North-Western European Journal of Mathematics

Generalized three and four person hat game

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Received: November 11, 2022/Accepted: March 15, 2024/Online: March 25, 2024

Abstract

Ebert's hat problem with two colors and equal probabilities has, remarkable, the same optimal winning probability for three and four players. This paper studies Ebert's hat problem for three and four players, where the probabilities of the two colors may be different for each player. Our goal is to maximize the probability of winning the game and to describe winning strategies. We obtain different results for games with three and four players. We use the concept of an adequate set. The construction of adequate sets is independent of underlying probabilities and we can use this fact in the analysis of our general case. The computational complexity of the adequate set method is dramatically lower than by standard methods.

Keywords: information theory, coding theory, cooperative games, computational complexity.

MSC: 91A12, 68Q11.

1 Introduction

Gardner (1961) formulated hat puzzles. They have got an impulse by Ebert (1998). Buhler (2002) stated: "It is remarkable that a purely recreational problem comes so close to the research frontier". Also articles: Robinson (2001), Blum (2001) and Poulos (2001) about this subject got broad attention. This paper studies generalized Ebert's hat problem for three and four players. The probabilities of the two colors may be different for each player, but known to all the players. All players guess simultaneously the color of their own hat observing only the hat colors of the other players. It is also allowed for each player to pass: no color is guessed. The team wins if at least one player guesses his or her hat color correctly and none of the players has an incorrect guess. Our goal is to maximize the probability of winning the game and to describe winning strategies. The symmetric two color hat problem (equal probability 0.5 for each color) with $N = 2^k - 1$ players is solved in Ebert, Merkle, and Vollmer (2003), using Hamming codes, and with $N = 2^k$ players in Cohen et al. (1997) using extended Hamming codes. Burke, Gustafson, and Kendall (2002)

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try to solve the symmetric hat problem with $N = 3,4,5,7$ players using genetic programming. Their conclusion: The N-prisoners puzzle (alternative names: hat problem, hat game) gives evolutionary computation and genetic programming a new challenge to overcome. Lenstra and Seroussi (2005) show that in the symmetric case of two hat colors, and for any value of N, playing strategies are equivalent to binary covering codes of radius one.

Krzywkowski (2010) describes applications of the hat problem and its variations, and their connections to different areas of science. We have an open problem[2] : If the hat colors are not equally likely, how will the optimal strategy be affected? We will answer this question and our method gives also interesting results in the symmetric case. In section 2 we define our main tool: an adequate set. In sections 3 and 6 we obtain results for three and four person two color hat game, where each player i may have different probabilities (p_i, q_i) to get a specific colored hat. In sections 4 and 7 we obtain results for the asymmetric three and four person two color hat game, where each player has the same set of probabilities (p,q) to get a specific colored hat, but the probabilities are different. In sections 5 and 8 we find old and new results for the well known symmetric case $p = q = \frac{1}{2}$. Section 9 gives a comparison between generalized three and four person hat game. Section 10 handles with computational complexity. Central in all our investigations are adequate sets.

2 Adequate set

In this section we have N players and q colors. The N persons in our game are distinguishable, so we can label them from 1 to N. We label the q colors $0,1,\ldots,q-1$. The probabilities of the colors are known to all players. The probability that color i will be on a hat of player j is $p_{i,j}$ ($\sum_{i=0}^{q-1} p_{i,j} = 1$). Each possible configuration of the hats can be represented by an element of

$$B = \{b_1 b_2 \ldots b_N \mid b_i \in \{0,1,\ldots,q-1\}, i = 1,2,\ldots,N\}.$$

The S-code represents what the N different players sees. Player i sees q-ary code $b_1 \ldots b_{i-1} b_{i+1} \ldots b_N$ with decimal value $s_i = \sum_{k=1}^{i-1} b_k q^{N-k-1} + \sum_{k=i+1}^{N} b_k q^{N-k}$, a value between 0 and $q^{N-1} - 1$.

Let S be the set of all S-codes:

$$S = \left\{ s_1 s_2 \ldots s_N \mid s_i = \sum_{k=1}^{i-1} b_k q^{N-k-1} + \sum_{k=i+1}^{N} b_k q^{N-k}, b_i \in \{0,1,\ldots,q-1\}, i = 1,2,\ldots,N \right\}.$$

Each player has to make a choice out of $q+1$ possibilities: 0='guess color 0', 1='guess color 1', ..., $q-1$ ='guess color $q-1$', q='pass'.

[2] Johnson, 2001, *The Hat Problem*.

We define a decision matrix $D = (a_{i,s})_{1\le i\le N,\, 0\le s\le q^{N-1}-1}$ where $a_{i,s} \in \{0,1,\dots,q\}$. The meaning of $a_{i,s}$ is: player i sees S-code s and takes decision $a_{i,s}$ (guess a color or pass). We observe the total probability (sum) of our guesses. For each $b_1b_2\dots b_N$ in B we have:

$$\text{If } \left(a_{1,s_1} \in \{q,b_1\}\right) \wedge \cdots \wedge \left(a_{N,s_N} \in \{q,b_N\}\right) \wedge \neg\left(a_{1,s_1} = \cdots = a_{N,s_N} = q\right) \text{ then}$$

$$sum = sum + p_{b_1,1}p_{b_2,2}\cdots p_{b_N,N}.$$

Any choice of the $a_{i,s}$ in the decision matrix determines which cases $b_1b_2\dots b_N$ have a positive contribution to sum (we call them *good cases*) and which cases don't contribute positive to sum (we call them *bad cases*).

Definition 1 – Let $A \subset B$. A is adequate to $B \setminus A$ if for each q-ary element x in $B \setminus A$ there are $q-1$ elements in A which are equal to x up to one fixed q-ary position.

Theorem 1 – *Bad cases are adequate to good cases.*

Proof. Any good case has at least one $a_{i,s}$ not equal to q. Let this specific $a_{i,s}$ have value b_{i_0}. Then our good case generates $q-1$ bad cases by only changing the value b_{i_0} in any value of $0,1,\dots,q-1$ except b_{i_0}. □

The notion of an adequate set is the same idea as the concept of strong covering, introduced by Lenstra and Seroussi (2005). The number of elements in an adequate set will be written as *das* (dimension of adequate set). Adequate sets are generated by an adequate set generator (ASG). See Appendix A for an implementation in a VBA/Excel program. Given an adequate set, we obtain a decision matrix $D = (a_{i,s})$ by the following procedure.

Procedure DMG (Decision Matrix Generator):
Begin Procedure
For each element in the adequate set:

- Determine the q-ary representation $b_1b_2\dots b_N$
- Calculate S-codes $s_i = \sum_{k=1}^{i-1} b_k q^{N-k-1} + \sum_{k=i+1}^{N} b_k q^{N-k}$ $(i = 1,\dots,N)$
- For each player i: fill decision matrix with $a_{i,s_i} = b_i$ $(i = 1,\dots,N)$, where each cell may contain several values.

Matrix D is filled with bad colors. We can extract the good colors by considering all $a_{i,s}$ with $q-1$ different bad colors and then choose the only missing color. In all situations with less than $q-1$ different bad colors we pass. When there is an $a_{i,s}$ with q different bad colors all colors are bad, so the first option is to pass. But when we choose any color, we get a situation with $q-1$ colors. So in case of q bad colors we are free to choose any color or pass. The code for pass is q, but in our decision

matrices we prefer a blank, which supports readability. The code for 'any color or pass will do' is defined $q+1$, but we prefer a "$\star$" for readability.
End Procedure.

This procedure is implemented in the VBA/Excel program DMG in Appendix B.

3 Generalized three person two color hat game

Three distinguishable players are randomly fitted with a white (code 0) or black (code 1) hat. Each player i has his own probabilities p_i and q_i to get a white respectively a black hat, where $0 < p_i < 1,\ \ p_i + q_i = 1$ $(i = 1, 2, 3)$. All probabilities are known to all players. Part of the strategy is that the players give themselves an identification: 1, 2 and 3.

Our goal is to maximize the probability of winning the game and to describe winning strategies.

Let X be an adequate set and $P(X)$ is the probability generated by the adequate set. The adequate set X dominates the adequate set Y if $P(X) \leq P(Y)$. We also define: X dominates the adequate set Y absolutely if $P(X) < P(Y)$. We use the abbreviation DOM for domination. An adequate set A is non-dominated by a collection C of adequate sets when A dominates each element of C. Adequate sets X and Y are isomorphic when there is a bijection from {1,2,3} to itself which transforms X into Y. The decision matrices are then also isomorphic.

A player i with $p_i < q_i$ gets an asterix: when observing such a player we have to flip the colors: white becomes black and vice versa. In such a way we have without loss of generality $p_i \geq q_i$ $(i = 1, 2, 3)$.

The next step is to renumber the players in such a way that $\frac{p_1}{q_1} \geq \frac{p_2}{q_2} \geq \frac{p_3}{q_3}$, which is equivalent to $p_1 \geq p_2 \geq p_3$. So: $1 > p_1 \geq p_2 \geq p_3 \geq \frac{1}{2}$ or, equivalently: $0 < q_1 \leq q_2 \leq q_3 \leq \frac{1}{2}$. We define decision matrices:

α :

00	01	10	11
1			0
1			0
1			0

δ :

00	01	10	11
0			1
	0	1	
	0	1	

ϵ :

00	01	10	11
0	0	0	0
		$\star$	$\star$
		$\star$	$\star$

Theorem 2 – *Given* $0 < q_1 \le q_2 \le q_3 \le \frac{1}{2}$ *we have:*

Case	$q_2 = \frac{1}{2}$	$q_2 < \frac{1}{2}$
$\frac{1}{q_1} > \frac{1}{q_2} + \frac{1}{q_3}$	$p_1 > \frac{3}{4}$ ϵ	$p_1 > \frac{3}{4}$ ϵ
$\frac{1}{q_1} < \frac{1}{q_2} + \frac{1}{q_3}$	$\frac{3}{4}$ α, δ	$p_1 + q_1 q_2 q_3 \left(\frac{1}{q_2} + \frac{1}{q_3} - \frac{1}{q_1}\right) > p_1 > \frac{3}{4}$ δ
$\frac{1}{q_1} = \frac{1}{q_2} + \frac{1}{q_3}$	$\frac{3}{4}$ α, δ, ϵ	$p_1 > \frac{3}{4}$ δ, ϵ

where in each case we give the optimal probability in the first line and the optimal decision matrices in the second line.

Proof. We shall show that the following sets are absolute dominant:

- $\{4,5,6,7\}$ when $\frac{1}{q_1} > \frac{1}{q_2} + \frac{1}{q_3}$,
- $\{3,4\}$ and $\{0,7\}$ when $\frac{1}{q_1} < \frac{1}{q_2} + \frac{1}{q_3}$ with $p_2 = \frac{1}{2}$,
- $\{3,4\}$ when $\frac{1}{q_1} < \frac{1}{q_2} + \frac{1}{q_3}$ with $p_2 > \frac{1}{2}$,
- $\{4,5,6,7\}$, $\{3,4\}$ and $\{0,7\}$ when $\frac{1}{q_1} = \frac{1}{q_2} + \frac{1}{q_3}$ with $p_2 = \frac{1}{2}$,
- $\{4,5,6,7\}$ and $\{3,4\}$ when $\frac{1}{q_1} = \frac{1}{q_2} + \frac{1}{q_3}$ with $p_2 > \frac{1}{2}$.

Let *das* be the dimension of an adequate set (number of elements in the set). Obviously, there are no adequate sets with $das < 2$. When $das = 2$, we find 4 adequate sets (use ASG), independent of the underlying probabilities: $\{0,7\}, \{1,6\}, \{2,5\}$ and $\{3,4\}$. We notice that $\{1,6\}, \{2,5\}$ and $\{3,4\}$ are isomorphic: they can be obtained from any of the three by renumbering the players. E.g. interchanging players 1 and 3 in binary codes of $\{1,6\}$ gives $\{3,4\}$ and interchanging players 2 and 3 in $\{1,6\}$ gives $\{2,5\}$.

We are looking for optimal adequate sets. An adequate set consist of bad cases. We want to maximize the winning probability, so we minimize the adequate set probability.

The next table shows the 4 adequate sets and probabilities:

$\{0,7\}$	$p_1 p_2 p_3 + q_1 q_2 q_3 = A$
$\{1,6\}$	$p_1 p_2 q_3 + q_1 q_2 p_3 = B$
$\{2,5\}$	$p_1 q_2 p_3 + q_1 p_2 q_3 = C$
$\{3,4\}$	$p_1 q_2 q_3 + q_1 p_2 p_3 = D$

We have:

$$A - B = q_1 q_2 q_3 \left(\frac{p_1}{q_1}\frac{p_2}{q_2} - 1\right)\left(\frac{p_3}{q_3} - 1\right) \qquad B - C = q_1 q_2 q_3 \left(\frac{p_2}{q_2} - \frac{p_3}{q_3}\right)\left(\frac{p_1}{q_1} - 1\right)$$
$$A - C = q_1 q_2 q_3 \left(\frac{p_1}{q_1}\frac{p_3}{q_3} - 1\right)\left(\frac{p_2}{q_2} - 1\right) \qquad B - D = q_1 q_2 q_3 \left(\frac{p_1}{q_1} - \frac{p_3}{q_3}\right)\left(\frac{p_2}{q_2} - 1\right)$$
$$A - D = q_1 q_2 q_3 \left(\frac{p_3}{q_3}\frac{p_2}{q_2} - 1\right)\left(\frac{p_1}{q_1} - 1\right) \qquad C - D = q_1 q_2 q_3 \left(\frac{p_1}{q_1} - \frac{p_2}{q_2}\right)\left(\frac{p_3}{q_3} - 1\right)$$

So we have: $A \geq B \geq C \geq D$: the adequate set {3,4} dominates all other adequate sets when *das*=2.

When $das = 3$, we get (using the adequate set generator) 24 adequate sets (see Appendix C), all absolutely dominated by $\{0,7\},\{1,6\},\{2,5\}$ or $\{3,4\}$.

When $das = 4$ we get the situation in Appendix D.

When $das > 4$ then always one or more of the sets $\{0,7\},\{1,6\},\{2,5\}$, $\{3,4\}$ is included (we don't need the ASG).

Adequate set $\{0,7\}$ has value A and decision matrix α. Adequate set $\{3,4\}$ has value D and decision matrix δ. Adequate set $\{4,5,6,7\}$ has value p_1 and decision matrix ϵ. Using DMG (Appendix B) we find α, δ and ϵ.

We first consider the battle between $\{3,4\}$ and $\{4,5,6,7\}$. $\{3,4\}$ is the winner when $p_1 q_2 q_3 + q_1 p_2 p_3 < q_1$, so: $\frac{1}{q_1} < \frac{1}{q_2} + \frac{1}{q_3}$. $\{4,5,6,7\}$ is the winner when $\frac{1}{q_1} > \frac{1}{q_2} + \frac{1}{q_3}$.

Case 1: $\frac{1}{q_1} > \frac{1}{q_2} + \frac{1}{q_3}$. So $q_1 < q_2$; we get: $1 > p_1 > p_2 \geq p_3 \geq \frac{1}{2} \geq q_3 \geq q_2 > q_1 > 0$.

Consider Appendix D. To obtain absolute dominance we have to examine the adequate sets $\{0,3,5,6\}$ and $\{1,2,4,7\}$. For the set $\{0,3,5,6\}$, the probability is $q_1 + (p_1 - q_1)(p_2 p_3 + q_2 q_3) > q_1$.

Similarly, for the set $\{1,2,4,7\}$, its probability is also greater than q_1.

So $\{4,5,6,7\}$ is absolute dominant with winning probability $1 - q_1 = p_1$ and decision matrix ϵ, where $\frac{1}{q_1} > \frac{1}{q_2} + \frac{1}{q_3} \geq 4$, so $p_1 > \frac{3}{4}$.

Case 2: $\frac{1}{q_1} < \frac{1}{q_2} + \frac{1}{q_3}$. There are 4 potential dominant sets: $\{0,7\},\{1,6\},\{2,5\},\{3,4\}$. The last three are isomorphic and because of $A \leq B \leq C \leq D$ we analyze the battle between A and D.

Case 2.1: α and δ are both optimal: $A = D$. So:

$$\left(\frac{p_3}{q_3}\frac{p_2}{q_2} - 1\right)\left(\frac{p_1}{q_1} - 1\right) = 0$$
$$\Leftrightarrow \left(\left(p_2 = q_2 = \frac{1}{2}\right) \wedge \left(p_3 = q_3 = \frac{1}{2}\right)\right) \vee \left(p_1 = q_1 = \frac{1}{2}\right)$$

(Cont. next page)

$$\Leftrightarrow\left(\frac{1}{2}=p_3=p_2\le p_1<1\right)\vee\left(p_1=p_2=p_3=\frac{1}{2}\right)$$
$$\Leftrightarrow\left(\frac{1}{2}=p_3=p_2\le p_1<1\right)$$
$$\Leftrightarrow p_2=\frac{1}{2}$$

and optimal probability is $1-(p_1q_2q_3+q_1p_2p_3)=\frac{3}{4}$.

Case 2.2: Only δ optimal: $A<D$. So: $p_2>\frac{1}{2}$. We note: $\{1,6\},\{2,5\}$ and $\{3,4\}$ are isomorphic, but this doesn't imply $B=C=D$: the probabilities are not always invariant by a bijection of the players. We have $A<D$ and $B\le C\le D$. When B or C are equal to D then D is absolute dominant (up to isomorphic) otherwise $B\le C<D$ and D is absolute dominant. Optimal probability: $1-(p_1q_2q_3+q_1p_2p_3)=p_1+q_1q_2q_3(\frac{1}{q_2}+\frac{1}{q_3}-\frac{1}{q_1})>p_1$. And $\frac{1}{q_1}=\frac{1}{q_2}+\frac{1}{q_3}<4$, so $p_1>\frac{3}{4}$.

Case 3: $\frac{1}{q_1}=\frac{1}{q_2}+\frac{1}{q_3}$.
Optimal probability: $p_1+q_1q_2q_3(\frac{1}{q_2}+\frac{1}{q_3}-\frac{1}{q_1})=p_1$.

Case 3.1: $p_2=\frac{1}{2}$.
Optimal probability: $p_1=\frac{3}{4}$.
Optimal decision matrices: $\frac{1}{q_1}\ge\frac{1}{q_2}+\frac{1}{q_3}$ gives ϵ and $\frac{1}{q_1}\le\frac{1}{q_2}+\frac{1}{q_3}$ gives α and δ.

Case 3.2: $p_2>\frac{1}{2}$.
Optimal probability: p_1 and $\frac{1}{q_1}=\frac{1}{q_2}+\frac{1}{q_3}<4$, so $p_1>\frac{3}{4}$.
Optimal decision matrices: $\frac{1}{q_1}\ge\frac{1}{q_2}+\frac{1}{q_3}$ gives ϵ and $\frac{1}{q_1}\le\frac{1}{q_2}+\frac{1}{q_3}$ gives δ. □

Note 1. Instead of $\frac{1}{q}$ we can also use $\frac{p}{q}$. E.g. : we get $\frac{p_1}{q_1}=\frac{p_2}{q_2}+\frac{p_3}{q_3}+1$ instead of $\frac{1}{q_1}=\frac{1}{q_2}+\frac{1}{q_3}$.

Note 2. We observe that the well-known strategy α is only dominant when $(\frac{1}{q_1}\le\frac{1}{q_2}+\frac{1}{q_3})\wedge(q_2=\frac{1}{2})\Leftrightarrow(p_2=p_3=\frac{1}{2})\wedge(\frac{1}{2}\le p_1\le\frac{3}{4})$.

4 Asymmetric three person two color hat game

In this section we study three person two color asymmetric hat game. For each player let p be the probability to get a white hat and q be the probability to get a black hat. Without loss of generality we may assume (asymmetric case): $\frac{1}{2}<p<1$.

Theorem 3 – *In asymmetric three person two color hat game we have maximal probability* $1 - pq$ *of winning the game, with decision matrix:*

00	01	10	11
0			1
	0	1	
	0	1	

Proof. Use the result of Case 2.2 $(\frac{1}{q_1} < \frac{1}{q_2} + \frac{1}{q_3}) \wedge (p_2 > \frac{1}{2})$ in Theorem 2. Optimal probability is $1 - (p_1 q_2 q_3 + q_1 p_2 p_3) = 1 - pq$. □

5 Symmetric two color three person hat game

In this section we focus on the symmetric hat game with two colors and three players. Each player has a white hat with probability $\frac{1}{2}$ and a black hat with probability $\frac{1}{2}$.

Theorem 4 – *For symmetric three person two color hat game the maximal probability is* $\frac{3}{4}$ *and the optimal decision matrices are:*

00	01	10	11
0			1
	0	1	
	0	1	

and

00	01	10	11
1			0
1			0
1			0

Proof. Use result in Case 2.1 $(\frac{1}{q_1} < \frac{1}{q_2} + \frac{1}{q_3}) \wedge (p_2 = \frac{1}{2})$ in Theorem 2. □

6 Generalized four person two color hat game

Four distinguishable players are randomly fitted with a white or black hat. The code for a white hat is 0 and for a black hat is 1. Each player i has his own probabilities p_i and q_i to get a white respectively a black hat, where $0 < p_i < 1$, $p_i + q_i = 1$ $(i = 1, 2, 3, 4)$. All probabilities are known to all players. Part of the strategy is that the players give themselves an identification: 1, 2, 3 and 4.

Our goal is to maximize the probability of winning the game and to describe winning strategies.

A player i with $p_i < q_i$ gets an asterix: when observing such a player we have to flip the colors: white becomes black and vice versa. In such a way we have without loss of generality $p_i \geq q_i$ $(i = 1, 2, 3, 4)$.

The next step is to renumber the players in such a way that $\frac{p_1}{q_1} \geq \frac{p_2}{q_2} \geq \frac{p_3}{q_3} \geq \frac{p_4}{q_4}$, which is equivalent to $p_1 \geq p_2 \geq p_3 \geq p_4$. So: $1 > p_1 \geq p_2 \geq p_3 \geq p_4 \geq \frac{1}{2}$ or, equivalently: $0 < q_1 \leq q_2 \leq q_3 \leq q_4 \leq \frac{1}{2}$. Using ASG with $das < 4$ we get no adequate sets.

Lemma 1 – $\{6,7,8,9\}$ *dominates all adequate sets with* $das = 4$.

Proof. When $das = 4$ we get, using ASG, 40 adequate sets, see appendix E. In the same Appendix we see that all sets are dominated by another set, except $\{6,7,8,9\}$. The proof of each dominance relation proceeds along the same lines. We give one example: nr 2. dominates nr 1.
$\{0,2,13,15\}\, DOM\, \{0,1,14,15\}$
$1,14 \geq 2,13 \quad 0001\ 1110 \geq 0010\ 1101$
$\left(\frac{p_1}{q_1}.\frac{p_2}{q_2} - 1\right)\left(\frac{p_1}{q_1} - \frac{p_2}{q_2}\right) \geq 0.$ □

Lemma 2 – *When* $das = 5$ *we get 8 non-dominated adequate sets:*

S_0	0	11	13	14	15
S_1	1	10	12	14	15
S_2	2	9	12	13	15
S_3	3	8	12	13	14
S_4	4	9	10	11	15
S_5	5	8	10	11	14
S_6	6	8	9	11	13
S_7	7	8	9	10	12

where S_x *is the shortcut for the adequate set starting with an* x.

Proof. When $das = 5$ we get, using ASG, 560 adequate sets. We use the inclusion principle: we look for subsets of these 560 sets in the set of 40 adequate sets with $das = 4$. This procedure is realized in the program ASG45 (see appendix F). In this way we eliminate 480 dominated adequate sets. The output of ASG45 (80 non-dominated sets) is shown in appendix G, where we also show that S_x $(x = 0,1,\ldots,7)$ are the only non-dominated sets when $das = 5$. □

Lemma 3 – *When* $das > 5$ *we find one non-dominated adequate set:*

$$\{8,9,10,11,12,13,14,15\}.$$

Proof. When $das > 5$ then only dominated sets are found, with one exception: $das = 8$. Use the inclusion principle: a subset is found in the $das = 5$ adequate sets; this procedure can be automated in programs ASG56, ASG57,..., ASG516, analogue to program ASG45. Using ASG58 we get 10 (out of 10310) non-dominated sets :

0	1	2	3	4	5	6	7
0	1	2	3	8	9	10	11
0	1	4	5	8	9	12	13
0	2	4	6	8	10	12	14
0	3	5	6	9	10	12	15
1	2	4	7	8	11	13	14
1	3	5	7	9	11	13	15
2	3	6	7	10	11	14	15
4	5	6	7	12	13	14	15
8	9	10	11	12	13	14	15

They are all dominated (use from top to bottom the four different positions principle) by the last one. For example first and second element:

$\{0,1,2,3,8,9,10,11\}\,DOM\,\{0,1,2,3,4,5,6,7\}$

$4\ 5\ 6\ 7 \geq 8\ 9\ 10\ 11$

$0100\ 0101\ 0110\ 0111\ \geq\ 1000\ 1001\ 1010\ 1011$

$(01-10)00+(01-10)01+(01-10)10+(01-10)11 \geq 0$

which results in: $\frac{p_1}{q_1} - \frac{p_2}{q_2} \geq 0$.
We abbreviate the last one by its first element: S_8. □

The optimal set when $das = 4$: $\{6,7,8,9\}$ is noted by its last element: S_9. Our goal is to prove: $\{S_7, S_8, S_9\}$ dominates all adequate sets. We make use of the following Lemmas:

Lemma 4 – $\{S_7, S_9\}$ *dominates* S_6.

Proof. $S_7 = \{7,8,9,10,12\}$ dominates $S_6 = \{6,8,9,11,13\}$ when:

$7\ 10\ 12\ \leq\ 6\ 11\ 13$

$0111\ 1010\ 1100\ \leq\ 0110\ 1011\ 1101$

$(-p_1q_2q_3 + q_1p_2q_3 + q_1q_2p_3)(p_4 - q_4) \leq 0$

$$\left(-\frac{p_1}{q_1} + \frac{p_2}{q_2} + \frac{p_3}{q_3}\right)\left(\frac{p_4}{q_4} - 1\right) \leq 0.$$

S_9=$\{6,7,8,9\}$ dominates S_6= $\{6,8,9,11,13\}$ when:

$7\ \leq\ 11\ 13$

$0111\ \leq\ 1011\ 1101$

$(p_1q_2q_3 - q_1p_2q_3 - q_1q_2p_3)q_4 \leq 0$

$$\frac{p_1}{q_1} - \frac{p_2}{q_2} - \frac{p_3}{q_3} \leq 0.$$ □

Lemma 5 – $\{S_7, S_9\}$ *dominates* S_5.

Proof. $S_7 = \{7,8,9,10,12\}$ dominates $S_5 = \{5,8,10,11,14\}$ when:

$$7\ 9\ 12\ \leq 5\ 11\ 14$$
$$0111\ 1001\ 1100\ \leq\ 0101\ 1011\ 1110$$
$$\left(-\frac{p_1}{q_1}+\frac{p_2}{q_2}+\frac{p_4}{q_4}\right)\left(\frac{p_3}{q_3}-1\right)\leq 0.$$

S_9= $\{6,7,8,9\}$ dominates S_5= $\{5,8,10,11,14\}$ when:

$$6\ 7\ 9\ \leq\ 5\ 10\ 11\ 14$$
$$0110\ 0111\ 1001\ \leq\ 0101\ 1010\ 1011\ 1110$$
$$\left(\frac{p_1}{q_1}-\frac{p_2}{q_2}\right)\left(\frac{p_3}{q_3}-\frac{p_4}{q_4}\right)\geq \frac{p_1}{q_1}-\frac{p_2}{q_2}-\frac{p_4}{q_4}.$$

□

Lemma 6 – $\{S_7, S_9\}$ *dominates* S_4.

Proof. $S_7 = \{7,8,9,10,12\}$ dominates $S_4 = \{4,9,10,11,15\}$ when:

$$7\ 8\ 12\ \leq\ 4\ 11\ 15$$
$$0111\ 1000\ 1100\ \leq\ 0100\ 1011\ 1111$$
$$\left(\frac{p_1}{q_1}-\frac{p_2}{q_2}-1\right)\left(\frac{p_3}{q_3}\frac{p_4}{q_4}-1\right)\geq 0.$$

S_9=$\{6,7,8,9\}$ dominates S_4= $\{5,9,10,11,15\}$ when:

$$6\ 7\ 8\ \leq\ 4\ 10\ 11\ 15$$
$$0110\ 0111\ 1000\ \leq\ 0100\ 1010\ 1011\ 1111$$
$$\left(\frac{p_1}{q_1}-\frac{p_2}{q_2}\right)\left(\frac{p_3}{q_3}-1\right)\frac{p_4}{q_4}\geq \frac{p_1}{q_1}-\frac{p_2}{q_2}-1.$$

□

Lemma 7 – $\{S_7, S_9\}$ *dominates* S_3.

Proof. S_7=$\{7,8,9,10,12\}$ dominates S_3= $\{3,8,12,13,14\}$ when:

$$7\ 9\ 10\ \leq\ 3\ 13\ 14$$
$$0111\ 1001\ 1010\ \leq\ 0011\ 1101\ 1110$$
$$\left(\frac{p_1}{q_1}-\frac{p_3}{q_3}-\frac{p_4}{q_4}\right)\left(\frac{p_2}{q_2}-1\right)\geq 0.$$

S_9= $\{6,7,8,9\}$ dominates S_3=$\{3,8,12,13,14\}$ when:

$$6\ 7\ 9 \ \leq\ 3\ 12\ 13\ 14$$
$$0110\ 0111\ 1001 \ \leq\ 0011\ 1100\ 1101\ 1110$$
$$\left(\frac{p_1}{q_1}-\frac{p_3}{q_3}\right)\left(1+\frac{q_4}{p_4}-\frac{p_2}{q_2}\frac{q_4}{p_4}\right)\leq 1.$$

When $\frac{p_1}{q_1}-\frac{p_3}{q_3}-\frac{p_4}{q_4}\geq 0$ then S_7 dominates S_3. So we consider $\frac{p_1}{q_1}-\frac{p_3}{q_3}-\frac{p_4}{q_4}\leq 0$ and get in the S_9 dominates S_3 case:

$$\left(\frac{p_1}{q_1}-\frac{p_3}{q_3}\right)\left(1+\frac{q_4}{p_4}-\frac{p_2}{q_2}\frac{q_4}{p_4}\right)\leq \frac{p_4}{q_4}(1+\frac{q_4}{p_4}-\frac{p_2}{q_2}\frac{q_4}{p_4})=\frac{p_4}{q_4}-\frac{p_2}{q_2}+1\leq 1.$$ □

Lemma 8 – $\{S_7,S_9\}$ *dominates* S_2.

Proof. After some calculations we get: S_7 dominates S_2 when

$$\left(\frac{p_1}{q_1}-\frac{p_3}{q_3}-1\right)\left(\frac{p_2}{q_2}\frac{p_4}{q_4}-1\right)\geq 0.$$

S_9 dominates S_2 when

$$\left(\frac{p_1}{q_1}-\frac{p_3}{q_3}\right)\left(1+\frac{p_4}{q_4}-\frac{p_2}{q_2}\frac{p_4}{q_4}\right)\leq 1.$$

When $\frac{p_1}{q_1}-\frac{p_3}{q_3}\geq 1$ then S_7 dominates S_2. So we consider $\frac{p_1}{q_1}-\frac{p_3}{q_3}\leq 1$ and get in the S_9 dominates S_2 case:

$$\left(\frac{p_1}{q_1}-\frac{p_3}{q_3}\right)\left(1+\frac{p_4}{q_4}-\frac{p_2}{q_2}\frac{p_4}{q_4}\right)\leq 1+\frac{p_4}{q_4}\left(1-\frac{p_2}{q_2}\right)\leq 1.$$ □

Lemma 9 – $\{S_7,S_9\}$ *dominates* S_1.

Proof. After some calculations we get: S_7 dominates S_1 when

$$\left(\frac{p_1}{q_1}-\frac{p_4}{q_4}-1\right)\left(\frac{p_2}{q_2}\frac{p_3}{q_3}-1\right)\geq 0.$$

S_9 dominates S_1 when

$$\left(\frac{p_1}{q_1}-\frac{p_4}{q_4}\right)\left(1-\frac{p_2}{q_2}\frac{p_3}{q_3}\right)+\left(\frac{p_1}{q_1}-\frac{p_2}{q_2}\right)\frac{p_4}{q_4}+\left(\frac{p_2}{q_2}-\frac{p_4}{q_4}\right)\frac{p_3}{q_3}\geq 1.$$

When $\frac{p_1}{q_1} - \frac{p_4}{q_4} \geq 1$ then S_7 dominates S_1. So we consider $\frac{p_1}{q_1} - \frac{p_4}{q_4} \leq 1$ and get in the S_9 dominates S_1 case:

$$\begin{aligned}\left(\frac{p_1}{q_1} - \frac{p_4}{q_4}\right)\left(1 - \frac{p_2}{q_2}\frac{p_3}{q_3}\right) + \left(\frac{p_1}{q_1} - \frac{p_2}{q_2}\right)\frac{p_4}{q_4} + \left(\frac{p_2}{q_2} - \frac{p_4}{q_4}\right)\frac{p_3}{q_3} \\ &\leq \left(\frac{p_1}{q_1} - \frac{p_4}{q_4}\right)\left(1 - \frac{p_2}{q_2}\frac{p_3}{q_3}\right) + \left(\frac{p_1}{q_1} - \frac{p_2}{q_2}\right)\frac{p_3}{q_3} + \left(\frac{p_2}{q_2} - \frac{p_4}{q_4}\right)\frac{p_3}{q_3} \\ &= \left(\frac{p_1}{q_1} - \frac{p_4}{q_4}\right)\left(1 - \frac{p_2}{q_2}\frac{p_3}{q_3} + \frac{p_3}{q_3}\right) \\ &\leq 1 + \left(1 - \frac{p_2}{q_2}\right)\frac{p_3}{q_3} \leq 1. \qquad \square\end{aligned}$$

In the next lemma we use$\{S_6, S_9\}$ instead of $\{S_7, S_9\}$, which gives no problems for $\{S_7, S_9\}$ dominates S_6.

Lemma 10 – *$\{S_6, S_9\}$ dominates S_0.*

Proof. After some calculations we get: S_6 dominates S_0 when

$$\left(\frac{p_4}{q_4}(\frac{p_1}{q_1} - 1) - 1\right)\left(\frac{p_2}{q_2}\frac{p_3}{q_3} - 1\right) \geq 0.$$

S_9 dominates S_0 when

$$\left(\frac{p_1}{q_1} - 1\right)\left(1 - \frac{p_2}{q_2}\frac{p_3}{q_3}\right)\frac{p_4}{q_4} + \left(\frac{p_1}{q_1} - \frac{p_2}{q_2}\right) + \left(\frac{p_2}{q_2} - 1\right)\frac{p_3}{q_3} \leq 1.$$

When $\frac{p_4}{q_4}(\frac{p_1}{q_1} - 1) \geq 1$ then S_6 dominates S_0. So we consider $\frac{p_4}{q_4}\left(\frac{p_1}{q_1} - 1\right) \leq 1$ and get in the S_9 dominates S_0 case:

$$\begin{aligned}\left(\frac{p_1}{q_1} - 1\right)\left(1 - \frac{p_2}{q_2}\frac{p_3}{q_3}\right)\frac{p_4}{q_4} + \left(\frac{p_1}{q_1} - \frac{p_2}{q_2}\right) + \left(\frac{p_2}{q_2} - 1\right)\frac{p_3}{q_3} \\ &\leq \left(\frac{p_1}{q_1} - 1\right)\left(1 - \frac{p_2}{q_2}\frac{p_3}{q_3}\right)\frac{p_4}{q_4} + \left(\frac{p_1}{q_1} - \frac{p_2}{q_2}\right)\frac{p_3}{q_3} + \left(\frac{p_2}{q_2} - 1\right)\frac{p_3}{q_3} \\ &= \left(\frac{p_1}{q_1} - 1\right)\frac{p_4}{q_4} + \left(\frac{p_1}{q_1} - 1\right)\frac{p_3}{q_3}\left(1 - \frac{p_2}{q_2}\frac{p_4}{q_4}\right) \leq 1. \qquad \square\end{aligned}$$

All these Lemmas leads us to

Theorem 5 – *When $\frac{1}{q_1} \leq \frac{1}{q_2} + \frac{1}{q_3} - 1$ then maximal winning probability is*

$$p_1 + q_1 q_2 q_3 \left(\frac{1}{q_2} + \frac{1}{q_3} - \frac{1}{q_1}\right) \geq p_1 + q_1 q_2 q_3$$

with optimal decision matrix:

000	001	010	011	100	101	110	111
0	0					1	1
		0	0	1	1		
		0	0	1	1		
			⋆	⋆			

($\star$ means: any color or pass; stars are independent)

When $\frac{1}{q_2}+\frac{1}{q_3}-1 \le \frac{1}{q_1} \le \frac{1}{q_2}+\frac{1}{q_3}+\frac{1}{q_4}-1$ *then maximal winning probability is*

$$p_1 + q_1q_2q_3q_4\left(\frac{1}{q_2}+\frac{1}{q_3}+\frac{1}{q_4}-\frac{1}{q_1}-1\right),$$

a value between p_1 *and* $p_1+q_1q_2q_3$*, with optimal decision matrix:*

000	001	010	011	100	101	110	111
0	0	0		0			1
			0	⋆	1	1	
			0	⋆	1	1	
			0	⋆	1	1	

When $\frac{1}{q_1} \ge \frac{1}{q_2}+\frac{1}{q_3}+\frac{1}{q_4}-1$ *then maximal winning probability is* p_1 *with optimal decision matrix:*

000	001	010	011	100	101	110	111
0	0	0	0	0	0	0	0
				⋆	⋆	⋆	⋆
				⋆	⋆	⋆	⋆
				⋆	⋆	⋆	⋆

Proof. The preceding Lemmas invite us to make a comparison between S_7, S_8 and S_9. After some calculations we get:

S_9 is winner when (S_9 dominates S_8) and (S_9 dominates S_7):
$(\frac{1}{q_1} \le \frac{1}{q_2}+\frac{1}{q_3}) \wedge (\frac{1}{q_1} \le \frac{1}{q_2}+\frac{1}{q_3}-1) = (\frac{1}{q_1} \le \frac{1}{q_2}+\frac{1}{q_3}-1)$
Maximal probability: $1-(p_1q_2q_3+q_1p_2p_3) = p_1+q_1q_2q_3(\frac{1}{q_2}+\frac{1}{q_3}-\frac{1}{q_1}) \ge p_1+q_1q_2q_3$.

S_7 is winner when (S_7 dominates S_9) and (S_7 dominates S_8):
$(\frac{1}{q_1} \ge \frac{1}{q_2}+\frac{1}{q_3}-1) \wedge (\frac{1}{q_1} \le \frac{1}{q_2}+\frac{1}{q_3}+\frac{1}{q_4}-1) = (\frac{1}{q_2}+\frac{1}{q_3}-1 \le \frac{1}{q_1} \le \frac{1}{q_2}+\frac{1}{q_3}+\frac{1}{q_4}-1)$
Maximal probability:
$1-[p_1q_2q_3q_4+q_1p_2p_3+q_1p_4(p_2q_3+q_2p_3)] = p_1+q_1q_2q_3q_4(\frac{1}{q_2}+\frac{1}{q_3}+\frac{1}{q_4}-\frac{1}{q_1}-1)$, a value between p_1 and $p_1+q_1q_2q_3$.

S_8 is winner when (S_8 dominates S_9) and (S_8 dominates S_7):

$(\frac{1}{q_1} \geq \frac{1}{q_2} + \frac{1}{q_3}) \wedge (\frac{1}{q_1} \geq \frac{1}{q_2} + \frac{1}{q_3} + \frac{1}{q_4} - 1) = (\frac{1}{q_1} \geq \frac{1}{q_2} + \frac{1}{q_3} + \frac{1}{q_4} - 1)$
Maximal probability: p_1. □

Note 1. Conditions can also be formulated in the form $\frac{p}{q}$, e.g. $\frac{1}{q_2} + \frac{1}{q_3} - 1 \leq \frac{1}{q_1} \leq \frac{1}{q_2} + \frac{1}{q_3} + \frac{1}{q_4} - 1$ becomes $\frac{p_2}{q_2} + \frac{p_3}{q_3} \leq \frac{p_1}{q_1} \leq \frac{p_2}{q_2} + \frac{p_3}{q_3} + \frac{p_4}{q_4} + 1$.

Note 2. The domination used in this section is not absolutely, so there may be more optimal decision matrices with the same maximal probability. In sections 7 and 8 we shall use absolute domination and get all non isomorphic optimal decision matrices.

7 Asymmetric four person two color hat game

Theorem 6 – *In asymmetric four person (two color) hat game we have maximal probability* $1 - pq$ *of winning the game, with two optimal decision matrices:*

000	001	010	011	100	101	110	111
	1		1	0		0	
	1		1	0		0	
	★					★	
0	0					1	1

and:

000	001	010	011	100	101	110	111
	1	0			0	1	
	1	0			0	1	
	1	0		0			1
0			1		1	0	

Proof. Partial results can be obtained as a special case of the preceding section, but we want all (non isomorphic) optimal decision matrices and therefore we use absolute domination.

We use ASG with parameters n=4, p=0.9, das = 4 and get 40 adequate sets. Minimum sum is 0.09 and 24 adequate sets are optimal. Appendix H shows a sorted list of all 40 adequate sets. By definition, the construction of an adequate set is independent of p. In Appendix H we use patterns. E.g. pattern 01210 correspondents with probability $0 * q^4 + 1 * pq^3 + 2 * p^2q^2 + 1 * p^3q + 0 * p^4$. We get:

pattern	probability
01210 (sum : 0.09)	$pq^3 + 2p^2q^2 + p^3q$
10120 (sum : 0.154)	$q^4 + p^2q^2 + 2p^3q$
02101 (sum : 0.666)	$2pq^3 + p^2q^2 + p^4$
11011 (sum : 0.73)	$q^4 + pq^3 + p^3q + p^4$

It is not difficult to prove that 01210 absolutely dominates all other patterns when $p>q$: the probability of 01210 is less than all the other probabilities.

Let $\Psi(N,p)$ be the maximum probability of correct guessing in our asymmetric hat game with N players.

$$\Psi(4,p) = 1 - \left(pq^3 + 2\, p^2 q^2 + \ p^3 q \right) = 1 - pq = 1 - p + \ p^2$$

We remark that $\Psi(4,p) = \Psi(3,p)$.

All 24 adequate sets with the 01210 pattern generates optimal decision matrices. Procedure DMG gives as result with adequate set $\{1,3,12,14\}$:

000	001	010	011	100	101	110	111
	1		1	0		0	
	1		1	0		0	
	⋆					⋆	
0	0					1	1

where ⋆ means: any color or pass will do. This happens when player 3 sees 001 or 110, which corresponds to situations 0001, 0011, 1100, 1110. In all these situations player 1 guesses wrong, so the guess of player 3 is irrelevant.

We concentrate on the 24 optimal decision matrices and observe 12 matrices where one player can always pass and 12 matrices of a different structure. Appendix H shows the two groups of 12 elements, the position of the player who can always PASS and the CYCLE to obtain isomorphic relation with the first element of each group: the 24 optimal adequate sets can be divided in two groups of each 12 isomorphic elements. So the first 12 rows are isomorphic to the adequate set {1,3,12,14} and the next 12 rows are isomorphic to the adequate set {1,6,10,13} with decision matrix:

000	001	010	011	100	101	110	111
	1	0			0	1	
	1	0			0	1	
	1	0		0			1
0			1		1	0	

Two adequate sets are equivalent when they have the same probability function. All 24 optimal sets are equivalent (probability function $pq^3 + 2p^2q^2 + p^3q$). Two adequate sets are isomorphic when one set can be obtained from the other set by renumbering the players. We notice that equivalency doesn't imply isomorphic behavior. So in the asymmetric case we have two different optimal solutions. Both solutions have probability $1 - pq$ for success.

The last point is to convince ourselves that any adequate set with $das > 4$ doesn't yield better solutions. This can be done by running the program ASG with $n = 4$, $das = 5, 6, \ldots, 16$: all adequate sets have probabilities greater than $pq^3 + 2p^2q^2 + p^3q$.□

8 Symmetric four person two color hat game

Theorem 7 – *For symmetric four person two color hat game we have: maximal probability is $\frac{3}{4}$ with 5 optimal (non isomorphic) decision matrices:*

000	001	010	011	100	101	110	111
	1		1	0		0	
	1		1	0		0	
	⋆					⋆	
0	0					1	1

000	001	010	011	100	101	110	111
	1	0			0	1	
	1	0			0	1	
	1	0		0			1
0			1		1	0	

000	001	010	011	100	101	110	111
	1	1		0			0
	1	1		0			0
0	1					1	0
0	1					1	0

000	001	010	011	100	101	110	111
1			1		0	0	
1			1		0	0	
1	0					0	1
1	0					0	1

000	001	010	011	100	101	110	111
1	1					0	0
1	1					0	0
1	1					0	0
⋆							⋆

(Remark: When taking 'pass' for ⋆ in the last matrix, we get the solution where one player passes and the other three go for the well known solution of the three person game)

Proof. Appendix H gives an overview of 40 optimal adequate sets. There are 5 non-isomorphic sets. The first two base decision matrices (corresponding to row 1 and row 13) are given in section 7. The last three decision matrices, corresponding to rows 25, 31 and 37 are generated by adequate sets {1 2 12 15} , {0 3 13 14} and {0 1 14 15}. They can be found with DMG . □

9 Computational complexity

We consider the number of strategies to be examined to solve the hat problem with N players and two colors. Each of the N players has 2^{N-1} possible situations to observe and in each situation there are three possible guesses: white, black or pass. So we have $(3^{2^{N-1}})^N$ possible strategies. Krzywkowski [14] shows that is suffices to examine $(3^{2^{N-1}-2})^N$ strategies.

The adequate set method has to deal where $\{i_1, i_2, .., i_{das}\}$ with $0 \le i_1 < i_2 < \ldots < i_{das} \le 2^N - 1$.

The number of strategies for fixed das is the number of subsets of dimension das of $\{0, 1, \ldots, 2^N - 1\}$: $\binom{2^N}{das}$. But we have to test all possible values of das. So the correct expression is: $\sum_{das} \binom{2^N}{das} = 2^{(2^N)}$. To get an idea of the power of the adequate set method, we compare the number of strategies (brute force, Krzywkowski and adequate set method):

N	$(3^{2^{N-1}})^N$	$(3^{2^{N-1}-2})^N$	$2^{(2^N)}$
4	$1.80E+15$	$2.80E+11$	65536

A ASG three persons

```
Sub adequate_sets()
Dim c() As Integer: Dim d() As Integer: Dim i() As Integer: Dim j() As Integer: Dim check() As Integer
n = 3: H = 2 ^ n - 1: das = 2
ReDim c(0 To H, 1 To n) As Integer: ReDim d(1 To n) As Integer
ReDim i(1 To das) As Integer: ReDim j(1 To das, 1 To n) As Integer: ReDim check(0 To H) As Integer
' for each number from 0 to H: first calculate binary digits 000 ..... 111 and put it in matrix c:
For k = 0 To H: g = k
   For Z = 1 To n:  c(k, Z) = g Mod 2:  g = g \ 2
   Next Z
Next k
x = 0 ' x: row in Excel where result is displayed
For i1 = 0 To H - das + 1  ' adequate set: {i_1,i_2..,i_das}
i(1) = i1  ' VBA-EXCEL can't handle with an array in for to next
For i2 = i1 + 1 To H - das + 2: i(2) = i2
For k = 1 To das: g = i(k) 'binary digits for adequate set:
      For Z = 1 To n: j(k, Z) = g Mod 2: g = g \ 2: Next Z
Next k
' check on adequate set property: each element of B has distance 0 or 1 to A
For k = 0 To H:   check(k) = 0
   For m = 1 To das  ' distance<2
   If Abs(c(k, 1) - j(m, 1)) + Abs(c(k, 2) - j(m, 2)) + Abs(c(k, 3) - j(m, 3)) < 2 Then check(k) = 1
   Next m
Next k
State = 1
For k = 0 To H:   State = State * check(k)
Next k
If State = 1 Then x = x + 1  '  state=1 means: we found an adequate set; go to next row in Excel sheet
 For k = 1 To das:   If State = 1 Then Cells(x, k) = i(k): ' shows elements of adequate set
 Next k
Next i2: Next i1
End Sub
```

B DMG three persons

```
Sub decision_matrix_generator()
Dim d() As Integer: Dim i() As Integer: Dim a() As Integer: Dim b() As Integer
n = 3: m = 2 ^ (n - 1) - 1: das = 2
ReDim d(1 To n) As Integer: ReDim i(1 To das) As Integer
ReDim a(1 To n, 0 To m) As Integer: ReDim b(1 To n, 0 To m) As Integer
i(1) = 1: i(2) = 6  'adequate set:
For s = 1 To n: For k = 0 To m
a(s, k) = 0    'counts number of zero's in cell (s,k)
b(s, k) = 0     'count number of one's in cell (s,k)
Next k: Next s
For k = 1 To das:  g = i(k)
        For Z = n To 1 Step -1: d(Z) = g Mod 2: g = g \ 2
        Next Z  'd(1)d(2)d(3) is binary representation of g
    If d(1) = 0 Then a(1, 2 * d(2) + d(3)) = a(1, 2 * d(2) + d(3)) + 1 'update a(.,.) and b(.,.):
    If d(1) = 1 Then b(1, 2 * d(2) + d(3)) = b(1, 2 * d(2) + d(3)) + 1
    If d(2) = 0 Then a(2, 2 * d(1) + d(3)) = a(2, 2 * d(1) + d(3)) + 1
    If d(2) = 1 Then b(2, 2 * d(1) + d(3)) = b(2, 2 * d(1) + d(3)) + 1
    If d(3) = 0 Then a(3, 2 * d(1) + d(2)) = a(3, 2 * d(1) + d(2)) + 1
    If d(3) = 1 Then b(3, 2 * d(1) + d(2)) = b(3, 2 * d(1) + d(2)) + 1
Next k
For k = 1 To das: Cells(1, k + 2) = i(k): Next k
For s = 1 To n
For k = 0 To m
If a(s, k) + b(s, k) = 0 Then Cells(x + s, k + 1) = 2   'code 2: pass
If a(s, k) >= 1 And b(s, k) = 0 Then Cells(x + s, k + 1) = 1  '0's are bad cases, so we need a 1
If b(s, k) >= 1 And a(s, k) = 0 Then Cells(x + s, k + 1) = 0
If a(s, k) >= 1 And b(s, k) >= 1 Then Cells(x + s, k + 1) = " *" 'any guess or pass will do
Next k
Next s
End Sub
```

C Adequate sets, three players, $das = 3$

1	0	1	6
2	0	1	7
3	0	2	5
4	0	2	7
5	0	3	4
6	0	3	7
7	0	4	7
8	0	5	7
9	0	6	7
10	1	2	5
11	1	2	6
12	1	3	4
13	1	3	6
14	1	4	6
15	1	5	6
16	1	6	7
17	2	3	4
18	2	3	5
19	2	4	5
20	2	5	6
21	2	5	7
22	3	4	5
23	3	4	6
24	3	4	7

D Dominated sets, three persons, two colors, $das = 4$

1	3	5	7	q_3
2	3	5	7	$> P(\{2,5\})$
3	4	5	7	$> P(\{3,4\})$
1	3	5	6	$> P(\{1,6\})$
1	2	5	7	$> P(\{2,5\})$
1	4	6	7	$> P(\{1,6\})$
1	2	3	5	$> P(\{2,5\})$
1	3	4	5	$> P(\{3,4\})$
2	3	4	5	$> P(\{3,4\})$
1	2	4	7	$\geq q_1$
1	2	3	4	$> P(\{3,4\})$
0	5	6	7	$> P(\{0,7\})$
0	1	5	7	$> P(\{0,7\})$
0	2	5	7	$> P(\{0,7\})$
0	4	5	7	$> P(\{0,7\})$
0	1	5	6	$> P(\{1,6\})$
0	3	4	5	$> P(\{3,4\})$
0	1	4	7	$> P(\{0,7\})$
0	1	2	5	$> P(\{2,5\})$
0	1	4	5	p_2
0	2	4	5	$> P(\{2,5\})$

1	3	6	7	$> P(\{1,6\})$
2	3	6	7	q_2
3	4	6	7	$> P(\{3,4\})$
2	3	5	6	$> P(\{2,5\})$
1	2	6	7	$> P(\{1,6\})$
2	3	4	7	$> P(\{3,4\})$
1	2	3	6	$> P(\{1,6\})$
1	3	4	6	$> P(\{3,4\})$
2	3	4	6	$> P(\{3,4\})$
1	2	4	5	$> P(\{2,5\})$
0	3	5	7	$> P(\{0,7\})$
0	3	5	6	$\geq q_1$
0	1	6	7	$> P(\{0,7\})$
0	2	6	7	$> P(\{0,7\})$
0	4	6	7	$> P(\{0,7\})$
0	2	3	5	$> P(\{2,5\})$
0	3	4	6	$> P(\{3,4\})$
0	2	4	7	$> P(\{0,7\})$
0	1	2	6	$> P(\{1,6\})$
0	1	4	6	$> P(\{1,6\})$
0	2	4	6	p_3

1	5	6	7	$> P(\{1,6\})$
2	5	6	7	$> P(\{2,5\})$
4	5	6	7	q_1
3	4	5	6	$> P(\{3,4\})$
1	3	4	7	$> P(\{3,4\})$
2	4	5	7	$> P(\{2,5\})$
1	2	5	6	$> P(\{1,6\})$
1	4	5	6	$> P(\{1,6\})$
2	4	5	6	$> P(\{2,5\})$
1	2	4	6	$> P(\{1,6\})$
0	3	6	7	$> P(\{0,7\})$
0	1	3	7	$> P(\{0,7\})$
0	2	3	7	$> P(\{0,7\})$
0	3	4	7	$> P(\{0,7\})$
0	1	3	6	$> P(\{1,6\})$
0	2	5	6	$> P(\{2,5\})$
0	1	2	7	$> P(\{0,7\})$
0	1	2	3	p_1
0	1	3	4	$> P(\{3,4\})$
0	2	3	4	$> P(\{3,4\})$

We have 62 adequate sets, where 54 are absolutely dominated by $\{0,7\}$, $\{1,6\}$, $\{2,5\}$ or $\{3,4\}$.
Probability of $\{4,5,6,7\}$ is $q_1p_2p_3 + q_1p_2q_3 + q_1q_2p_3 + q_1q_2q_3 = q_1$.
For the set $\{0,3,5,6\}$, we obtain that its probability is

$$\begin{aligned} p_1p_2p_3 + p_1q_2q_3 + q_1p_2q_3 + q_1q_2p_3 &\\ &= q_1(p_2q_3 + q_2p_3 + p_2p_3 + q_2q_3) + (p_1 - q_1)(p_2p_3 + q_2q_3) \\ &= q_1 + (p_1 - q_1)(p_2p_3 + q_2q_3) \geq q_1. \end{aligned}$$

Similarly, for the set $\{1,2,4,7\}$, its probability verifies

$$\begin{aligned} p_1p_2q_3 + p_1q_2p_3 + q_1p_2p_3 + q_1q_2q_3 &\\ &= q_1(p_2p_3 + q_2q_3 + p_2q_3 + q_2p_3) + (p_1 - q_1)(p_2q_3 + q_2p_3) \\ &= q_1 + (p_1 - q_1)(p_2q_3 + q_2p_3) \geq q_1. \end{aligned}$$

E Four persons, $das = 4$

Nr.	adequate set				dominated by nr.:
1	0	1	14	15	2
2	0	2	13	15	3
3	0	3	13	14	12
4	0	4	11	15	5
5	0	5	11	14	6
6	0	6	11	13	18
7	0	7	8	15	8
8	0	7	9	14	9
9	0	7	10	13	10
10	0	7	11	12	19
11	1	2	12	15	12
12	1	3	12	14	20
13	1	4	10	15	14
14	1	5	10	14	17
15	1	6	8	15	22
16	1	6	9	14	17
17	1	6	10	13	18
18	1	6	11	12	19
19	1	7	10	12	27
20	2	3	12	13	24
21	2	4	9	15	22
22	2	5	8	15	23
23	2	5	9	14	24
24	2	5	10	13	25
25	2	5	11	12	31
26	2	6	9	13	27
27	2	7	9	12	34
28	3	4	8	15	29
29	3	4	9	14	30
30	3	4	10	13	31
31	3	4	11	12	34
32	3	5	8	14	33
33	3	6	8	13	34
34	3	7	8	12	40
35	4	5	10	11	36
36	4	6	9	11	37
37	4	7	9	10	39
38	5	6	8	11	40
39	5	7	8	10	40
40	6	7	8	9	x

F Program: four persons, $das = 4$ versus $das = 5$

```
Sub ASG45()
  Dim m5() As Integer: Dim m4() As Integer: Dim c() As Integer: Dim check() As Integer
  Dim i() As Integer: Dim j() As Integer: Dim d() As Integer
  n = 4    'four players
  H = 2 ^ n - 1    '2^n elements in B
  ReDim c(0 To H, 1 To n) As Integer: ReDim check(0 To H) As Integer
  ReDim i(1 To 5) As Integer: ReDim j(0 To H, 1 To n) As Integer
  ReDim m5(1 To 560, 1 To 5) As Integer
  ReDim d(1 To n) As Integer
  ReDim m4(1 To 40, 1 To 4) As Integer
  das = 5
  For k = 0 To H
    g = k
    For Z = 1 To n: c(k, Z) = g Mod 2:  g = g \ 2: Next Z
  Next k
  x = 0
  For i1 = 0 To H + 1 - das: i(1) = i1
  For i2 = i1 + 1 To H + 2 - das: i(2) = i2
  For i3 = i2 + 1 To H + 3 - das: i(3) = i3
  For i4 = i3 + 1 To H + 4 - das: i(4) = i4
  For i5 = i4 + 1 To H + 5 - das: i(5) = i5
    For k = 1 To das
    g = i(k)  'binary digits for adequate set:
    For Z = 1 To n: j(k, Z) = g Mod 2: g = g \ 2: Next Z
  Next k
  ' check on adequate set property; each element of B has distance 0 or 1 to A
  For k = 0 To H
    check(k) = 0
    For m = 1 To das   ' distance<2
      If Abs(c(k, 1) - j(m, 1)) + Abs(c(k, 2) - j(m, 2)) + Abs(c(k, 3) - j(m, 3)) + Abs(c(k, 4) - j(m, 4)) < 2
Then check(k) = 1
```

```
    Next m
  Next k
    State = 1   'potential adequate set
  For k = 0 To H
    State = State * check(k)
  Next k
  If State = 1 Then
    x = x + 1
    For k = 1 To das
      m5(x, k) = i(k)
    Next k
  End If
  Next i5: Next i4: Next i3: Next i2: Next i1
  das = 4:    x = 0
  For i1 = 0 To H + 1 - das: i(1) = i1
  For i2 = i1 + 1 To H + 2 - das: i(2) = i2
  For i3 = i2 + 1 To H + 3 - das: i(3) = i3
  For i4 = i3 + 1 To H + 4 - das: i(4) = i4
  For k = 1 To das
    g = i(k)  'binary digits for adequate set:
    For Z = 1 To n: j(k, Z) = g Mod 2: g = g \ 2: Next Z
  Next k
  ' check on adequate set property; each element of B has distance 0 or 1 to A
  For k = 0 To H
    check(k) = 0
    For m = 1 To das   ' distance<2
      If Abs(c(k, 1) - j(m, 1)) + Abs(c(k, 2) - j(m, 2)) + Abs(c(k, 3) - j(m, 3)) + Abs(c(k, 4) - j(m, 4)) < 2
Then check(k) = 1
    Next m
  Next k
    State = 1   'potential adequate set
```

```
  For k = 0 To H
    State = State * check(k)
  Next k
  If State = 1 Then
    x = x + 1
    For k = 1 To das
      m4(x, k) = i(k)
    Next k
  End If
   Next i4: Next i3: Next i2: Next i1
  Z = 0
  For x = 1 To 560
    Max = 0
  For y = 1 To 40
    t = 0
    For n = 1 To 4
      If (m4(y, n) - m5(x, 1)) = 0 Or (m4(y, n) - m5(x, 2)) = 0 Or (m4(y, n) - m5(x, 3)) = 0 Or (m4(y, n) -
m5(x, 4)) = 0 Or (m4(y, n) - m5(x, 5)) = 0 Then t = t + 1
    Next n
    If t = 4 Then Max = 4
  Next y
  If Max = 0 Then
    Z = Z + 1
    For k = 1 To 5
      Cells(Z, k) = m5(x, k)
    Next k
  End If
Next x
End Sub
```

G Output: four persons, $das = 4$ versus $das = 5$

Nr.	adequate set:					non-dom	d(80)	d(560)	X	Nr.	adequate set:					non-dom	d(80)	d(560)
1	0	1	2	4	15		2		X	48	2	4	6	9	14		49	
2	0	1	2	8	15		5		X	49	2	4	7	9	14		58	
3	0	1	3	5	14		4		X	50	2	5	8	10	11		51	
4	0	1	3	9	14		6		X	51	2	5	8	10	14		52	
5	0	1	4	8	15		9		X	52	2	5	8	11	14		56	
6	0	1	5	9	14		11		X	53	2	5	9	12	13		54	
7	0	2	3	6	13		8		X	54	2	5	9	12	15		55	
8	0	2	3	10	13		10		X	55	2	5	9	13	15		57	
9	0	2	4	8	15		28		X	56	2	5	10	11	14		78	
10	0	2	6	10	13		12		X	57	2	5	12	13	15		59	
11	0	3	5	9	14		30		X	58	2	6	7	9	14		75	
12	0	3	6	10	13		45		X	59	2	9	12	13	15	S_2		
13	0	4	5	6	11		14		X	60	3	4	8	12	13		61	
14	0	4	5	11	12		15		X	61	3	4	8	12	14		62	
15	0	4	6	11	12		16		X	62	3	4	8	13	14		67	
16	0	5	6	11	12		73		X	63	3	4	9	10	11		64	
17	0	7	8	9	10		18		X	64	3	4	9	10	15		65	
18	0	7	8	9	12		19		X	65	3	4	9	11	15		66	
19	0	7	8	10	12		20		X	66	3	4	10	11	15		76	
20	0	7	9	10	12		80		X	67	3	4	12	13	14		72	
21	0	7	11	13	14		22		X	68	3	5	6	7	8		69	
22	0	7	11	13	15		23		X	69	3	5	6	8	15		70	
23	0	7	11	14	15		24		X	70	3	5	7	8	15		71	
24	0	7	13	14	15		25		X	71	3	6	7	8	15		77	
25	0	11	13	14	15	S_0			X	72	3	8	12	13	14	S_3		
26	1	2	3	7	12		27		X	73	4	5	6	11	12			{4,5,10,11,12}
27	1	2	3	11	12		29		X	74	4	5	7	10	13			{4,5,10,11,13}
28	1	2	4	8	15			{1,2,4,9,15}	X	75	4	6	7	9	14			{4,6,9,11,14}
29	1	2	7	11	12		31		X	76	4	9	10	11	15	S_4		
30	1	3	5	9	14			{1,3,5,10,14}	X	77	5	6	7	8	15			{5,6,8,11,15}
31	1	3	7	11	12		46		X	78	5	8	10	11	14	S_5		
32	1	4	5	7	10		33		X	79	6	8	9	11	13	S_6		
33	1	4	5	10	13		34		X	80	7	8	9	10	12	S_7		
34	1	4	7	10	13		35		X									
35	1	5	7	10	13		74		X									
36	1	6	8	9	11		37		X									
37	1	6	8	9	13		38		X									
38	1	6	8	11	13		39		X									
39	1	6	9	11	13		79		X									
40	1	6	10	12	14		41		X									
41	1	6	10	12	15		42		X									
42	1	6	10	14	15		43		X									
43	1	6	12	14	15		44		X									
44	1	10	12	14	15	S_1			X									
45	2	3	6	10	13			{2,3,7,12,13}	X									
46	2	3	7	11	12			{2,3,7,12,13}	X									
47	2	4	6	7	9		48		X									

Legenda:
In column "non-dom" we have the non-dominated sets

Column "d(80)" gives the domination relations in the set of 80 adequate sets (set 1 is dominated by set 2 etc.)

Column "d(560)" gives the domination relations using in the original 560 adequate sets (set 28 is dominated by {1,2,4,9,15})

Note: We have comparised sets with differs in one element.
Most of them can be found in the 80-set.
The elements in the last column can easily be found by running ASG55 with max=4 (4 hits).

H Four persons, $das = 4$, sorted list, $p = 0.9$

row	adequate set				prob.	pattern					PASS	CYCLE	BASE
1	1	3	12	14	0.09	0	1	2	1	0	3	(1)	{1,3,12,14}
2	1	5	10	14	0.09	0	1	2	1	0	2	(23)	
3	1	6	9	14	0.09	0	1	2	1	0	1	(13)	
4	2	3	12	13	0.09	0	1	2	1	0	4	(34)	
5	2	5	10	13	0.09	0	1	2	1	0	1	(143)	
6	2	6	9	13	0.09	0	1	2	1	0	2	(243)	
7	3	4	11	12	0.09	0	1	2	1	0	1	(143)	
8	3	7	8	12	0.09	0	1	2	1	0	2	(23)(14)	
9	4	5	10	11	0.09	0	1	2	1	0	4	(234)	
10	4	6	9	11	0.09	0	1	2	1	0	3	(24)	
11	5	7	8	10	0.09	0	1	2	1	0	3	(14)	
12	6	7	8	9	0.09	0	1	2	1	0	4	(134)	
13	1	6	10	13	0.09	0	1	2	1	0		(1)	{1,6,10,13}
14	1	6	11	12	0.09	0	1	2	1	0		(23)	
15	1	7	10	12	0.09	0	1	2	1	0		(13)	
16	2	5	9	14	0.09	0	1	2	1	0		(34)	
17	2	5	11	12	0.09	0	1	2	1	0		(243)	
18	2	7	9	12	0.09	0	1	2	1	0		(143)	
19	3	4	9	14	0.09	0	1	2	1	0		(234)	
20	3	4	10	13	0.09	0	1	2	1	0		(24)	
21	3	5	8	14	0.09	0	1	2	1	0		(134)	
22	3	6	8	13	0.09	0	1	2	1	0		(14)	
23	4	7	9	10	0.09	0	1	2	1	0		(24)(13)	
24	5	6	8	11	0.09	0	1	2	1	0		(14)(23)	
25	1	2	12	15	0.154	1	0	1	2	0		(1)	{1,2,12,15}
26	1	4	10	15	0.154	1	0	1	2	0		(23)	
27	1	6	8	15	0.154	1	0	1	2	0		(13)	
28	2	4	9	15	0.154	1	0	1	2	0		(234)	
29	2	5	8	15	0.154	1	0	1	2	0		(14)	
30	3	4	8	15	0.154	1	0	1	2	0		(14)(23)	
31	0	3	13	14	0.666	0	2	1	0	1		(1)	{0,3,13,14}
32	0	5	11	14	0.666	0	2	1	0	1		(23)	
33	0	6	11	13	0.666	0	2	1	0	1		(234)	
34	0	7	9	14	0.666	0	2	1	0	1		(13)	
35	0	7	10	13	0.666	0	2	1	0	1		(14)	
36	0	7	11	12	0.666	0	2	1	0	1		(13)(24)	
37	0	1	14	15	0.73	1	1	0	1	1	4	(1)	{0,1,14,15}
38	0	2	13	15	0.73	1	1	0	1	1	3	(34)	
39	0	4	11	15	0.73	1	1	0	1	1	2	(24)	
40	0	7	8	15	0.73	1	1	0	1	1	1	(14)	

Acknowledgments

The author thanks the referee for clear and constructive comments.

References

Blum, W. (2001). "Denksport für Hutträger". *Die Zeit* (cit. on p. 25).
Buhler, J. (2002). "Hat tricks". *Math. Intelligencer 24, 44-49* (cit. on p. 25).
Burke, E., S. Gustafson, and G. Kendall (2002). "A puzzle to challenge genetic programming, Genetic Programming, 136-147". *Lecture Notes in Computer Science, Springer* (cit. on p. 25).
Cohen, G. et al. (1997). "Covering Codes". *North-Holland, Mathematical Library 54* (cit. on p. 25).
Ebert, T. (1998). "Applications of recursive operators to randomness and complexity". *Ph.D. Thesis, University of California, Santa Barbara* (cit. on p. 25).
Ebert, T., W. Merkle, and H. Vollmer (2003). "On the autoreducibility of random sequences". *SIAM J. Comp. 32, 1542-1569* (cit. on p. 25).
Gardner, M. (1961). *The 2nd Scientific American Book of Mathematical Puzzles & Diversions, Simon and Schuster, New York* (cit. on p. 25).
Johnson, B. (2001). *The Hat Problem*. `https://mathstat.slu.edu/~johnson/public/maths/hatproblem.pdf` (cit. on p. 26).
Krzywkowski, M. (2010). "On the hat problem, its variations, and their applications". *Annales Universitatis Paedagogicae Cracoviensis Studia Mathematica 9 (1), 55-67* (cit. on p. 26).
Lenstra, H. and G. Seroussi (2005). "On hats and other covers, (Extended Summary), Arxiv cs/0509045v1 [cs.IT]" (cit. on pp. 26, 27).
Poulos, J. (2001). "Could you solve this 1 million hat trick?" *abcNews, November 29, 2001* (cit. on p. 25).
Robinson, S. (2001). "Why mathematicians now care about their hat color". *The New York Times, Science Times Edition, page D5, April 10, 2001* (cit. on p. 25).

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