could refer to Ukai and Asano (1982) and Yang and Yu (2016) for more detail about soft potential near Maxwellian.

Now we turn to review some results for the partial differential equations in modulation spaces. There are many mathematicians working on this research area extensively for decades. For instance, Wang, Han, and Huang (2009) investigated the global well-posedness and scattering for the derivative nonlinear Schr*ö*dinger equation with small rough data in modulation spaces $M_{2,1}^{\frac{3}{2}}$. Also, Wang and Huang (2007) studied the Cauchy problems for the generalized BO, KDV and NLS equations and obtained the global well-posedness of solution with small rough data in certain modulation spaces. For the nonlinear heat equation and Navier-Stokes equation, Iwabuchi[8] obtained solutions in modulation spaces with negative derivative indices. For more about the study of partial differential equations in modulation spaces, see Baoxiang, Lifeng, and Boling (2006) and Wang, Huo, et al. (2011) and the references therein.

[5] Gamba, Panferov, and Villani, 2004, "On the Boltzmann equation for diffusively excited granular media".

[6] Caflisch, 1980, "The Boltzmann equation with a soft potential".

[7] Duan and Yu, 2017, "The relativistic Boltzmann equation for soft potentials".

[8] Iwabuchi, 2010, "Navier–Stokes equations and nonlinear heat equations in modulation spaces with negative derivative indices".

However, there are few literature studying on Boltzmann equation in modulation spaces. In this paper, we try to make an effort to step forward in this direction. Compared with the above literature, which worked on Boltzmann equation in Sobolev spaces near Maxwellian, while our interest focus on the homogenous Boltzmann equation with soft potential, and the Cauchy problems with initial data near vacuum in modulation space.

Our strategy is to apply the Banach fixed-point theorem on $\Psi(f)$ which is defined by

$$\Psi(f)(t) =: e^{t\Delta} f_0 + \int_0^t e^{(t-\tau)\Delta} Q(f,f) d\tau.$$

To do so, we need to prove $\Psi(f)$ is a contraction mapping. Firstly, the boundedness of Ψ in modulation space is one of major parts to prove the contraction. In turn, the boundedness of Boltzmann operator in modulation space becomes fundamentally important. Actually, we establish our own estimates of Boltzmann operator which are off-diagonal estimates, i.e. L^p-L^q and l^σ-l^ν, whose proofs are shown in Proposition 3. With this in hand, we make great effort to compute the time integral estimate, see (44) and (61), which leads to the boundeness of Ψ in modulation space globally in time.

We firstly set our notations and definitions.

1.1 General notations and definitions

- Given $f \in \mathscr{S}$ Schwartz class, its Fourier transform $\mathscr{F} f = \hat{f}$ is defined by

  $$\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x) dx,$$

  and its inverse Fourier transform is defined by $\mathscr{F}^{-1} f(x) = \hat{f}(-x)$.

- $\|f\|_{L^p_x} = \left(\int_{\mathbb{R}^n} |f|^p dx \right)^{\frac{1}{p}}$.

- $\|\{a_j\}\|_{l^r} = \left(\sum_{j\in\mathbb{Z}^n} |a_j|^r \right)^{\frac{1}{r}}$, $|\xi|_\infty = \max|\xi_i|, \xi = (\xi_1, \cdots, \xi_n) \in \mathbb{R}^n$.

- Let $A > 0, B > 0$, $A \lesssim B$ means there exists a positive constant c independent of the main parameters such that $A \leq cB$. $A \sim B$ means $A \lesssim B$ and $B \lesssim A$.

1.2 Definitions of Q(g, f) (see Gamba, Panferov, and Villani (2004) and Glassey (1996))

In this part, we introduce the definition of Boltzmann operator which takes the form:

$$\begin{aligned} Q(g,f)(v,t) &= \int_{\mathbb{R}^n}\int_{S^{n-1}} f(v',t)g(v'_*,t)\frac{1}{|v-v_*|^\gamma}b(v-v_*,\sigma)d\sigma dv_* \\ &\quad - \int_{\mathbb{R}^n}\int_{S^{n-1}} f(v,t)g(v_*,t)\frac{1}{|v-v_*|^\gamma}b(v-v_*,\sigma)d\sigma dv_* \\ &=: \ Q^+(g,f)(v,t) - Q^-(g,f)(v,t), \end{aligned} \tag{2}$$

where

$$v' = \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\sigma, \quad v'_* = \frac{v+v_*}{2} - \frac{|v-v_*|}{2}\sigma,$$

and $-1 \le \gamma < n$, $b(u,\sigma) \overset{def}{=} b\left(\frac{u}{|u|}\cdot\sigma\right)$, let $\cos\theta = \frac{u}{|u|}\cdot\sigma$, assume $0 \le b(\cos\theta) \le c\,|\cos\theta|$ (angular cutoff), c is a generic constant. Note $\int_{S^{n-1}} b(v-v_*,\sigma)d\sigma \sim 1$.

- We call $Q^+(g,f)$ the gain term and $Q^-(g,f)$ the loss term separately.
- $Q^+(g,f)$ has another expression:

$$Q^+(g,f)(v,t) = \int_{\mathbb{R}^n}\int_{S^{n-1}} f(v+\frac{u'-u}{2},t)g(v-\frac{u'+u}{2},t)\frac{1}{|u|^\gamma}b(u,\sigma)d\sigma du,$$

  where $u' = |u|\sigma$.
- Also, $Q^-(g,f) \sim f\cdot(g * \frac{1}{|v|^\gamma})$.
- Usually, we call it hard potential if $\gamma \in [-1,0]$, especially, when $\gamma = -1$, we call it hard sphere case; and soft potential if $\gamma \in (0,n)$, which is the case we are going to focus on in this paper.

For more information about Boltzmann operator, see Chen and He (2019) , Duan and Yu (2017), He, Chen, Fang, et al. (2021a) , Ukai and Asano (1982) , and Yang and Yu (2016) .

1.3 Definition of modulation space

Let us recall the definition of modulation spaces $M^s_{p,q}(\mathbb{R}^n)$[9] .

[9] Wang, Huo, et al., 2011, *Harmonic analysis method for nonlinear evolution equations, I.*

1. Introduction

Let $\rho \in \mathscr{S}(\mathbb{R}^n)$ which is Schwartz space, $\rho : \mathbb{R}^n \to [0,1]$ be a smooth function verifying $\rho(\xi) = 1$ for $|\xi|_\infty \le 1/2$ and $\rho(\xi) = 0$ for $|\xi|_\infty \ge 1$. Let ρ_k be the translation of ρ,

$$\rho_k(\xi) = \rho(\xi - k), \quad k \in \mathbb{Z}^n, \tag{3}$$

Denote

$$\sigma_k(\xi) = \rho_k(\xi)\Big(\sum_{k\in\mathbb{Z}^n} \rho_k(\xi)\Big)^{-1}, \quad k \in \mathbb{Z}^n, \tag{4}$$

and

$$\Box_k := \mathscr{F}^{-1}\sigma_k\mathscr{F}, \quad k \in \mathbb{Z}^n, \tag{5}$$

The operators $\{\Box_k\}_{k\in\mathbb{Z}^n}$ are said to be frequency-uniform decomposition operators. For $k \in \mathbb{Z}^n$, we denote $|k| = |k_1| + \cdots + |k_n|, \langle k\rangle = 1 + |k|$.

Let $-\infty < s < \infty, 0 \le p, q \le \infty$, we define

$$M^s_{p,q}(\mathbb{R}^n) = \Big\{ f \in \mathscr{S}'(\mathbb{R}^n) : \|f\|_{M^s_{p,q}} < \infty \Big\}, \tag{6}$$

$$\|f\|_{M^s_{p,q}} := \Big(\sum_{k\in\mathbb{Z}^n} \langle k\rangle^{sq} \|\Box_k f\|_p^q \Big)^{1/q}, \tag{7}$$

$M^s_{p,q}(\mathbb{R}^n)$ is said to be the modulation space. In this paper, we consider only $M^0_{p,q}(\mathbb{R}^n)$ type space.

Now we introduce other notations and assumptions which will be used in our proof of main theorem before we narrate it.

Denote

- $\alpha =: \frac{n}{2}(\frac{1}{\nu} - \frac{1}{\sigma}), \quad \beta =: \frac{n}{2}(\frac{1}{p} - \frac{1}{r}).$
- $\|f\|_Y =: \sup_{t\in[0,\infty)} \|f(t)\|_{M^0_{p,\sigma}(\mathbb{R}^n)}.$
- $\|f\|_{Z_{r,\nu}} =: \sup_{t\in[0,\infty)} t^\alpha(1+t)^{\beta-\alpha}\|f(t)\|_{M^0_{r,\nu}(\mathbb{R}^n)}.$
- $\|f\|_X =: \sup_{t\in[0,\infty)} \|f(t)\|_{M^0_{p,\sigma}(\mathbb{R}^n)} + \sup_{t\in[0,\infty)} t^\alpha(1+t)^{\beta-\alpha}\|f(t)\|_{M^0_{r,\nu}(\mathbb{R}^n)},$
  i.e., $\|f\|_X = \|f\|_Y + \|f\|_{Z_{r,\nu}}.$

We also need to make the following assumptions.
Assumption: Let $0 < \alpha \le \frac{1}{2}, \beta \ge 1, n \ge 3, 0 < \gamma < n, 1 < p < r, 1 < \nu < \sigma$, and there exist $(p_1, p_2, q, \tilde{p}, \tilde{r}, \tilde{\nu})$ such that

$$\begin{cases} \frac{1}{p_2} + \frac{n-\gamma}{n} \le \frac{1}{r}, \\ \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{\tilde{r}}, \ \ p_1 \ge r > 1, \\ \frac{\tilde{r}}{q} = \frac{1}{\tilde{p}} - \frac{n-\gamma\tilde{r}}{n}, \tilde{p}\tilde{r} \ge r, 1 < \tilde{p} < \frac{n}{n-\gamma\tilde{r}}, \\ q' \ge r, \frac{1}{q} + \frac{1}{q'} = 1, \\ 1 < \tilde{r} < p < r < \infty, \end{cases} \tag{8}$$

and

$$\begin{cases} 1 < \nu < \sigma < \tilde{\nu} < \infty, \\ \frac{1}{\tilde{\nu}} = \frac{2}{\nu} - 1, \\ \frac{n}{2}(\frac{1}{\sigma} - \frac{1}{\tilde{\nu}}) + 2\alpha \le 1, \\ \frac{n}{2}(\frac{1}{\nu} - \frac{1}{\tilde{\nu}}) + \alpha \le 1. \end{cases} \tag{9}$$

Remark 1 – It is not hard to see that the indices set satisfying (8) and (9) is nonempty. For instance, one can take

$$(n, \gamma, \alpha, \sigma, r, \nu) = (15, \frac{19}{2}, \frac{1}{16}, \frac{38}{35}, \frac{3}{2}, \frac{16}{15}),$$

and

$$(p_1, p_2, q, \tilde{p}, \tilde{r}, \tilde{\nu}) = (\frac{3}{2}, \frac{15}{2}, 3, \frac{8}{5}, \frac{5}{4}, \frac{8}{7}).$$

Now we are in the position to state the main theorem.

Theorem 1 – *Under the above assumption, there exists sufficiently small $\delta > 0$ such that for any f_0 with $\|f_0\|_{M^0_{p,\sigma}(\mathbb{R}^n)} \le \delta$, (1) possesses a unique global solution in X, where*

$$X =: \left\{ f \in C([0,\infty), M^0_{p,\sigma}(\mathbb{R}^n) : \|f\|_X \le c_1\delta, \textit{ for some constant } c_1. \right\}$$

Remark 2 – Compared with Iwabuchi (2010) , the nonlinear term $Q(f,g)$ in this paper, which represents the phenomenon of collision of microscopic particles and has applications in statistical background, is quite different from the one in Iwabuchi (2010) . Besides, the estimate of the nonlinear term $Q(f,g)$ is more complicated than the one in Iwabuchi (2010) .

2 Some lemmas

In this section, we would like to cite some useful lemmas. The first one is about the properties of $e^{t\Delta}$ in modulation spaces.

Lemma 1 – *(Iwabuchi (2010)) Let $1 \le q, r, \sigma, \nu \le \infty, s \in \mathbb{R}$.*
(i) If $q \ge r$, there exists a constant $c > 0$ such that

$$\|e^{t\Delta} f\|_{M^s_{q,\sigma}} \le c(1+t)^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{q})}\|f\|_{M^s_{r,\sigma}}. \tag{10}$$

(ii) If $\sigma \le \nu$, there exists a constant $c > 0$ such that

$$\|e^{t\Delta} f\|_{M^0_{q,\sigma}} \le c(1+t^{-\frac{n}{2}(\frac{1}{\sigma}-\frac{1}{\nu})})\|f\|_{M^0_{q,\nu}}. \tag{11}$$

Also, the following property for modulation space is useful.

Lemma 2 – *(Wang, Huo, et al. (2011)) There exists a constant $c > 0$ which is independent of p, q, such that*

$$\|\Box_k f\|_{L^q} \le c\|\Box_k f\|_{L^p}, \quad 1 < p \le q \le \infty. \tag{12}$$

Remark 3 – Comparing with the counterpat in Besov space,

$$\|\Delta_k f\|_{L^q} \le C2^{nk(\frac{1}{p}-\frac{1}{q})}\|\Delta_k f\|_{L^p}, \quad 1 < p \le q < \infty, \tag{13}$$

we see that the operator Δ_k (see Bergh and Löfström (2012) , He and Chen (2021) , He, Chen, Fang, et al. (2021b)) results in the increase of the derivative, i.e. we need higher order derivative to control L^q. However, for the operator $\Box_k$, we do not need higher order derivative to control higher integral index which is an advantage in modulation space.

Finally, we need the following notations and property of $S_\pm[\psi]$ which will be used in the estimate of the gain term $Q^+(g,f)$ in modulation space.
Denote

$$S[\psi](v,v_*) = \int_{S^{n-1}} \psi(v')b(u,\sigma)d\sigma, \quad u = v - v_*, \; v' = \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\sigma, \tag{14}$$

and

$$S_\pm[\psi](v,v_*) = \int_{\{\pm u\cdot\sigma>0\}} \psi(v')b(u,\sigma)d\sigma, \tag{15}$$

then we have the following boundedness estimate of $S_\pm$ in Lebesgue spaces.

Lemma 3 – *(Gamba, Panferov, and Villani (2004)) The operators*

$$S_+ : \quad L^q(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n_{v_*}, L^q(\mathbb{R}^n_v)),$$

$$S_- : \quad L^q(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n_v, L^q(\mathbb{R}^n_{v_*})),$$

are bounded for every $1 \le q \le \infty$.

3 Estimate of Boltzmann operator in modulation spaces

In this section, we would like to establish estimates of Boltzmann operator with soft potential in modulation spaces. We will split the procedure into two parts which are the estimate of gain term $Q^+(g,f)$ and the estimate of the loss term $Q^-(g,f)$.

3.1 Estimate of the gain term $Q^+(g,f)$

To estimate the gain term, we will prove the off-diagonal estimates L^p-L^q and l^σ-l^ν, which are very useful to deal with the nonlinear term estimate in section 4.

Lemma 4 – *For* $1 < \tilde{p} < \frac{n}{n-p\gamma}, 1 < p < \infty, \frac{1}{p} + \frac{1}{p'} = 1, \frac{1}{q} + \frac{1}{q'} = 1,$ *we have the following estimate for the gain term* $Q^+(g,f)$

$$|\langle Q^+(g,f),\psi\rangle| \lesssim \|g\|_{L^{q'}} \cdot \|f\|_{L^{p\tilde{p}}} \cdot \|\psi\|_{L^{p'}}, \tag{16}$$

i.e.

$$\|Q^+(g,f)\|_{L^p} \lesssim \|g\|_{L^{q'}} \cdot \|f\|_{L^{p\tilde{p}}}. \tag{17}$$

Proof. By the definitions of $Q^+(g,f)$ and $S_\pm[\Psi]$, following Gamba, Panferov, and Villani (2004) (see (4.2) in page 519), we can write the dual form as below:

$$\begin{aligned}
|\langle Q^+(g,f),\psi\rangle| =& \left|\int_{\mathbb{R}^n} Q^+(g,f)\psi dv\right| \\
=& \left|\int_{\mathbb{R}^n_v}\int_{\mathbb{R}^n_{v_*}} f(v)g(v_*)\frac{1}{|v-v_*|^\gamma}\int_{S^{n-1}}\psi(v')b(v-v_*,\sigma)d\sigma dv dv_*\right| \\
=& \left|\int_{\mathbb{R}^n_v} f(v)\int_{\mathbb{R}^n_{v_*}} g(v_*)\frac{1}{|v-v_*|^\gamma}(S_+[\psi](v,v_*)+S_-[\psi](v,v_*))dv dv_*\right|.
\end{aligned} \tag{18}$$

In order to get an estimate of $|\langle Q^+(g,f),\psi\rangle|$, we only need to estimate the term associated with $S_+[\psi](v,v_*)$ which satisfies

$$\begin{aligned}
&\int_{\mathbb{R}^n_v} f(v)\int_{\mathbb{R}^n_{v_*}} g(v_*)\frac{1}{|v-v_*|^\gamma}S_+[\psi](v,v_*)dv dv_* \\
\lesssim& \int_{\mathbb{R}^n_{v_*}} g(v_*)dv_* \cdot \left\|\frac{f(v)}{|v-v_*|^\gamma}\right\|_{L^p_v} \cdot \left\|S_+[\psi](v,v_*)\right\|_{L^\infty(\mathbb{R}^n_{v_*},L^{p'}(\mathbb{R}^n_v))} \\
\lesssim& \int_{\mathbb{R}^n_{v_*}} g(v_*)dv_* \cdot \left\|\frac{f(v)}{|v-v_*|^\gamma}\right\|_{L^p_v} \cdot \|\psi\|_{L^{p'}} \\
\lesssim& \|g\|_{L^{q'}_{v_*}} \cdot \left\|\frac{f(v)}{|v-v_*|^\gamma}\right\|_{L^q_{v_*}L^p_v} \cdot \|\psi\|_{L^{p'}},
\end{aligned} \tag{19}$$

where we applied Hölder's inequality in v with $\frac{1}{p}+\frac{1}{p'}=1$ and v_* with $\frac{1}{q}+\frac{1}{q'}=1$ separately in the second line and the last line, we also applied Lemma 3 with $\frac{1}{p}+\frac{1}{p'}=1$ in the third line.

We now continue to estimate the term $\left\|\frac{f(v)}{|v-v_*|^\gamma}\right\|_{L^q_{v_*}L^p_v}$.
Rewrite

$$\left\|\frac{f(v)}{|v-v_*|^\gamma}\right\|_{L^q_{v_*}L^p_v}=\left(\int_{\mathbb{R}^n_{v_*}}\left(\int_{\mathbb{R}^n_v}\frac{|f(v)|^p}{|v-v_*|^{p\gamma}}dv\right)^{\frac{q}{p}}dv_*\right)^{\frac{p}{q}\cdot\frac{1}{p}}, \tag{20}$$

by Lemma 5, let $h=|f|^p$, we have

$$\left\|I_\alpha h\right\|_{L^{\frac{q}{p}}_{v_*}}\lesssim\|h\|_{L^{\tilde{p}}}, \tag{21}$$

where $\frac{p}{q}=\frac{1}{\tilde{p}}-\frac{n-p\gamma}{n}$.
Thus,

$$\left\|\frac{f(v)}{|v-v_*|^\gamma}\right\|_{L^q_{v_*}L^p_v}\lesssim\||f|^p\|^{\frac{1}{p}}_{L^{\tilde{p}}}=\|f\|_{L^{p\tilde{p}}}, \tag{22}$$

then the desired result is immediate. □

Remark 4 – We imposed the conditions on the indices in Lemma 4 as follows:

$$\begin{cases}\frac{p}{q}=\frac{1}{\tilde{p}}-\frac{n-p\gamma}{n},\\ 1<\tilde{p}<\frac{n}{n-p\gamma},\end{cases} \tag{23}$$

which implies $\frac{q}{p}>\tilde{p}\Leftrightarrow q>p\tilde{p}>p$.

For the gain term $Q^+(g,f)$, we also have the following estimate in the modulation context.

Proposition 1 – *With the same assumption as in Lemma 4, and additionally,* $\frac{1}{\sigma}=\frac{2}{\nu}-1$, *then*

$$\|Q^+(g,f)\|_{M^0_{p,\sigma}}\lesssim\|g\|_{M^0_{q',\nu}}\cdot\|f\|_{M^0_{\tilde{p}p,\nu}}. \tag{24}$$

Proof. Note by the duality property for modulation space[10] ,

$$\|Q^+(g,f)\|_{M^0_{p,\sigma}}=\sup_{\|\psi\|_{M^0_{p',\sigma'}}\le 1}|\langle Q^+(g,f),\psi\rangle|,$$

we have

$$
\begin{aligned}
&|\langle Q^+(g,f),\psi\rangle| \\
=&\Big|\sum_{k\in\mathbb{Z}^n}\sum_{|l|_\infty\le 1}\langle \Box_k Q^+(g,f),\Box_{k+l}\psi\rangle\Big| \\
\lesssim&\sum_{k\in\mathbb{Z}^n}\sum_{|l|_\infty\le 1}\sum_{i\in\mathbb{Z}^n}\sum_{j\in\mathbb{Z}^n}\Big|\langle \Box_k Q^+(\Box_i g,\Box_j f),\Box_{k+l}\psi\rangle\Big|\cdot 1_{|k-i-j|\le k_0(n)} \\
\lesssim&\sum_{k\in\mathbb{Z}^n}\sum_{|l|_\infty\le 1}\sum_{i\in\mathbb{Z}^n}\sum_{j\in\mathbb{Z}^n}\Big|\langle Q^+(\Box_i g,\Box_j f),\Box_k\Box_{k+l}\psi\rangle\Big|\cdot 1_{|k-i-j|\le k_0(n)},
\end{aligned}
\tag{25}
$$

where we have used a fact: when $|k-i-j|\ge k_0(n)$, where $k_0(n)$ depends only on dimension n, we have $\Box_k Q^+(\Box_i g,\Box_j f)=0$, due to

$$
\begin{aligned}
&\Box_k Q^+(\Box_i g,\Box_j f) \\
=&\int_{\mathbb{R}^n}\int_{S^{n-1}}\Box_k[\Box_j f(v+\frac{u'-u}{2})\cdot\Box_i g(v-\frac{u'+u}{2})]\cdot\frac{1}{|u|^\gamma}b(u,\sigma)d\sigma du,
\end{aligned}
\tag{26}
$$

where $\Box_k,\Box_i,\Box_j$ are all about v variable, i.e. $\Box_k^v,\Box_i^v,\Box_j^v$, and $u'=|u|\sigma$.
Applying Lemma 4 yields that

$$
\begin{aligned}
&|\langle Q^+(g,f),\psi\rangle| \\
\lesssim&\sum_{k\in\mathbb{Z}^n}\sum_{i\in\mathbb{Z}^n}\sum_{j\in\mathbb{Z}^n}\|\Box_i g\|_{L^{q'}}\cdot\|\Box_j f\|_{L^{p\tilde{p}}}\cdot\|\Box_k\psi\|_{L^{p'}}\cdot 1_{|k-i-j|\le k_0(n)}.
\end{aligned}
\tag{27}
$$

Let $\|\Box_k\psi\|_{L^{p'}}=:c_k,\|\Box_j f\|_{L^{p\tilde{p}}}=:b_j,\|\Box_i g\|_{L^{q'}}=:a_i$,
then

$$
\begin{aligned}
&\sum_{k\in\mathbb{Z}^n}\sum_{i\in\mathbb{Z}^n}\sum_{j\in\mathbb{Z}^n}c_k\cdot b_j\cdot a_i\cdot 1_{|k-i-j|\le k_0(n)} \\
\lesssim&|c_k|_{l^{\sigma'}}\Big(\sum_{k\in\mathbb{Z}^n}(\sum_{i\in\mathbb{Z}^n}\sum_{j\in\mathbb{Z}^n}a_i\cdot b_j\cdot 1_{|k-i-j|\le k_0(n)})^\sigma\Big)^{\frac{1}{\sigma}} \\
\lesssim&|c_k|_{l^{\sigma'}}\cdot|a_i|_{l^\nu}\cdot|b_j|_{l^\nu},
\end{aligned}
\tag{28}
$$

where we applied Hölder's inequality with $\frac{1}{\sigma}+\frac{1}{\sigma'}=1$ in the second line and Young's inequality with $\frac{1}{\sigma}=\frac{2}{\nu}-1$ in the third line separately.
Thus,

$$
|\langle Q^+(g,f),\psi\rangle|\lesssim\|\psi\|_{M^0_{p',\sigma'}}\cdot\|g\|_{M^0_{q',\nu}}\cdot\|f\|_{M^0_{\tilde{p}p,\nu}},
\tag{29}
$$

i.e.,

$$\|Q^+(g,f)\|_{M^0_{p,\sigma}} \lesssim \|g\|_{M^0_{q',\nu}} \cdot \|f\|_{M^0_{\tilde{p}p,\nu}}. \tag{30}$$

□

3.2 Estimate of the loss term $Q^-(g,f)$

Let us turn to estimate the loss term $Q^-(g,f) = f \cdot (g * \frac{1}{|v|^\gamma})$. The estimate of the loss term is straightforward. We mainly make use of the product formula and Young's inequality. In addition, the property of Riesz potential is exploited as well.

Proposition 2 – *If* $\frac{1}{p_1} + \frac{1}{p_2} = 1, \frac{1}{p_2} + \frac{n-\gamma}{n} \le \frac{1}{r}, \frac{1}{\sigma} = \frac{2}{\nu} - 1, p, p_1, p_2, r > 1, 0 < \gamma < n$, *then the following inequality holds:*

$$\|Q^-(g,f)\|_{M^0_{p,\sigma}} \lesssim \|f\|_{M^0_{p_1,\nu}} \cdot \|g\|_{M^0_{r,\nu}}. \tag{31}$$

Proof. First of all,

$$\begin{aligned}
&\|\Box_k Q^-(g,f)\|_{L^p} \\
=&\left\|\Box_k (f \cdot (g * \frac{1}{|v|^\gamma}))\right\|_{L^p} \\
\lesssim& \sum_{i,j\in\mathbb{Z}^n} \left\|\Box_k (\Box_i f \cdot (\Box_j g * \frac{1}{|v|^\gamma}))\right\|_{L^p} \\
\lesssim& \sum_{i,j\in\mathbb{Z}^n} \|\Box_i f\|_{L^{p_1}} \cdot \left\|\Box_j g * \frac{1}{|v|^\gamma}\right\|_{L^{p_2}} \cdot 1_{|k-i-j|\le k_0(n)},
\end{aligned} \tag{32}$$

where $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$.
Note $\frac{1}{p_2} + \frac{n-\gamma}{n} \le \frac{1}{r}$, then by Lemma 5 and Lemma 2, we get

$$\|\Box_k Q^-(g,f)\|_{L^p} \lesssim \sum_{i,j\in\mathbb{Z}^n} \|\Box_i f\|_{L^{p_1}} \cdot \|\Box_j g\|_{L^r} \cdot 1_{|k-i-j|\le k_0(n)}. \tag{33}$$

Denote $a_i =: \|\Box_i f\|_{L^{p_1}}, b_j =: \|\Box_j g\|_{L^r}, c_i =: \sum_{j\in\mathbb{Z}^n} b_j \cdot 1_{|i-j|\le k_0(n)}$,

[10] Wang and Huang, 2007, "Frequency-uniform decomposition method for the generalized BO, KdV and NLS equations".

then it follows that

$$
\begin{aligned}
&\Big(\sum_{k\in\mathbb{Z}^n}\big(\sum_{i,j\in\mathbb{Z}^n} a_i\cdot b_j\cdot 1_{|k-i-j|\le k_0(n)}\big)^{\sigma}\Big)^{\frac{1}{\sigma}}\\
\lesssim&\Big(\sum_{k\in\mathbb{Z}^n}\big(\sum_{i\in\mathbb{Z}^n} a_{k-i}\cdot\sum_{j\in\mathbb{Z}^n} b_j\cdot 1_{|i-j|\le k_0(n)}\big)^{\sigma}\Big)^{\frac{1}{\sigma}} \qquad (34)\\
\lesssim&|\{a_i\}*\{c_i\}|_{l^{\sigma}}\\
\lesssim&|a_i|_{l^{\nu}}|c_i|_{l^{\nu}},
\end{aligned}
$$

where $\frac{1}{\sigma}=\frac{2}{\nu}-1$.
Thus,

$$
\|Q^-(g,f)\|_{M^0_{p,\sigma}}\lesssim\|f\|_{M^0_{p_1,\nu}}\cdot\|g\|_{M^0_{r,\nu}}. \qquad (35)
$$

□

Combining Lemma 2 , Proposition 1 and Proposition 2 leads to the following key proposition.

Proposition 3 – *Assume* $p>1, p_2>1, \frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{p}, \frac{1}{q}+\frac{1}{q'}=1, q, q', p_1, p_2, p>1$, $0<\gamma<n$ *and*

$$
\begin{cases}
\frac{1}{p_2}+\frac{n-\gamma}{n}\le\frac{1}{r}, p_1\ge r>1,\\
\frac{p}{q}=\frac{1}{\tilde p}-\frac{n-p\gamma}{n},\ \ 1<\tilde p<\frac{n}{n-p\gamma},\ \ \tilde p p\ge r,\\
q'\ge r,\\
\frac{1}{\sigma}=\frac{2}{\nu}-1,
\end{cases} \qquad (36)
$$

then we have

$$
\|Q(g,f)\|_{M^0_{p,\sigma}}\lesssim\|g\|_{M^0_{r,\nu}}\cdot\|f\|_{M^0_{r,\nu}}. \qquad (37)
$$

In particular,

$$
\|Q(f,f)\|_{M^0_{p,\sigma}}\lesssim\|f\|^2_{M^0_{r,\nu}}. \qquad (38)
$$

4 Global existence

With the boundedness of Boltzmann operator with soft potential in modulation space, we are ready to prove the global existence of (1). In this process, we adopt

Banach fixed-point theorem[11] . Concretely, we will show $\Psi(f)$ is a contraction mapping, and off-diagonal estimates of Boltzmann operator in Proposition 3 come into play when dealing with the nonlinear Boltzmann term, see (40). Besides, the time integral is carefully calculated to prove the boundedness of Ψ globally in time.

Proof. Inspired by Iwabuchi (2010) , we consider the solution in the following integral form:

$$\Psi(f)(t) =: e^{t\Delta} f_0 + \int_0^t e^{(t-\tau)\Delta} Q(f,f) d\tau.$$

We are going to show

$$\|\Psi(f)\|_X \lesssim \|f_0\|_{M^0_{p,\sigma}(\mathbb{R}^n)} + \|f\|_X^2. \tag{39}$$

Firstly, we would like to estimate $\|\Psi(f)\|_Y$.
Estimate $\|\Psi(f)\|_Y$.
Let $\tilde{\nu}$ satisfy $\frac{1}{\tilde{\nu}} = \frac{2}{\nu} - 1$. Applying Lemma 1, we have

$$\begin{aligned} &\|\Psi(f)\|_Y \\ \lesssim &\|f_0\|_{M^0_{p,\sigma}(\mathbb{R}^n)} + \sup_{t>0} \int_0^t \|e^{(t-\tau)\Delta} Q(f,f)\|_{M^0_{p,\sigma}(\mathbb{R}^n)} d\tau \\ \lesssim &\|f_0\|_{M^0_{p,\sigma}(\mathbb{R}^n)} + \sup_{t>0} \int_0^t \left(1 + \frac{1}{(t-\tau)^{\frac{n}{2}(\frac{1}{\sigma}-\frac{1}{\tilde{\nu}})}}\right) \frac{1}{(1+t-\tau)^{\frac{n}{2}(\frac{1}{\tilde{r}}-\frac{1}{p})}} \|Q(f,f)\|_{M^0_{\tilde{r},\tilde{\nu}}(\mathbb{R}^n)} d\tau, \end{aligned} \tag{40}$$

where $\frac{1}{\sigma} - \frac{1}{\tilde{\nu}} > 0$, i.e., $1 < \sigma < \tilde{\nu}$, and $\frac{1}{\tilde{r}} - \frac{1}{p} > 0$, i.e., $1 < \tilde{r} < p$, $\frac{1}{\tilde{\nu}} = \frac{2}{\nu} - 1$.
Applying Proposition 3 with $\tilde{r}, \tilde{\nu}$, i.e., $\tilde{r}, \tilde{\nu}$ play the same role as p, σ accordingly in Proposition 3, we get

$$\|Q(f,f)\|_{M^0_{\tilde{r},\tilde{\nu}}} \lesssim \|f\|^2_{M^0_{r,\nu}}, \tag{41}$$

where the indices satisfy the assumptions as below:

$$\begin{cases} \frac{1}{p_2} + \frac{n-\gamma}{n} \le \frac{1}{r}, \\ \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{\tilde{r}}, \quad p_1 \ge r > 1, \\ \frac{\tilde{r}}{q} = \frac{1}{\tilde{p}} - \frac{n-\gamma\tilde{r}}{n}, \quad \tilde{p}\tilde{r} \ge r, \quad 1 < \tilde{p} < \frac{n}{n-\gamma\tilde{r}}, \\ q' \ge r, \quad \frac{1}{q} + \frac{1}{q'} = 1, \\ \frac{1}{\tilde{\nu}} = \frac{2}{\nu} - 1, \end{cases} \tag{42}$$

[11] Brezis, 2011, *Functional analysis, Sobolev spaces and partial differential equations.*

where the intermediate indices $(p_1, p_2, q, \tilde{p})$ are defined in section 3.
Combining (40) and (41) yields that

$$
\begin{aligned}
&\|\Psi(f)\|_Y \\
\lesssim&\|f_0\|_{M^0_{p,\sigma}(\mathbb{R}^n)} + \sup_{t>0}\int_0^t \Big(1+\frac{1}{(t-\tau)^{\frac{n}{2}(\frac{1}{\sigma}-\frac{1}{\nu})}}\Big)\frac{1}{(1+t-\tau)^{\frac{n}{2}(\frac{1}{\tilde{r}}-\frac{1}{p})}}\|f(\tau)\|^2_{M^0_{r,\nu}(\mathbb{R}^n)}d\tau \\
\lesssim&\|f_0\|_{M^0_{p,\sigma}(\mathbb{R}^n)} + \|f\|^2_{Z_{r,\nu}}\cdot\sup_{t>0}\int_0^t \Big(1+\frac{1}{(t-\tau)^{\frac{n}{2}(\frac{1}{\sigma}-\frac{1}{\nu})}}\Big)\frac{1}{(1+t-\tau)^{\frac{n}{2}(\frac{1}{\tilde{r}}-\frac{1}{p})}}\cdot\frac{1}{\tau^{2\alpha}}\cdot\frac{1}{(1+\tau)^{2(\beta-\alpha)}}d\tau.
\end{aligned}
\tag{43}
$$

Before we proceed, we need the following claim.
Claim 1: If $\frac{n}{2}(\frac{1}{\sigma}-\frac{1}{\nu})+2\alpha\le 1$, and $\beta\ge 1$, then

$$
\sup_{t>0}\int_0^t \Big(1+\frac{1}{(t-\tau)^{\frac{n}{2}(\frac{1}{\sigma}-\frac{1}{\nu})}}\Big)\frac{1}{(1+t-\tau)^{\frac{n}{2}(\frac{1}{\tilde{r}}-\frac{1}{p})}}\cdot\frac{1}{\tau^{2\alpha}}\cdot\frac{1}{(1+\tau)^{2(\beta-\alpha)}}d\tau<\infty. \tag{44}
$$

Remark 5 – Since $\frac{1}{\sigma}-\frac{1}{\nu}>0$, we have in fact $0<\alpha\le\frac{1}{2}$.

Proof of Claim 1. Note that we can rewrite the integral term in (44) as the sum of the following four terms:

$$
\begin{aligned}
&\int_0^t \Big(1+\frac{1}{(t-\tau)^{\frac{n}{2}(\frac{1}{\sigma}-\frac{1}{\nu})}}\Big)\frac{1}{(1+t-\tau)^{\frac{n}{2}(\frac{1}{\tilde{r}}-\frac{1}{p})}}\cdot\frac{1}{\tau^{2\alpha}}\cdot\frac{1}{(1+\tau)^{2(\beta-\alpha)}}d\tau \\
=&\int_0^{\frac{t}{2}}\frac{1}{(t-\tau)^{\frac{n}{2}(\frac{1}{\sigma}-\frac{1}{\nu})}}\cdot\frac{1}{(1+t-\tau)^{\frac{n}{2}(\frac{1}{\tilde{r}}-\frac{1}{p})}}\cdot\frac{1}{\tau^{2\alpha}}\cdot\frac{1}{(1+\tau)^{2(\beta-\alpha)}}d\tau \\
&+\int_{\frac{t}{2}}^t\frac{1}{(t-\tau)^{\frac{n}{2}(\frac{1}{\sigma}-\frac{1}{\nu})}}\cdot\frac{1}{(1+t-\tau)^{\frac{n}{2}(\frac{1}{\tilde{r}}-\frac{1}{p})}}\cdot\frac{1}{\tau^{2\alpha}}\cdot\frac{1}{(1+\tau)^{2(\beta-\alpha)}}d\tau \\
&+\int_0^{\frac{t}{2}}\frac{1}{(1+t-\tau)^{\frac{n}{2}(\frac{1}{\tilde{r}}-\frac{1}{p})}}\cdot\frac{1}{\tau^{2\alpha}}\cdot\frac{1}{(1+\tau)^{2(\beta-\alpha)}}d\tau \\
&+\int_{\frac{t}{2}}^t\frac{1}{(1+t-\tau)^{\frac{n}{2}(\frac{1}{\tilde{r}}-\frac{1}{p})}}\cdot\frac{1}{\tau^{2\alpha}}\cdot\frac{1}{(1+\tau)^{2(\beta-\alpha)}}d\tau \\
=:&J_1+J_2+J_3+J_4.
\end{aligned}
\tag{45}
$$

Case 1: $0 < t \leq 100$.
For J_1, note that $(1+\tau)^{-2(\beta-\alpha)} \lesssim 1, (1+t-\tau)^{-\frac{n}{2}(\frac{1}{\tilde{r}}-\frac{1}{p})} \lesssim 1$, we get

$$\begin{aligned} J_1 \lesssim & \int_0^{\frac{t}{2}} \frac{1}{\tau^{2\alpha}} \cdot \frac{1}{(t-\tau)^{\frac{n}{2}(\frac{1}{\sigma}-\frac{1}{\tilde{\nu}})}} d\tau \\ \lesssim & t^{-\frac{n}{2}(\frac{1}{\sigma}-\frac{1}{\tilde{\nu}})} \frac{\tau^{1-2\alpha}}{1-2\alpha}\Big|_0^{\frac{t}{2}} \\ \lesssim & t^{-\frac{n}{2}(\frac{1}{\sigma}-\frac{1}{\tilde{\nu}})+1-2\alpha} \\ \lesssim & 1, \end{aligned} \tag{46}$$

since $\frac{n}{2}(\frac{1}{\sigma}-\frac{1}{\tilde{\nu}})+2\alpha \leq 1$.
For J_2, similarly, using the fact $(1+\tau)^{-2(\beta-\alpha)} \lesssim 1, (1+t-\tau)^{-\frac{n}{2}(\frac{1}{\tilde{r}}-\frac{1}{p})} \lesssim 1$, we have

$$\begin{aligned} J_2 \lesssim & t^{-2\alpha} \int_{\frac{t}{2}}^{t} \frac{1}{(t-\tau)^{\frac{n}{2}(\frac{1}{\sigma}-\frac{1}{\tilde{\nu}})}} d\tau \\ \lesssim & t^{1-\frac{n}{2}(\frac{1}{\sigma}-\frac{1}{\tilde{\nu}})-2\alpha} \\ \lesssim & 1, \end{aligned} \tag{47}$$

since $1-\frac{n}{2}(\frac{1}{\sigma}-\frac{1}{\tilde{\nu}})-2\alpha \geq 0, 0 < t \leq 100$.
For J_3, again note $(1+\tau)^{-2(\beta-\alpha)} \lesssim 1, (1+t-\tau)^{-\frac{n}{2}(\frac{1}{\tilde{r}}-\frac{1}{p})} \lesssim 1$, it follows that

$$\begin{aligned} J_3 \lesssim & \int_0^{\frac{t}{2}} 1 \cdot \frac{1}{\tau^{2\alpha}} \cdot 1 \, d\tau \\ \lesssim & \frac{\tau^{1-2\alpha}}{1-2\alpha}\Big|_0^{\frac{t}{2}} \\ \sim & t^{1-2\alpha} \\ \lesssim & 1, \end{aligned} \tag{48}$$

since $1-2\alpha \geq 0$, i.e., $\alpha \leq \frac{1}{2}$.
For J_4, note $1-2\alpha \geq \frac{n}{2}(\frac{1}{\sigma}-\frac{1}{\tilde{\nu}}) > 0, (1+\tau)^{-2(\beta-\alpha)} \lesssim 1, (1+t-\tau)^{-\frac{n}{2}(\frac{1}{\tilde{r}}-\frac{1}{p})} \lesssim 1$, then we have

$$\begin{aligned} J_4 \lesssim & \int_{\frac{t}{2}}^{t} \frac{1}{t^{2\alpha}} d\tau \\ \sim & \frac{\tau^{1-2\alpha}}{1-2\alpha}\Big|_{\frac{t}{2}}^{t} \\ \lesssim & t^{1-2\alpha} \\ \lesssim & 1. \end{aligned} \tag{49}$$

Case 2: $t > 100$.
For J_1, note when $0 < \tau \le \frac{t}{2}$, we have $1 + t - \tau \ge 1 + \frac{t}{2} \ge \frac{1}{2}(1+t)$, $t - \tau \ge \frac{t}{2} \ge \frac{1}{4}(1+t)$, it follows that

$$\begin{aligned}
&\int_0^{\frac{t}{2}} \frac{1}{(t-\tau)^{\frac{n}{2}(\frac{1}{\sigma}-\frac{1}{\nu})}} \frac{1}{(1+t-\tau)^{\frac{n}{2}(\frac{1}{r}-\frac{1}{p})}} \cdot \frac{1}{\tau^{2\alpha}} \cdot \frac{1}{(1+\tau)^{2(\beta-\alpha)}} d\tau \\
\lesssim &(1+t)^{-\frac{n}{2}(\frac{1}{\sigma}-\frac{1}{\nu})}(1+t)^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{p})} \int_0^{\frac{t}{2}} \frac{1}{\tau^{2\alpha}} \cdot \frac{1}{(1+\tau)^{2(\beta-\alpha)}} d\tau.
\end{aligned} \tag{50}$$

For the integral term in the second line of (50), we have

$$\begin{aligned}
&\int_0^{\frac{t}{2}} \frac{1}{\tau^{2\alpha}} \cdot \frac{1}{(1+\tau)^{2(\beta-\alpha)}} d\tau \\
= &\int_0^{1} \frac{1}{\tau^{2\alpha}} \cdot \frac{1}{(1+\tau)^{2(\beta-\alpha)}} d\tau + \int_1^{\frac{t}{2}} \frac{1}{\tau^{2\alpha}} \cdot \frac{1}{(1+\tau)^{2(\beta-\alpha)}} d\tau \\
= &J_1^1 + J_1^2.
\end{aligned} \tag{51}$$

For J_1^1, we have

$$\begin{aligned}
J_1^1 \lesssim &\int_0^1 \frac{1}{\tau^{2\alpha}} d\tau \\
= &\frac{\tau^{1-2\alpha}}{1-2\alpha}\Big|_0^1 \\
\lesssim &1,
\end{aligned} \tag{52}$$

where we used the fact $1 - 2\alpha > 0$, i.e., $\alpha < \frac{1}{2}$.
For J_1^2, we have

$$\begin{aligned}
J_1^2 \lesssim &\int_1^{\frac{t}{2}} \frac{1}{\tau^{2\alpha}} \cdot \frac{1}{(1+\tau)^{2(\beta-\alpha)}} d\tau \\
\lesssim &\int_1^{\frac{t}{2}} \frac{1}{\tau^{2\beta}} d\tau \\
= &\frac{\tau^{1-2\beta}}{1-2\beta}\Big|_1^{\frac{t}{2}} \\
\sim &1 - \left(\frac{t}{2}\right)^{1-2\beta} \\
\lesssim &1,
\end{aligned} \tag{53}$$

where we used the fact $1-2\beta<0$, i.e., $\beta>\frac{1}{2}$.
For J_2, agian using the fact that $(1+t-\tau)^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{p})}\lesssim 1$, we have

$$\begin{aligned}J_2 \lesssim & t^{-2\alpha}\cdot(1+t)^{-2(\beta-\alpha)}\int_{\frac{t}{2}}^{t}\frac{1}{(t-\tau)^{\frac{n}{2}(\frac{1}{\sigma}-\frac{1}{\nu})}}d\tau\\ \lesssim & t^{-2\beta}\int_0^{\frac{t}{2}}\frac{1}{\tau^{\frac{n}{2}(\frac{1}{\sigma}-\frac{1}{\nu})}}d\tau\\ \lesssim & t^{-2\beta+1-\frac{n}{2}(\frac{1}{\sigma}-\frac{1}{\nu})}\\ \lesssim & 1,\end{aligned} \tag{54}$$

since $1-\frac{n}{2}(\frac{1}{\sigma}-\frac{1}{\nu})-2\beta<0$ and $t>100$.
For J_3, similarly, note $(1+t-\tau)^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{p})}\lesssim 1$, we get

$$J_3\lesssim\int_0^{\frac{t}{2}}\frac{1}{\tau^{2\alpha}}\cdot\frac{1}{(1+\tau)^{2(\beta-\alpha)}}d\tau\lesssim 1. \tag{55}$$

For J_4, note when

$$\frac{t}{2}<\tau<t,(1+t-\tau)^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{p})}\lesssim 1,(1+\tau)^{-2(\beta-\alpha)}\sim(1+t)^{-2(\beta-\alpha)},\tau^{-2\alpha}\sim t^{-2\alpha},$$

then it follows that

$$\begin{aligned}J_4\lesssim & t^{-2\alpha}\cdot t^{-2(\beta-\alpha)}\cdot\frac{t}{2}\\ \sim & t^{1-2\beta}\\ \lesssim & 1,\end{aligned} \tag{56}$$

since $\beta>\frac{1}{2}$ and $t>100$.
Collecting all the above estimates involving J_1 to J_4, Claim 1 is immediate.
Consequently,

$$\|\Psi(f)\|_Y\lesssim\|f_0\|_{M^0_{p,\sigma}}+\|f\|^2_{Z_{r,\nu}}. \tag{57}$$

Now we turn to estimate the other term related to $Z_{r,\nu}$.
Estimate of $\|\Psi(f)\|_{Z_{r,\nu}}$.
On the one hand , by Lemma 1, we have

$$\|e^{t\Delta}f_0\|_{Z_{r,\nu}}\lesssim\sup_{t\in(0,\infty)}t^{\alpha}(1+t)^{\beta-\alpha}(1+\frac{1}{t})^{\alpha}\cdot\frac{1}{(1+t)^{\beta}}\|f_0\|_{M^0_{p,\sigma}(\mathbb{R}^n)}\lesssim\|f_0\|_{M^0_{p,\sigma}(\mathbb{R}^n)}, \tag{58}$$

where $1 < p < r$, and $1 < \nu < \sigma$.
On the other hand, for the integral term, note $\frac{1}{\nu} - \frac{1}{\tilde{\nu}} > 0$, $\frac{1}{\tilde{r}} - \frac{1}{r} > 0$, we have

$$\begin{aligned}&\Big\| \int_0^t e^{(t-\tau)\Delta} Q(f,f) d\tau \Big\|_{Z_{r,\nu}} \\ \lesssim &\sup_{t>0} t^{\alpha}(1+t)^{\beta-\alpha} \int_0^t \Big(1 + \frac{1}{(t-\tau)^{\frac{n}{2}(\frac{1}{\nu}-\frac{1}{\tilde{\nu}})}}\Big) \frac{1}{(1+t-\tau)^{\frac{n}{2}(\frac{1}{\tilde{r}}-\frac{1}{r})}} \|Q(f,f)\|_{M^0_{\tilde{r},\tilde{\nu}}(\mathbb{R}^n)} d\tau.\end{aligned} \tag{59}$$

Note by Proposition 3, see also (41), assume (42), we get

$$\|Q(f,f)\|_{M^0_{\tilde{r},\tilde{\nu}}(\mathbb{R}^n)} \lesssim \|f\|^2_{M^0_{r,\nu}(\mathbb{R}^n)}, \quad \frac{1}{\tilde{\nu}} = \frac{2}{\nu} - 1,$$

which implies that

$$\begin{aligned}&\Big\| \int_0^t e^{(t-\tau)\Delta} Q(f,f) d\tau \Big\|_{Z_{r,\nu}} \\ \lesssim &\|f\|^2_{Z_{r,\nu}} \sup_{t>0} t^{\alpha}(1+t)^{\beta-\alpha} \int_0^t \Big(1 + \frac{1}{(t-\tau)^{\frac{n}{2}(\frac{1}{\nu}-\frac{1}{\tilde{\nu}})}}\Big) \frac{1}{(1+t-\tau)^{\frac{n}{2}(\frac{1}{\tilde{r}}-\frac{1}{r})}} \frac{1}{\tau^{2\alpha}} \frac{1}{(1+\tau)^{2(\beta-\alpha)}} d\tau.\end{aligned} \tag{60}$$

Now we need another claim.
Claim 2: If $\frac{n}{2}(\frac{1}{\nu} - \frac{1}{\tilde{\nu}}) + \alpha \le 1, \nu < \tilde{\nu}, \tilde{r} < r$ and $\beta \ge 1$, then

$$\sup_{t>0} t^{\alpha}(1+t)^{\beta-\alpha} \int_0^t \Big(1 + \frac{1}{(t-\tau)^{\frac{n}{2}(\frac{1}{\nu}-\frac{1}{\tilde{\nu}})}}\Big) \frac{1}{(1+t-\tau)^{\frac{n}{2}(\frac{1}{\tilde{r}}-\frac{1}{r})}} \frac{1}{\tau^{2\alpha}} \frac{1}{(1+\tau)^{2(\beta-\alpha)}} d\tau < \infty. \tag{61}$$

Proof of Claim 2. Note that we can rewrite the integral term in (61) as

$$
\begin{aligned}
&t^{\alpha}(1+t)^{\beta-\alpha}\int_0^t \Big(1+\frac{1}{(t-\tau)^{\frac{n}{2}(\frac{1}{\nu}-\frac{1}{\tilde{\nu}})}}\Big)\frac{1}{(1+t-\tau)^{\frac{n}{2}(\frac{1}{\tilde{r}}-\frac{1}{r})}}\frac{1}{\tau^{2\alpha}}\frac{1}{(1+\tau)^{2(\beta-\alpha)}}d\tau\\
=&t^{\alpha}(1+t)^{\beta-\alpha}\int_0^{\frac{t}{2}}\frac{1}{(t-\tau)^{\frac{n}{2}(\frac{1}{\nu}-\frac{1}{\tilde{\nu}})}}\frac{1}{(1+t-\tau)^{\frac{n}{2}(\frac{1}{\tilde{r}}-\frac{1}{r})}}\frac{1}{\tau^{2\alpha}}\frac{1}{(1+\tau)^{2(\beta-\alpha)}}d\tau\\
&+t^{\alpha}(1+t)^{\beta-\alpha}\int_{\frac{t}{2}}^{t}\frac{1}{(t-\tau)^{\frac{n}{2}(\frac{1}{\nu}-\frac{1}{\tilde{\nu}})}}\frac{1}{(1+t-\tau)^{\frac{n}{2}(\frac{1}{\tilde{r}}-\frac{1}{r})}}\frac{1}{\tau^{2\alpha}}\frac{1}{(1+\tau)^{2(\beta-\alpha)}}d\tau\\
&+t^{\alpha}(1+t)^{\beta-\alpha}\int_0^{\frac{t}{2}}\frac{1}{(1+t-\tau)^{\frac{n}{2}(\frac{1}{\tilde{r}}-\frac{1}{r})}}\frac{1}{\tau^{2\alpha}}\frac{1}{(1+\tau)^{2(\beta-\alpha)}}d\tau\\
&+t^{\alpha}(1+t)^{\beta-\alpha}\int_{\frac{t}{2}}^{t}\frac{1}{(1+t-\tau)^{\frac{n}{2}(\frac{1}{\tilde{r}}-\frac{1}{r})}}\frac{1}{\tau^{2\alpha}}\frac{1}{(1+\tau)^{2(\beta-\alpha)}}d\tau\\
=:&J_5+J_6+J_7+J_8.
\end{aligned}
\tag{62}
$$

Case 1: $0<t\le 100$.
For J_5, note

$$
(t-\tau)^{-\frac{n}{2}(\frac{1}{\nu}-\frac{1}{\tilde{\nu}})}\lesssim t^{-\frac{n}{2}(\frac{1}{\nu}-\frac{1}{\tilde{\nu}})},(1+\tau)^{-2(\beta-\alpha)}\lesssim 1,(1+t-\tau)^{-\frac{n}{2}(\frac{1}{\tilde{r}}-\frac{1}{r})}\lesssim 1,
$$

we have

$$
\begin{aligned}
J_5\lesssim&t^{\alpha}\cdot 1\cdot t^{-\frac{n}{2}(\frac{1}{\nu}-\frac{1}{\tilde{\nu}})}\int_0^{\frac{t}{2}}\frac{1}{\tau^{2\alpha}}d\tau\\
\lesssim&t^{\alpha}\cdot t^{-\frac{n}{2}(\frac{1}{\nu}-\frac{1}{\tilde{\nu}})}\frac{\tau^{1-2\alpha}}{1-2\alpha}\Big|_0^{\frac{t}{2}}\\
\lesssim&t^{1-\alpha-\frac{n}{2}(\frac{1}{\nu}-\frac{1}{\tilde{\nu}})}\\
\lesssim&1,
\end{aligned}
\tag{63}
$$

since $1-2\alpha>0$, i.e., $\alpha<\frac{1}{2}$, and $1-\alpha-\frac{n}{2}(\frac{1}{\nu}-\frac{1}{\tilde{\nu}})\ge 0, 0<t\le 100$.
For J_6, note

$$
(1+\tau)^{-2(\beta-\alpha)}\lesssim 1,(1+t-\tau)^{-\frac{n}{2}(\frac{1}{\tilde{r}}-\frac{1}{r})}\lesssim 1,(1+t)^{\beta-\alpha}\lesssim 1,
$$

we have

$$\begin{aligned} J_6 \lesssim & t^{\alpha}(1+t)^{\beta-\alpha}\int_{\frac{t}{2}}^{t}\frac{1}{(t-\tau)^{\frac{n}{2}(\frac{1}{\nu}-\frac{1}{\tilde{\nu}})}}\frac{1}{\tau^{2\alpha}}d\tau \\ \lesssim & t^{\alpha}\cdot 1\cdot t^{-2\alpha}\int_{\frac{t}{2}}^{t}\frac{1}{(t-\tau)^{\frac{n}{2}(\frac{1}{\nu}-\frac{1}{\tilde{\nu}})}}d\tau \\ \lesssim & t^{-\alpha}\int_{0}^{\frac{t}{2}}\frac{1}{\tau^{\frac{n}{2}(\frac{1}{\nu}-\frac{1}{\tilde{\nu}})}}d\tau \\ \lesssim & t^{-\alpha+1-\frac{n}{2}(\frac{1}{\nu}-\frac{1}{\tilde{\nu}})} \\ \lesssim & 1, \end{aligned} \tag{64}$$

since $\frac{n}{2}(\frac{1}{\nu}-\frac{1}{\tilde{\nu}})+\alpha\le 1, 0<t\le 100$, i.e.,

$$t^{\alpha}(1+t)^{\beta-\alpha}\int_{\frac{t}{2}}^{t}\frac{1}{(t-\tau)^{\frac{n}{2}(\frac{1}{\nu}-\frac{1}{\tilde{\nu}})}}\frac{1}{(1+t-\tau)^{\frac{n}{2}(\frac{1}{\tilde{r}}-\frac{1}{r})}}\cdot\frac{1}{\tau^{2\alpha}}\cdot\frac{1}{(1+\tau)^{2(\beta-\alpha)}}d\tau\lesssim 1. \tag{65}$$

For J_7, by the fact that $(1+t-\tau)^{-\frac{n}{2}(\frac{1}{\tilde{r}}-\frac{1}{r})}\lesssim 1$, we have

$$J_7\lesssim t^{\alpha}\cdot 1\cdot\int_{0}^{\frac{t}{2}}\frac{1}{\tau^{2\alpha}}d\tau\lesssim t^{\alpha}\cdot t^{1-2\alpha}\sim t^{1-\alpha}\lesssim 1, \tag{66}$$

since $\alpha<1$.
For J_8, note $(1+\tau)^{-2(\beta-\alpha)}\lesssim 1, (1+t-\tau)^{-\frac{n}{2}(\frac{1}{\tilde{r}}-\frac{1}{r})}\lesssim 1$, we get

$$J_8\lesssim t^{\alpha}\cdot 1\cdot t^{-2\alpha}\int_{\frac{t}{2}}^{t}d\tau\lesssim t^{1-\alpha}\lesssim 1, \tag{67}$$

since $1-\alpha>0$.
Case 2: $t>100$.
For J_5, note when $0<\tau\le\frac{t}{2}$,

$$1+t-\tau\ge 1+\frac{t}{2}\ge\frac{1}{2}(1+t), \qquad t-\tau\ge\frac{t}{2}\ge\frac{1}{4}(1+t),$$

it follows that

$$\begin{aligned} J_5 \lesssim & t^{\alpha}(1+t)^{\beta-\alpha}(1+t)^{-\frac{n}{2}(\frac{1}{\nu}-\frac{1}{\tilde{\nu}})}(1+t)^{-\frac{n}{2}(\frac{1}{\tilde{r}}-\frac{1}{r})}\int_{0}^{\frac{t}{2}}\frac{1}{\tau^{2\alpha}}\cdot\frac{1}{(1+\tau)^{2(\beta-\alpha)}}d\tau \\ \lesssim & t^{\beta-\frac{n}{2}(\frac{1}{\nu}-\frac{1}{\tilde{\nu}})}(1+t)^{-\frac{n}{2}(\frac{1}{\tilde{r}}-\frac{1}{r})} \\ \lesssim & 1, \end{aligned} \tag{68}$$

where $\beta - \frac{n}{2}(\frac{1}{\nu} - \frac{1}{\tilde{\nu}}) - \frac{n}{2}(\frac{1}{\tilde{r}} - \frac{1}{r}) \le 0$, since $1 < \tilde{r} < p < r$.
For J_6, we have

$$\begin{aligned} J_6 \lesssim &(1+t)^{\beta}(1+t)^{-\frac{n}{2}(\frac{1}{\nu}-\frac{1}{\tilde{\nu}})} \int_{\frac{t}{2}}^{t} \frac{1}{(1+\tau)^{2\beta}} d\tau \\ \lesssim &(1+t)^{\beta}(1+t)^{-\frac{n}{2}(\frac{1}{\nu}-\frac{1}{\tilde{\nu}})} t^{-2\beta+1} \\ \lesssim &t^{1-\beta-\frac{n}{2}(\frac{1}{\nu}-\frac{1}{\tilde{\nu}})} \\ \lesssim &1, \end{aligned} \tag{69}$$

where $1 - \beta - \frac{n}{2}(\frac{1}{\nu} - \frac{1}{\tilde{\nu}}) \le 0$.
For J_7, note $1 + t - \tau \ge \frac{1}{2}(1+t)$, we have

$$J_7 \lesssim (1+t)^{\beta}(1+t)^{-\frac{n}{2}(\frac{1}{\tilde{r}}-\frac{1}{r})} \lesssim 1, \tag{70}$$

where $\beta - \frac{n}{2}(\frac{1}{\tilde{r}} - \frac{1}{r}) \le 0$, since $1 < \tilde{r} < p < r$.
For J_8, note that $(1+t-\tau)^{-\frac{n}{2}(\frac{1}{\tilde{r}}-\frac{1}{r})} \lesssim 1$, we get

$$J_8 \lesssim (1+t)^{\beta} t^{-2\beta+1} \lesssim t^{1-\beta} \lesssim 1, \tag{71}$$

where $\beta \ge 1$.
Collecting all the above estimates involving J_5 to J_8, Claim 2 is immediate.
Consequently,

$$\|\Psi(f)\|_{Z_{r,\nu}} \lesssim \|f_0\|_{M^0_{p,\sigma}} + \|f\|^2_{Z_{r,\nu}}. \tag{72}$$

Combining (57) and (72), and recalling that $\|f\|_X = \|f\|_X + \|f\|_{Z_{r,\nu}}$, the desired result (39) follows immediately.

With the boundedness of estimates obtained so far, finally we arrive at proving $\Psi(f)$ is a contraction mapping.
Note

$$Q(f,f) - Q(g,g) = Q(f, f-g) + Q(f-g, g),$$

we have

$$\Psi(f) - \Psi(g) = \int_0^t e^{(t-\tau)\Delta}(Q(f,f-g) + Q(f-g,g))d\tau.$$

Similar to (43) and (60), recalling Claim 1 and Claim 2, we get

$$\|\Psi(f) - \Psi(g)\|_Y \lesssim \Big(\|f\|_{Z_{r,\nu}} + \|g\|_{Z_{r,\nu}}\Big) \cdot \|f-g\|_{Z_{r,\nu}}, \tag{73}$$

and

$$\|\Psi(f)-\Psi(g)\|_{Z_{r,\nu}} \lesssim \Big(\|f\|_{Z_{r,\nu}}+\|g\|_{Z_{r,\nu}}\Big)\cdot\|f-g\|_{Z_{r,\nu}}. \tag{74}$$

Note by (39), if $\|f\|_X \le c_1\delta$, for some constant c_1, we have

$$\begin{aligned}\|\Psi(f)\|_X \le & c\Big(\|f_0\|_{M^0_{p,\sigma}}+\|f\|_X^2\Big)\\ \le & c(\delta+(c_1\delta)^2)\\ \le & c_1\delta.\end{aligned} \tag{75}$$

Thus, as long as $c(1+c_1\delta)\le c_1$, i.e., $c_1 \ge \frac{c}{1-c\delta}$, δ is small enough, we have

$$\|\Psi(f)-\Psi(g)\|_X \le \frac{1}{2}\|f-g\|_X, \text{ if } \delta \text{ is sufficently small.} \tag{76}$$

Consequently, the contraction mapping principle[12] could be applied and the proof of the main theorem is complete. □

5 Appendix

For the completeness, we give the classcial results about the estimate of the Riesz potential.

For $0<\alpha<n$, we let I_α be the Riesz potential operator defined for locally integrable functions by

$$I_\alpha(f)(x)=\int_{\mathbb{R}^n}\frac{f(y)}{|x-y|^{n-\alpha}}dy.$$

Lemma 5 – *Stein (2016) Assume* $1<p<\frac{n}{\alpha}, f\in L^p(\mathbb{R}^n)$, *then*

$$\|I_\alpha f\|_{L^q(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)}, \tag{77}$$

where

$$\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{n}. \tag{78}$$

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[12]Brezis, 2011, *Functional analysis, Sobolev spaces and partial differential equations.*

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