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Hamiltonian delay equations for central configurations

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Abstract

We provide in this paper the hamiltonian delay equations of motion for the newtonian n-body problem deduced from the quantum calculus of variations developed in Cresson 2005; Cresson, Frederico, and Torres 2009; Ryckelynck and Smoch 2013, 2014. These equations are brought into the usual lagrangian and hamiltonian formulations of the dynamics and yield sampled functional equations involving generalized derivatives. We investigate especially homographic solutions to these equations that we obtain by solving algebraic systems of equations similar to the classical ones. When the potential forces are homogeneous, homographic solutions to the delayed and to the classical equations may be related through an explicit expansion factor that we provide. Consequently, perturbative equations both in lagrangian and hamiltonian formalisms are deduced.

Keywords: Functional equations, calculus of variations, n-body problem, homographic configurations.

MSC: 49J21, 70F10, 49J15, 70F07.

1 Introduction

This paper is devoted to the application of quantum calculus of variations (see Cresson (2005), Cresson, Frederico, and Torres (2009), Ryckelynck and Smoch (2013, 2014)) to newtonian dynamics and especially to celestial mechanics. After having presented the hamiltonian delay equations driven from the principle of least action for the quantum action on a convenient phase space, we focus on central configurations and especially on libration points or regular polygonal solutions El Mabsout (1988, 1991), which are the best well-known periodic solutions. These motions are entirely explicit provided we can solve specific algebraic equations for the coordinates in a rotating frame. Another family of periodic solutions consists in choreographic solutions Chenciner and Montgomery (2000) which are obtained

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through topological arguments. In this respect we have obtained in Ryckelynck and Smoch (2014) choreographic solutions to quadratic lagrangian systems either in classical and quantum contexts. Both families of solutions are infinite and give rise to a huge number of theoretical and numerical works.

Let us consider the motion in the euclidean space $\mathbb{R}^d$ of a generalized particle $\mathbf{x}(t) = (x_i(t)) \in \mathbb{R}^d$. A system of delay differential equations in $\mathbb{R}^d$ is a system of functional equations of the shape $x_i'(t) = f_i(\mathbf{x}(t-t_1), \dots \mathbf{x}(t-t_p))$ where the p fixed real numbers $t_1 < \dots < t_p$ are given. We may bring in this form a system of delay differential equations of higher order $x_i''(t) = f_i(\mathbf{x}(t-t_1), \dots \mathbf{x}(t-t_p), \mathbf{x}'(t-t_1'), \dots \mathbf{x}'(t-t_q))$ by embedding the state vector $\mathbf{x}$ into another one in higher dimension. Many examples may be found in science and engineering, see Erneux (2009). The book of Hale and Verduyn Lunel (1993) gives a general theory of those delay systems. Among delay differential systems, hamiltonian ones play an interesting role, due to their connection with classical mechanics and in particular celestial mechanics, as well as the calculus of variations. The problem when a system of delay equations on $\mathbb{R}^d$ is equivalent to the search of critical points of lagrangian involving delays is analyzed in Kolesnikova, Popov, and Savchin (2007) and more recently in Bakker and Scheel (2018).

Let us motivate the use of the formalism of quantum calculus of variations. First of all, the starting point of the quantum calculus of variations consists in substituting for derivatives of the dynamical variables $\mathbf{u}$ in the lagrangian density $\mathcal{L}(t, \mathbf{u}, \mathbf{u}')$ over the interval $[t_0, t_f]$ some conveniently chosen sampled operators. For instance, Cresson, Frederico, and Torres (2009) used the Nottale operator $\Box_q \mathbf{u}(t) = \frac{1-i}{2\varepsilon}\mathbf{u}(t+\varepsilon) + \frac{i}{\varepsilon}\mathbf{u}(t) - \frac{1+i}{2\varepsilon}\mathbf{u}(t-\varepsilon)$, deduced from requirements linked to Schroedinger's equation, and studied the minimization of the action integral $\int_{t_0-\varepsilon}^{t_f+\varepsilon} \mathcal{L}(t, \mathbf{u}, \Box_q \mathbf{u}) dt$. We deal in Ryckelynck and Smoch (2013, 2014) with sampling operators generalizing this choice by associating to $\mathbf{u}$ a linear combination of the samples of $\mathbf{u}$ cut by the characteristic function $\chi(t)$ of $[t_0, t_f]$. Formally we use

$$\Box \mathbf{u}(t) = \sum_{\ell=-N}^{N} \frac{\gamma_\ell}{\varepsilon} \chi(t+\ell\varepsilon) \mathbf{u}(t+\ell\varepsilon), \tag{1}$$

for all $t \in [t_0, t_f]$, for all $\mathbf{u} : [t_0, t_f] \to \mathbb{R}^d$, $d \in \mathbb{N}^*$. Therefore, we may express the least-action principle in a nutshell inside $[t_0, t_f]$ by considering the action $\int_{t_0}^{t_f} \mathcal{L}(t, \mathbf{u}, \Box \mathbf{u}) dt$ together with a properly-posed Dirichlet problem, which was our primary goal. By the way, generalizing the particular previous coefficients to a system $(c_k)_{-N \le k \le N}$ leads to interesting problems among which convergence and resonance. We may also control the numerical efficiency of the chosen operator $\Box$ by varying the coefficients c_k and introducing unusual difference schemes.

Second, we emphasize the fact that the equations of motion to be used, see (9) and (25) below, are functional difference equations and not differential nor recurrence ones. In a sense, we may consider these equations as a new intermediate level between classical and discretized equations of motion, the latter being obtained through usual numerical schemes. We have already compared the properties of various generalized difference operators and first-order necessary conditions in Ryckelynck and Smoch (2013).

Now, in order to compare the two previous frameworks, it is important to work with integrable systems. Indeed, integrable systems give rise to numerous enough independent integrals of motion defined help to Poisson brackets which are themselves operators on functions spaces. So, in this context, we may highlight three aspects of convergence. The first one is the convergence of the quantum operators occuring in the delay equations of motion to the corresponding continuous ones. The second property is the convergence of integrals of motion, especially integrals in involution with the hamiltonian function, for quantum framework to continuous ones. If these two aspects are considered, then the problem is in a sense convergent and consistent and we may settle the third question, namely the convergence of the solution of the delay systems of equations of motion to the corresponding continuous one. In general, those problems are difficult and not naturally related since for instance, several phenomenon can occur and prevent the convergence such as resonance, radiation, bifurcation. Those difficulties are handled in the two following works. In Ryckelynck and Smoch (2013), we used harmonic oscillators and some null lagrangians while, in Ryckelynck and Smoch (2014), we explored the quadratic choreographies. We focus in this paper on relative equilibria in celestial mechanics. The consideration of those configurations allows to compare the two sets of explicit quantities and equations, one in the classical and continuous settings, the other in the sampled setting. The forecoming sections will display the dynamical variables, lagrangians, hamiltonians, integrals of motion in classical or delayed versions, in galilean or rotating frames, in Lagrangian or Hamiltonian formulations. Lastly, another important feature of this research is the fact that sampling the action in the lagrangian picture is general while, in constrast, sampling the hamiltonian and the hamiltonian equations is far more difficult and in no way universal. The problem underlying the absence of a suitable hamiltonian framework is the lack of an appropriate Legendre's transform, in the quantum calculus of variation. One should argue that it is possible to discretize a priori the Hamilton equations, but if so they won't express the dynamical equations of motion in the phase space, and the equivalence with the Lagrange equations will fail.

The paper is organized as follows. In Section 2, we give the notation used throughout this work. In Section 3, we present the four systems of equations of motion for the n-body problem, either in the lagrangian or hamiltonian formulations and either classical or quantum settings. This being done, we discuss the existence of constants of motion and galilean equilibria. We emphasize on the non-covariance of the delay hamiltonian equations. In Section 4, we introduce the additional operators V_c, V_s, W_c, W_s, used when expressing the Euler-Lagrange and hamiltonian equations in a rotating frame to be further developed in Section 5. There we provide the convenient formulas for $\mathcal{L}_c, \mathcal{L}_d, \mathcal{H}_c, \mathcal{H}_d$, and the four corresponding sets of equations of motion. In Section 6, we give the proof of the main theorems of Section 3. In Section 7, we determine relative equilibria solutions to the n-body problem that is to say, the solutions (7) obtained so that the functions $a_i(t)$ and $b_i(t)$ are constant w.r.t. time. When the potential functions are homogeneous, we show that the solutions to the delay Euler-Lagrange functional equations are homothetic to those to the classical Euler-Lagrange equations. The homothety ratio $\varphi(\varepsilon)$ that we call the expansion factor, is determined and its convergence as ε tends to 0 is studied. Finally, in Section 8, we provide some numerical experiments and results which illustrate our analysis.

2 Delay lagrangian and hamiltonian for Newtonian systems of pairwise interacting particles

The notation $\square^\star$ shall denote the adjoint operator to $\square$ which is obtained from $\square$ by reversing its coefficients γ_ℓ. The investigation of the convergence of $\square\mathbf{u}$ and $-\square^\star\mathbf{u}$ to $\dot{\mathbf{u}}$ has been undertaken in Ryckelynck and Smoch (2013, Proposition 2.3). We have already pointed out the fact that for all $\mathbf{u} \in \mathcal{C}^2([t_0, t_f], \mathbb{R}^d)$ and for all $t \in]t_0, t_f[$,

$$\lim_{\varepsilon \to 0} \square\mathbf{u}(t) = \lim_{\varepsilon \to 0} -\square^\star\mathbf{u}(t) = \dot{\mathbf{u}}(t) \tag{2}$$

locally uniformly in $]t_0, t_f[$ if and only if

$$\sum_\ell \gamma_\ell = 0 \text{ and } \sum_\ell \ell\gamma_\ell = 1. \tag{3}$$

These notations and properties being introduced, we consider a system of n particles P_i, with mass m_i, located at points $\mathbf{x}_i = (x_{ik})_k \in \mathbb{R}^d$ where $i = 1, \ldots, n$ and $k = 1, \ldots, d$. The distance r_{ij} between P_i and P_j is defined by $r_{ij}^2 = \sum_{k=1}^{d} (x_{ik} - x_{jk})^2$. We assume that there exist $\frac{n(n-1)}{2}$ functions of forces $f_{ij}(r_{ij})$ determining the interactions between each pair of particles (P_i, P_j). So we set for all $\mathbf{x}, \mathbf{y} \in (\mathbb{R}^d)^n$:

$$T(\mathbf{y}) = \frac{1}{2} \sum_{i=1}^{n} \sum_{k=1}^{d} m_i y_{ik}^2, \; U(\mathbf{x}) = \sum_{i<j} f_{ij}(r_{ij}). \tag{4}$$

Accordingly, the Lagrangian and the Hamiltonian are defined for all configurations of particles $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_n) \in \mathcal{C}^0([t_0, t_f], (\mathbb{R}^d)^n)$ in the classical and quantum settings respectively by

$$\mathcal{L}_c = T(\dot{\mathbf{x}}) + U(\mathbf{x}),\ \mathcal{H}_c = T(\dot{\mathbf{x}}) - U(\mathbf{x}), \tag{5}$$
$$\mathcal{L}_d = T(\Box\mathbf{x}) + U(\mathbf{x}),\ \mathcal{H}_d = T(\Box\mathbf{x}) - U(\mathbf{x}). \tag{6}$$

The homogeneous potential functions of the shape $f_{ij}(r) = \mu_{ij} r^\beta$, with some common exponent $\beta \in \mathbb{Q} - \{0, 2\}$, constitute a case of special interest. We are particularly interested in the gravific interaction, described by $\beta = -1$ and $\mu_{ij} = g m_i m_j$, so that $f_{ij}(r) = g\frac{m_i m_j}{r}$.

Now, when working in a rotating frame with constant pulsation ω, we look for the homographic solutions to the n-body problem, i.e. the solutions to the equations of motion of the shape

$$x_{i1}(t) = a_i(t)\cos\omega t - b_i(t)\sin\omega t,\ x_{i2}(t) = a_i(t)\sin\omega t + b_i(t)\cos\omega t, \tag{7}$$

for some functions $a_i(t)$ and $b_i(t)$, according to the additional conditions $x_{ij} = 0$ for $j \geq 3$. The connection between homographic and central configurations is explored in Boccaletti and Pucacco (1996) and Smale (2000).

3 Main results

In this section we give the four systems of equations governing the system of pairwise interacting particles. The formalisms, which are classical or quantum, lagrangian or hamiltonian, give equations of motion for the n-body problem in a galilean frame, for which the covariance with respect to changes of frames is a central question. The proofs are collected in Section 6.

We recall the well-known classical Euler-Lagrange equations of motion for the newtonian n-body problem according to (5)

$$m_i \ddot{x}_{ik} = \sum_{j\neq i} f'_{ij}(r_{ij}) \frac{x_{ik} - x_{jk}}{r_{ij}}, \tag{8}$$

where $i \in \{1,\ldots,n\}, k \in \{1,\ldots,d\}$. By considering the quantum Euler-Lagrange equations introduced in Ryckelynck and Smoch (2013), we may deduce the analogous functional equations to (8).

Theorem 1 – *Let a system of n particles interacting according to (6) then the delay functional equations of motion are, for all i and k,*

$$-m_i \Box^\star \Box x_{ik} = \sum_{j\neq i} f'_{ij}(r_{ij}) \frac{x_{ik} - x_{jk}}{r_{ij}}. \tag{9}$$

In order to provide the hamiltonian equations equivalent to the Euler-Lagrange equations, we introduce the components of the momenta $p_{ik} = \frac{\partial \mathcal{L}_c}{\partial \dot{x}_{ik}} = \frac{\partial T}{\partial \dot{x}_{ik}}$. We suppose that the hessian matrix of T is invertible. Then the mapping defined by $\mathbb{R}^{2dn} \to \mathbb{R}^{2dn}$, $(\mathbf{x}_i, \dot{\mathbf{x}}_i)_{i\in\{1,\dots,n\}} \mapsto (\mathbf{x}_i, \mathbf{p}_i)_{i\in\{1,\dots,n\}}$ is locally one-to-one. So we may express $T(\dot{\mathbf{x}})$, $U(\mathbf{x})$ as functions of $\mathbf{x}$ and $\mathbf{p} = (\mathbf{p}_1, \dots, \mathbf{p}_n) \in \mathcal{C}^0([t_0, t_f], (\mathbb{R}^d)^n)$ to obtain $T - U = \mathcal{H}_c(\mathbf{x}, \mathbf{p})$. Similarly, in the quantum setting, the convenient coordinates of the momenta are $p_{ik} = \frac{\partial \mathcal{L}_d}{\partial \Box x_{ik}}$ and we obtain accordingly the Hamiltonian $\mathcal{H}_d(\mathbf{x}, \mathbf{p})$. Due to (5) and (6), we note that the two hamiltonian functions are formally the same function that we denote $\mathcal{H}(\mathbf{x}, \mathbf{p})$. As a well known result, the equations (8) are equivalent to the systems

$$\dot{p}_{ik} = -\frac{\partial \mathcal{H}}{\partial x_{ik}}, \ \dot{x}_{ik} = \frac{\partial \mathcal{H}}{\partial p_{ik}}, \tag{10}$$

We may easily state an analogue of (10) in the quantum setting.

Theorem 2 – *The equations (9) are equivalent to the systems of Hamilton's delay equations*

$$\Box^\star p_{ik} = \frac{\partial \mathcal{H}}{\partial x_{ik}}, \ \Box x_{ik} = \frac{\partial \mathcal{H}}{\partial p_{ik}}. \tag{11}$$

The third important result in this work deals with canonical transformations of coordinates in the plane hamiltonian n-body in a rotating frame as in (7). For the sake of clarity, we drop the index k from the letters x_{ik}, p_{ik} and use notations as x, y. So, in the classical setting, we obtain a change of coordinates sending (x_i, y_i) to (a_i, b_i). We shall indicate herafter, in Section 5, the canonical transformation of the momenta (x_i', y_i') to new coordinates (c_i, d_i) not being the derivatives of (a_i, b_i) but instead the covariant derivative of the impulsion expressed in the rotating frame. Accordingly, in the context of delay dynamics, the cartesian coordinates (x_i, y_i) give rise to coordinates (A_i, B_i) while momenta $(\Box x_i, \Box y_i)$ are canonically transformed in delay coordinates (C_i, D_i) not being the $\Box$ nor the $\Box^\star$ operators applied to components of the cartesian momenta.

Theorem 3 – *In the rotating frame, the delay hamiltonian function is*

$$\mathcal{H}_d = \frac{1}{2}\sum_{i=1}^n \frac{1}{m_i}(C_i^2 + D_i^2) - \omega \sum_{i=1}^n (A_i D_i - B_i C_i) - U(\mathbf{x}).$$

Next, the equations of motion in lagrangian form (25) are equivalent to

(Cont. next page)

$$V_c(A_i) - V_s(B_i) + \omega B_i = \frac{\partial \mathcal{H}_d}{\partial C_i}, \tag{12}$$

$$V_s(A_i) + V_c(B_i) - \omega A_i = \frac{\partial \mathcal{H}_d}{\partial D_i}, \tag{13}$$

$$m_i(W_c(A_i) - W_s(B_i)) - \omega D_i = \frac{\partial \mathcal{H}_d}{\partial A_i}, \tag{14}$$

$$m_i(W_s(A_i) + W_c(B_i)) + \omega C_i = \frac{\partial \mathcal{H}_d}{\partial B_i}. \tag{15}$$

Lastly, let us suppose that the operator $\Box$ satisfies (1) and (3). Then for all $\varepsilon_0 > 0$ and for all set of $4n$ functions $(A_i, B_i, C_i, D_i)_{1,\le i \le n} :]-\varepsilon_0, \varepsilon_0[\times [t_0, t_f] \to \mathbb{R}$, continuous w.r.t. (ε, t) and $\mathcal{C}^2$ w.r.t. t, the Lagrangian $\mathcal{L}_d$, the Hamiltonian $\mathcal{H}_d$ and the four Hamilton's equations (12) to (15) converge locally uniformly in $]t_0, t_f[$ respectively to the Lagrangian $\mathcal{L}_c$, the Hamiltonian $\mathcal{H}_c$ and the four Hamilton's equations (27) as ε tends to 0.

To conclude this section with important features of this formalism, we first remark that, although the delay hamiltonian equations in the cartesian frame look very similar to the classical ones (see formula (11)), they do not behave covariantly under general change of coordinates since we might expect, in a rotating frame, equations of the shape

$$\Box A_i = \frac{\partial \mathcal{H}_d}{\partial C_i}, \ \Box B_i = \frac{\partial \mathcal{H}_d}{\partial D_i}, \ -\Box^\star C_i = -\frac{\partial \mathcal{H}_d}{\partial A_i}, \ -\Box^\star D_i = -\frac{\partial \mathcal{H}_d}{\partial B_i}$$

which are not true. Hence, hamiltonian delay equations are not covariant in general, but however their transformation does hold the structure of delay equations. The covariance of Euler-Lagrange equations and of Hamilton equations is a very important property, and is a pillar of axiomatization of mathematical physics. It is often presented as an axiom of the theory of gravitation, see for instance the celebrated Treatise of Landau and Lifschitz (1970, 1973). Several drawbacks of loss of covariance are also presented in a clever but physical manner, for instance, we may cite adiabatic invariants, Larmor precession, shift to red frequencies or search for periodic solutions of equations of motion. A very interesting paper of Cremaschini and Tessarotto (2012) is devoted to hamiltonian formulation of the equations of motion for classical n-body systems in electromagnetic fields with finite propagation of interaction. This leads to both local and non-local actions describing electromagnetic interaction. Non locality of lagrangian implies in general non covariance of Hamilton equations, and gives rise to radiation-reaction phenomena.

The previous construction has three interesting features. The first one is obviously that $\mathcal{H}_d$ depends in an algebraic way of the variables A_i, B_i, C_i, D_i, and not of some additional derivative operators acting on the previous variables. Next, if $\omega = 0$, we recover (6). Another important feature is the possibility to provide the Hamilton-Jacobi partial differential equation in the quantum calculus of variations expressed as

$$\frac{1}{2}\sum_{i=1}^{n}\frac{1}{m_i}\left(\left(\frac{\partial S}{\partial A_i}\right)^2+\left(\frac{\partial S}{\partial B_i}\right)^2\right)-\omega\sum_{i=1}^{n}\left(A_i\frac{\partial S}{\partial B_i}-B_i\frac{\partial S}{\partial B_i}\right)-U(\mathbf{x})=cst,$$

for the unknown action $S=S((A_i,B_i)_i)$. As one knows, this equation is of a crucial importance when constructing variational integrators for approximating the solutions of the equations of motion. Numerical methods preserving the structure of Euler-Lagrange equations and of Hamilton's equations are known as variational integrators, and among them the methods which preserve conservation laws under discretization, i.e. the symplectic integrators, are of great importance. One may refer to the reference paper of Marsden and West (2001) as well as the monograph by Hairer, Lubich, and Wanner (2006), which all provide systematic methods of constructing variational and symplectic integrators. The starting point of their approach is to introduce a discrete Lagrangian $\mathcal{L}_d$ constructed from the continuous Lagrangian $\mathcal{L}_c$ by solving the problem of minimizing each infinitesimal action $\int_0^h \mathcal{L}(q(t),\dot{q}(t))dt$ on a small interval, and this in turn is equivalent to Jacobi's solution of the Hamilton-Jacobi equation. For more information, we refer to the very interesting and comprehensive paper Tran and Leok (2023) which contains an up-to-date bibliography.

4 The four functional operators V_c, V_s, W_c, W_s

In this section we introduce four continuous linear operators between the euclidean function space $\mathcal{C}^0([t_0,t_f],\mathbb{R}^d)$ and the function space of which the elements are the piecewise continuous functions vanishing outside $[t_0-N\varepsilon,t_f+N\varepsilon]$. The first space is equipped with the ordinary scalar product, i.e. $\langle\mathbf{f},\mathbf{g}\rangle_0=\int_{t_0}^{t_f}\langle\mathbf{f}(t),\mathbf{g}(t)\rangle dt$, while the second one is endowed with the same scalar product except for the domain of integration which is $[t_0-N\varepsilon,t_f+N\varepsilon]$. As usual, we use the notation $\star$ for denoting the adjoint of an operator.

Lemma 1 – *There exist four uniquely well-defined operators V_c, V_s, W_c, W_s such that*

$$\square(\mathbf{f}(t)\cos(\omega t))=V_c(\mathbf{f})(t)\cos(\omega t)-V_s(\mathbf{f})(t)\sin(\omega t), \tag{16}$$

$$\square(\mathbf{f}(t)\sin(\omega t))=V_s(\mathbf{f})(t)\cos(\omega t)+V_c(\mathbf{f})(t)\sin(\omega t), \tag{17}$$

$$\square^\star\square(\mathbf{f}(t)\cos(\omega t))=W_c(\mathbf{f})(t)\cos(\omega t)-W_s(\mathbf{f})(t)\sin(\omega t), \tag{18}$$

$$\square^\star\square(\mathbf{f}(t)\sin(\omega t))=W_s(\mathbf{f})(t)\cos(\omega t)+W_c(\mathbf{f})(t)\sin(\omega t). \tag{19}$$

for all mappings $\mathbf{f}:[t_0,t_f]\to\mathbb{R}^d$. The four operators V_c, V_s, W_c, W_s are connected through the formulas

$$W_c=V_c^\star V_c+V_s^\star V_s=W_c^\star,\quad W_s=V_c^\star V_s-V_s^\star V_c=-W_s^\star. \tag{20}$$

Lastly, let $(\gamma_i) \in \mathbb{R}^{2N+1}$ be such that $\sum_\ell \gamma_\ell = 0$ and $\sum_\ell \ell\gamma_\ell = 1$. Then, for all $\mathbf{f} \in \mathcal{C}^2([t_0, t_f], \mathbb{R}^d)$,

$$\lim_{\varepsilon\to 0} V_c(\mathbf{f}) = \dot{\mathbf{f}} \text{ and } \lim_{\varepsilon\to 0} V_s(\mathbf{f}) = \omega\mathbf{f}, \tag{21}$$

$$\lim_{\varepsilon\to 0} W_c(\mathbf{f}) = \omega^2\mathbf{f} - \ddot{\mathbf{f}} \text{ and } \lim_{\varepsilon\to 0} W_s(\mathbf{f}) = -2\omega\dot{\mathbf{f}} \tag{22}$$

locally uniformly in $]t_0, t_f[$.

Proof. Let us introduce the four operators

$$V_c(\mathbf{f}) = \frac{1}{\varepsilon}\sum_{|j|\le N} \gamma_j \cos(\omega j\varepsilon)\chi(t+j\varepsilon)\mathbf{f}(t+j\varepsilon),$$

$$V_s(\mathbf{f}) = \frac{1}{\varepsilon}\sum_{|j|\le N} \gamma_j \sin(\omega j\varepsilon)\chi(t+j\varepsilon)\mathbf{f}(t+j\varepsilon),$$

$$W_c(\mathbf{f}) = \frac{1}{\varepsilon^2}\sum_{\substack{|\ell|\le 2N\\ |j|\le N\\ |\ell+j|\le N}} \gamma_{\ell+j}\gamma_j\chi(t-j\varepsilon)\chi(t+\ell\varepsilon)\cos(\omega\ell\varepsilon)\mathbf{f}(t+\ell\varepsilon),$$

$$W_s(\mathbf{f}) = \frac{1}{\varepsilon^2}\sum_{\substack{|\ell|\le 2N\\ |j|\le N\\ |\ell+j|\le N}} \gamma_{\ell+j}\gamma_j\chi(t-j\varepsilon)\chi(t+\ell\varepsilon)\sin(\omega\ell\varepsilon)\mathbf{f}(t+\ell\varepsilon).$$

Straightforward computations show that equations (16) to (19) hold. Let us recall from Ryckelynck and Smoch (2014, Lemma 2.1) that the adjoint of an operator $\square$ defined by (1) is obtained by reversing its coefficients $(\gamma_j)_j$ to $(\gamma_{-j})_j$. Substituting $\square^\star$ for $\square$ in (16) and (17) implies two new operators $\tilde{V}_c$ and $\tilde{V}_s$, whose coefficients are the sequences $(\gamma_{-j}\cos(\omega j\varepsilon))_j$ and $(\gamma_{-j}\sin(\omega j\varepsilon))_j$ respectively, that is to say the reversed sequences of coefficients of the operators $V_c^\star$ and $-V_s^\star$ respectively. It follows that the two formulas $W_c = V_c^\star V_c + V_s^\star V_s$ and $W_s = V_c^\star V_s - V_s^\star V_c$ hold, as consequences of (16) and (17) when using the operators $\square$ and $\square^\star$ and mentioning the unicity of coefficients in (18) and (19). Let us remark that formulas (20) imply that W_c is symmetric and W_s is skew-symmetric. An inspection of the proof given in Ryckelynck and Smoch (2013, Proposition 2.3) shows that the result of convergence (2) extends to $\mathcal{C}^2$-piecewise functions $\mathbf{u}$. This being observed, we may deduce the two last results of the property. Since the proofs are similar, we focus especially on (22). Equations (18) and (19) may be rewritten as

$$\begin{pmatrix}\cos(\omega t) & -\sin(\omega t)\\ \sin(\omega t) & \cos(\omega t)\end{pmatrix}\begin{pmatrix}W_c(\mathbf{f})\\ W_s(\mathbf{f})\end{pmatrix} = \begin{pmatrix}\square^\star\square(\mathbf{f}\cos(\omega t))\\ \square^\star\square(\mathbf{f}\sin(\omega t))\end{pmatrix}$$

and, as a consequence, we get the identity

$$\begin{pmatrix}\cos(\omega t) & -\sin(\omega t)\\ \sin(\omega t) & \cos(\omega t)\end{pmatrix}\begin{pmatrix}(\ddot{\mathbf{f}}-\omega^2\mathbf{f})+W_c(\mathbf{f})\\ 2\omega\dot{\mathbf{f}}+W_s(\mathbf{f})\end{pmatrix}=\begin{pmatrix}(\mathbf{f}\cos(\omega t))''+\square^\star\square(\mathbf{f}\cos(\omega t))\\ (\mathbf{f}\sin(\omega t))''+\square^\star\square(\mathbf{f}\sin(\omega t))\end{pmatrix}. \quad (23)$$

The matrix of this system being invertible, the r.h.s of (23) tends to 0 as ε tends to 0 if and only if its l.h.s tends to 0, locally uniformly on each interval of the shape $[t_0+\delta, t_f-\delta]$, for all $\delta>0$. By using the formula (1) in the interval $[t_0+N\varepsilon, t_f-N\varepsilon]$, we see that $\square\mathbf{f}(t)=\frac{\gamma_{-N}}{\varepsilon}\mathbf{f}(t-N\varepsilon)+\ldots+\frac{\gamma_N}{\varepsilon}\mathbf{f}(t+N\varepsilon)$. Since $\mathbf{f}$ is C^2 in $[t_0,t_f]$, $\square\mathbf{f}$ is $\mathcal{C}^2$ in $[t_0+N\varepsilon, t_f-N\varepsilon]$ and we may apply Ryckelynck and Smoch (2013, Proposition 2.3) to state first that $-\square^\star\square\mathbf{f}$ tends to $\ddot{\mathbf{f}}$ as ε tends to 0 and next, help to (23), that (21) and (22) hold. □

We may interprete the four identities (16) to (19) as specialized Leibniz formulas of order 1 and 2. In Ryckelynck and Smoch (2013, Theorem 3.1), we already expressed the remainder $\square(\mathbf{fg})-\mathbf{f}\square\mathbf{g}-\mathbf{g}\square\mathbf{f}$ in a generalized Leibniz formula for a specific class of operators $\square$.

5 Equations of motion in a rotating frame

The aim of this section is to provide, when it is possible, the classical and quantum equations of motion in a rotating frame by using the lagrangian and hamiltonian formalisms and the Legendre transform. From now on, we drop t from the following dynamic variables since it is clear.

To begin with, we shall suppose that the cartesian coordinates $(x_{ik}(t))_{i,k}$ in the classical and quantum settings are of the shape (7), expressed respectively through the $2\times(2n)$ functions $a_i(t), b_i(t)$ and $A_i(t,\varepsilon), B_i(t,\varepsilon)$, which may be thought as perturbative variables around the relative equilibria. Therefore, the distances between the points lying in the plane (x_{i1}, x_{i2}), in the classical and quantum settings, are respectively equal to

$$r_{ij}^2=(a_i-a_j)^2+(b_i-b_j)^2 \text{ and } R_{ij}^2=(A_i-A_j)^2+(B_i-B_j)^2.$$

Let us provide now the perturbative Euler-Lagrange equations in the rotating frame. In the classical setting, we use (7) to compute $\dot{x}_{i1}, \dot{x}_{i2}, \ddot{x}_{i1}, \ddot{x}_{i2}$ and we plug these functions in (8). By using suitable linear combinations, we obtain

$$\begin{aligned} m_i(\ddot{a}_i-2\omega\dot{b}_i-\omega^2 a_i)&=\sum_{j\neq i}\frac{f_{ij}'(R_{ij})}{R_{ij}}(a_i-a_j),\\ m_i(\ddot{b}_i+2\omega\dot{a}_i-\omega^2 b_i)&=\sum_{j\neq i}\frac{f_{ij}'(R_{ij})}{R_{ij}}(b_i-b_j). \end{aligned} \quad (24)$$

Now we use (9), (18) and (19) to obtain

$$\Box^\star\Box x_{i1}(t) = (W_c(A_i) - W_s(B_i))(t)\cos(\omega t) - (W_s(A_i) + W_c(B_i))(t)\sin(\omega t),$$
$$\Box^\star\Box x_{i2}(t) = (W_s(A_i) + W_c(B_i))(t)\cos(\omega t) + (W_c(A_i) - W_s(B_i))(t)\sin(\omega t).$$

Help to the same linear combinations than in the classical case, we obtain the following equations of motion

$$\begin{aligned} -m_i(W_c(A_i) - W_s(B_i)) &= \sum_{j\neq i} \frac{f'_{ij}(R_{ij})}{R_{ij}}(A_i - A_j), \\ -m_i(W_s(A_i) + W_c(B_i)) &= \sum_{j\neq i} \frac{f'_{ij}(R_{ij})}{R_{ij}}(B_i - B_j). \end{aligned} \tag{25}$$

As a corollary of Proposition 1, we readily see that the operators in the l.h.s. of (25) converge to the operators of the l.h.s. of (24), as ε tends to 0, provided the coefficients (γ_ℓ) satisfy the assumptions of the proposition. Moreover, if we consider any family $(A_i(\varepsilon,t), B_i(\varepsilon,t))_{1\le i\le n}$ of $2n$ functions from $]-\varepsilon_0,\varepsilon_0[\times[t_0,t_f]$ to $\mathbb{R}$, and if we set $a_i(t) = A_i(0,t)$ and $b_i(t) = B_i(0,t)$, then the formulas (21) and (22) imply that both sides of each equation in (25) converge to the respective quantities in (24). Let us construct first the canonical coordinates in the rotating frame. Obviously, the decompositions in the galilean frame (5) and (6) do not longer hold when working in a rotating frame since there appears some inertial forces and effects. Let us consider the classical case. As in classical textbooks (for instance Boccaletti and Pucacco (1996, p. 266)), we choose as coordinates $a_i(t), b_i(t)$ and as momenta

$$c_i(t) = m_i(\dot{a}_i(t) - \omega b_i(t)), \quad d_i(t) = m_i(\dot{b}_i(t) + \omega a_i(t)). \tag{26}$$

By using the derivatives of $x_{i1}(t)$ and $x_{i2}(t)$ obtained from (7) and the expression of T given in (4), we easily find

$$\mathcal{L}_c = \frac{1}{2}\sum_{i=1}^n m_i[(\dot{a}_i - \omega b_i)^2 + (\dot{b}_i + \omega a_i)^2] + U(\mathbf{x}) = \frac{1}{2}\sum_{i=1}^n \frac{1}{m_i}(c_i^2 + d_i^2) + U(\mathbf{x}).$$

The Lagrangian $\mathcal{L}_c$ depends naturally on the variables $a_i, b_i, \dot{a}_i, \dot{b}_i$ while the hamiltonian function $\mathcal{H}_c$, obtained through the Legendre transform of $\mathcal{L}_c$, depends essentially on a_i, b_i, c_i, d_i. Its value is given by

$$\mathcal{H}_c = \sum_{i=1}^n \left(\dot{a}_i \frac{\partial \mathcal{L}_c}{\partial \dot{a}_i} + \dot{b}_i \frac{\partial \mathcal{L}_c}{\partial \dot{b}_i}\right) - \mathcal{L}_c = \frac{1}{2}\sum_{i=1}^n \frac{1}{m_i}(c_i^2 + d_i^2) - \omega \sum_{i=1}^n (a_i d_i - b_i c_i) - U(\mathbf{x}).$$

By using (24) and (26), we may easily show that the partial derivatives of $\mathcal{H}_c$ w.r.t. c_i, d_i, a_i, b_i are respectively equal to

$$\dot{a}_i = \frac{\partial \mathcal{H}_c}{\partial c_i}, \ \dot{b}_i = \frac{\partial \mathcal{H}_c}{\partial d_i}, \ \dot{c}_i = -\frac{\partial \mathcal{H}_c}{\partial a_i}, \ \dot{d}_i = -\frac{\partial \mathcal{H}_c}{\partial b_i}. \tag{27}$$

Plugging the various equations (7) in (16) and (17), we get

$$\begin{aligned}\Box x_{i1}(t) &= (V_c(A_i) - V_s(B_i))(t)\cos(\omega t) - (V_s(A_i) + V_c(B_i))(t)\sin(\omega t),\\ \Box x_{i2}(t) &= (V_s(A_i) + V_c(B_i))(t)\cos(\omega t) + (V_c(A_i) - V_s(B_i))(t)\sin(\omega t).\end{aligned}$$

Now, squaring, expanding and summing, we find that the delayed Lagrangian defined by (6) is given as follows

$$\mathcal{L}_d = \frac{1}{2}\sum_{i=1}^{n} m_i[(V_c(A_i) - V_s(B_i))^2 + (V_s(A_i) + V_c(B_i))^2] + U(\mathbf{x}).$$

Because of the formal similarity between $\mathcal{L}_c$ and $\mathcal{L}_d$, we choose as canonical coordinates $A_i(t), B_i(t)$, and as momenta

$$C_i(t) = m_i(V_c(A_i)(t) - V_s(B_i)(t)), \quad D_i(t) = m_i(V_s(A_i)(t) + V_c(B_i)(t)). \tag{28}$$

Hence, $\mathcal{L}_d$ may be rewritten as

$$\mathcal{L}_d = \frac{1}{2}\sum_{i=1}^{n} \frac{1}{m_i}(C_i^2 + D_i^2) + U(\mathbf{x}).$$

The Lagrangian $\mathcal{L}_d$ depends naturally on the variables A_i, B_i, but also on $V_c(A_i)$, $V_c(B_i)$, $V_s(A_i)$, $V_s(B_i)$. In contrast, $\mathcal{L}_d$ does not depend naturally on the variables C_i and D_i introduced a posteriori nor on the variables $\Box A_i$ and $\Box B_i$. This is a clear indication of the non-covariance of the discretization procedure when dealing with inertial frames. At this point, it is essential to note that the Legendre transform may not be generalized in a convenient way to the quantum case. A first reason for this, is that the derivation of $\mathcal{L}_d$ w.r.t. the variables $\Box A_i$ and $\Box B_i$ is a nonsense. A second deeper one is that the delay Euler-Lagrange functional equations are not covariant w.r.t. change of variables. A last reason is that the Hamilton's principle is not covariant in the quantum setting.

6 Proofs of the results.

Proof (Theorem 1). We deduced in Ryckelynck and Smoch (2013, Theorem 4.1) the delay Euler-Lagrange functional equations which may be written as

$$\Box^\star \frac{\partial \mathcal{L}_d}{\partial \Box x_{ik}}(t, \mathbf{x}(t), \Box\mathbf{x}(t)) + \frac{\partial \mathcal{L}_d}{\partial x_{ik}}(t, \mathbf{x}(t), \Box\mathbf{x}(t)) = 0, \tag{29}$$

for all $i \in \{1, \ldots, n\}$ and $k \in \{1, \ldots, d\}$. Equations (9) are an easy consequence of (29) applied to $\mathcal{L}_d$ given in (6). Now, since the explicit value of the left-hand side of (9) is given help to the following formula

$$\Box^\star \Box \mathbf{f}(t) = \frac{1}{\varepsilon^2} \sum_{\substack{|\ell|\leq 2N \\ |j|\leq N \\ |\ell+j|\leq N}} \gamma_{\ell+j}\gamma_j \chi(t-j\varepsilon)\chi(t+\ell\varepsilon)\mathbf{f}(t+\ell\varepsilon). \tag{30}$$

which is excerpt from Ryckelynck and Smoch (2013), we see that (9) are functional delay equations. □

Proof (Theorem 2). In this fairly simple setting we have $p_{ik} = \frac{\partial \mathcal{L}_d}{\partial \Box x_{ik}} = m_i \Box x_{ik}$. Thus, equation (29) may be rewritten as

$$\Box^\star p_{ik} = -\frac{\partial \mathcal{L}_d}{\partial x_{ik}} = -\frac{\partial U}{\partial x_{ik}} = \frac{\partial \mathcal{H}}{\partial x_{ik}}.$$

Next, since $T = \frac{1}{2}\sum_{i=1}^n \sum_{k=1}^d \frac{1}{m_i} p_{ik}^2$, it is obvious that

$$\frac{\partial \mathcal{H}}{\partial p_{ik}} = \frac{\partial T}{\partial p_{ik}} = \frac{1}{m_i} p_{ik} = \Box x_{ik}.$$

Thus, we have proved that (9) implies (11) and the converse is easy. The same reasoning as before shows that (11) are functional delay equations. □

Proof (Theorem 3). The two first equations (12) and (13) arise from the computation of the partial derivatives of $\mathcal{H}_d$ w.r.t. C_i or D_i and the definition (28) of these momenta. The two last ones (14) and (15) are easy consequences of the values of the partial derivatives of $\mathcal{H}_d$ w.r.t. A_i or B_i and the use of the r.h.s. of (25) to eliminate $\frac{\partial U}{\partial A_i}$ and $\frac{\partial U}{\partial B_i}$. Now let us prove that the respective left hand-sides of equations (12) to (15) converge to the left hand-sides of equations (27) as ε tends to 0, which depends mainly on (21) and (22).

Let (A, B, C, D) be four functions of class $\mathcal{C}^1$ w.r.t. t and let us denote

$$(a(t), b(t), c(t), d(t)) = (A(0,t), B(0,t), C(0,t), D(0,t)).$$

Convergence of first order functional delay operators hold:

$$V_c(A) - V_s(B) + \omega B \to \dot{A}(0,t) - \omega B + \omega B = \dot{a},$$
$$V_s(A) + V_c(B) - \omega A \to \omega A + \dot{B}(0,t) - \omega A = \dot{b},$$

while for second order functional delay operators, we get:

$$m(W_c(A) - W_s(B)) - \omega D \to m(\omega^2 A(0,t) - \ddot{A}(0,t) + 2\omega\dot{B}(0,t)) - \omega D(0,t) = -\dot{c}$$
$$m(W_s(A) + W_c(B)) + \omega C \to m(-2\omega\dot{A}(0,t) + \omega^2 B(0,t) + \ddot{A}(0,t)) + \omega C(0,t) = -\dot{d}$$

as ε tends to 0. From this we deduce that the schemes (12) to (15) converge to (27) and the analogous property for Lagrangian and Hamiltonian is obvious. □

7 Relative equilibria solutions to the generalized n-body problem in classical and quantum settings

We recall that a relative equilibrium solution of a generalized n-body problem is a configuration of n moving particles which are located at fixed points in a uniformly rotating plane. We mention the terminology used in Boccaletti and Pucacco (1996, pp. 217, pp. 219) as planar solution and homographic solution with dilatation equal to 1. Some other authors call those solutions Lagrange configurations.

We shall study first the case where (a_i, b_i) and (A_i, B_i) are constant w.r.t. time. We shall give some remarks at the end of this section in the case when the coordinates (a_i, b_i) and (A_i, B_i) are of the shape $(a_i, b_i) = (a_i^0 \lambda(t), b_i^0 \lambda(t))$ and $(A_i, B_i) = (A_i^0 \Lambda(t), B_i^0 \Lambda(t))$ where λ and Λ are some dilatation factors. Note that the mutual distances r_{ij} and R_{ij} are constant w.r.t. time.

7.1 Existence of equilibria in galilean frames

As usually done in the classical case, one may ask if there exist solutions of (9) which are constant w.r.t. time.

Proposition 1 – *Let us consider a galilean frame, let I be an interval included in $\mathbb{R}$. Let us suppose that one of the two following conditions holds:*

- $[t_0, t_f] \subset I$ *and* $\Box$ *is not defined by* $(0, \ldots, 0, \gamma_0, 0, \ldots, 0) \in \mathbb{R}^{2N+1}$,
- $I = \mathbb{R}$ *and* $\Box \neq 0$.

Then there do not exist solutions of (9) remaining constant w.r.t time in the interval I.

Proof. Let us prove the two points by contraposition. Let us suppose that there exists a solution $\{x_{ik}(t)\}$ of (9), i and k running from 1 to n and d respectively, which remains constant w.r.t. time inside I. Then formula (9) shows that $\Box^\star \Box 1$ must be constant, say cst, over I. In order to compute the value of $\Box^\star \Box f(t)$ for any function $f(t)$, we use (30) and apply it to $f(t) = 1$. We look for the coefficients $(\gamma_\ell)_\ell$ in order to check $\Box^\star \Box 1 = cst$ in I. In both cases, we shall assume for ease of exposition that $N = 1$ since the proof for arbitrary N is similar .

In the first case, when $[t_0, t_f] \subset I$, we consider the five explicit values of $\Box^\star \Box 1$ in the convenient intervals

- if $t \in [t_0, t_0 + \varepsilon[$, $\Box^\star \Box 1(t) = \frac{1}{\varepsilon^2}(\gamma_{-1}(\gamma_{-1} + \gamma_0 + \gamma_1) + \gamma_0(\gamma_0 + \gamma_1))$,
- if $t \in [t_0 + \varepsilon, t_0 + 2\varepsilon[$, $\Box^\star \Box 1(t) = \frac{1}{\varepsilon^2}((\gamma_{-1} + \gamma_0)(\gamma_{-1} + \gamma_0 + \gamma_1) + \gamma_1(\gamma_0 + \gamma_1))$,
- if $t \in [t_0 + 2\varepsilon, t_f - 2\varepsilon]$, $\Box^\star \Box 1(t) = \frac{1}{\varepsilon^2}(\gamma_{-1} + \gamma_0 + \gamma_1)^2 = (\Box 1(t))^2$,
- if $t \in]t_f - 2\varepsilon, t_f - \varepsilon]$, $\Box^\star \Box 1(t) = \frac{1}{\varepsilon^2}(\gamma_{-1}(\gamma_{-1} + \gamma_0) + (\gamma_0 + \gamma_1)(\gamma_{-1} + \gamma_0 + \gamma_1))$,

- if $t \in]t_f - \varepsilon, t_f]$, $\square^\star \square 1(t) = \frac{1}{\varepsilon^2}(\gamma_0(\gamma_{-1}+\gamma_0) + \gamma_1(\gamma_{-1}+\gamma_0+\gamma_1))$.

The second equation being identical to the fourth one, one sees easily that the system implies $(\gamma_{-1}, \gamma_0, \gamma_1) = (0, \gamma_0, 0)$. The case for arbitrary N is similar.

Now, let us deal with the second case $I = \mathbb{R}$. We recall that $\square \mathbf{x}$ is compactly supported for all $\mathbf{x}$ (see Ryckelynck and Smoch (2013, Proposition 2.1)). So we see that the l.h.s. of (9) vanishes, for all index i, outside the interval $[t_0 - N\varepsilon, t_f + N\varepsilon]$. Since the functions appearing in the rh.s. of (9) are obviously constant, the l.h.s. of (9) vanishes for all $t \in \mathbb{R}$. As a rule, since $\square^\star \square 1(t) = (\square 1(t))^2$ for $t \in [t_0 + 2N\varepsilon, t_f - 2N\varepsilon]$, we must have $\sum_\ell \gamma_\ell = 0$ or equivalently $\gamma_0 = -\sum_{\ell \neq 0} \gamma_\ell$ which implies that $\square = 0$ from the previous result, thus contradicting the assumption and this ends the proof. □

Remark 1 – The study of constant solutions in the newtonian case is much more simple since the functions f_{ij} are all increasing or all decreasing. Indeed, we may prove the result by considering for all $k \in \{1, \ldots, d\}$ the equations (8) or (9) of index $i \in \{1, \ldots, n\}$ maximizing x_{ik}, without studying $\square^\star \square$ in the various intervals of time.

Remark 2 – In contrast with Proposition 1 and Remark 1, it might exist constant solutions of (8) in $\mathbb{R}$. This occurs for instance when considering the Laplace-Sellinger or the London potentials f_{ij}.

7.2 The algebraic equations of relative equilibria

We introduce the system of $2n$ algebraic equations

$$-\lambda m_i x_i = \sum_{j \neq i} f'_{ij}(s_{ij}) \frac{x_i - x_j}{s_{ij}} \text{ and } -\lambda m_i y_i = \sum_{j \neq i} f'_{ij}(s_{ij}) \frac{y_i - y_j}{s_{ij}}, \tag{31}$$

where $s_{ij} = ((x_i - x_j)^2 + (y_i - y_j)^2)^{1/2}$, which constitute a slight generalization of the algebraic equations for relative equilibria to appear later on. The unknowns are the $2n+1$ real numbers x_i, y_i and λ. The number λ is related to the pulsation ω of the configuration. The $2n$ preceding equations are dependent because they sum to 0 so that $\sum_{k=1}^n m_k x_k = \sum_{k=1}^n m_k y_k = 0$ using an argument of symmetry. We note that if all the functions $f_{ij}(r)$ are algebraic w.r.t. r then the functions $f'_{ij}(r)/r$ are also algebraic and thus, the equations (31) are algebraic w.r.t. the coordinates a_i, b_i and A_i, B_i. In constrast, those equations are not algebraic w.r.t. ε or ω. We note also by the way that equations (31) are not invariant by translation. However, they are invariant by rotation in the plane, that is to say if (x_i, y_i) is a set of solutions, then for all $\alpha \in \mathbb{R}$, $(x_i \cos\alpha - y_i \sin\alpha, x_i \sin\alpha + y_i \cos\alpha)$ is another set of solutions of (31). The problem of finiteness of the quotient set of solutions by the orthogonal group $SO(2, \mathbb{R})$ remains open even in the newtonian case and is known as the Wintner's conjecture mentioned in Smale (2000). We may conjecture, help to Bezout theorem in algebraic geometry, that (31) is a system of algebraic equations of rank $2n-2$ with no common zero-hypersurfaces and thus have finitely many solutions up to rotations.

Proposition 2 – *Let $n = 3$. We assume that the three potential functions f_{ij} ($i,j \in \{1,2,3\}$, $i < j$) satisfy the following condition: there exists an injective function $\zeta : \mathbb{R}_+^* \to \mathbb{R}$ such that for all $s > 0$, one has $f_{ij}'(s) = s\frac{\zeta(s)}{m_k}$, where $\{i,j,k\} = \{1,2,3\}$. Then each configuration (x_i, y_i) satisfying (31) for some λ is either colinear or equilateral.*

Proof. Suppose that (x_i, y_i) is a solution of (31) for some λ. We deduce from (31) the following equations

$$\begin{pmatrix} \dfrac{m_1}{\dfrac{f_{13}'(s_{13})}{s_{13}}} & \dfrac{m_2}{\dfrac{f_{23}'(s_{23})}{s_{23}}} \end{pmatrix} \begin{pmatrix} x_1 - x_3 \\ x_2 - x_3 \end{pmatrix} = \begin{pmatrix} -(m_1 + m_2 + m_3)x_3 \\ \lambda m_3 x_3 \end{pmatrix} \tag{32}$$

and an entirely similar system for the vector ${}^{\mathsf{T}}(y_1 - y_3, y_2 - y_3)$. By permuting the indices $1,2,3$, we obtain six bidimensional linear systems, inducing only three different matrices. Let us suppose that one of these three matrices is regular, say the one occuring in (32). Then we obtain (x_1, x_2) and (y_1, y_2) as functions of x_3 and y_3 respectively and we check easily that the four points (x_i, y_i) and $(0,0)$ lie on the same straight line. Now, when all the three matrices are singular, one has the following system

$$\begin{cases} f_{13}'(s_{13}) m_2 s_{23} = f_{23}'(s_{23}) m_1 s_{13} \\ f_{12}'(s_{12}) m_3 s_{23} = f_{23}'(s_{23}) m_1 s_{12} \\ f_{13}'(s_{13}) m_2 s_{12} = f_{12}'(s_{12}) m_3 s_{13} \end{cases}$$

which yields simply $\zeta(s_{12}) = \zeta(s_{13}) = \zeta(s_{23})$. Due to the assumptions on ζ, we get $s_{12} = s_{13} = s_{23}$. □

Remark 3 – In contrast with the proof given in Boccaletti and Pucacco (1996), we do not use Galileo's law.

Remark 4 – If we choose for some fixed number $\beta \in \mathbb{R}^*$, $\zeta(s) = m_1 m_2 m_3 \beta s^{\beta - 2}$, we recover the homogeneous potential function occuring in Section 1.

Now, let us connect the system (31) to the search of constant solutions of (24), respectively (25). We shall say that a solution $\{(a_i, b_i)\}$ of (24) is a relative equilibrium if all coordinates (a_i, b_i) are independent on $t \in \mathbb{R}$. Similarly, we say that a solution $\{(A_i, B_i)\}$ of (25) is a relative equilibrium if all coordinates (A_i, B_i) are independent on $t \in [t_0 + 2N\varepsilon, t_f - 2N\varepsilon]$. This specific interval is chosen in such a way that relative equilibria exist in each setting and are closely connected. Let us note that, in the case the previous interval is replaced with $\mathbb{R}$, no solution would be found as an analysis similar to the proof of Proposition 1 shows.

We note that the function $W_c(1)(t)$ takes a constant value inside $[t_0 + 2N\varepsilon, t_f - 2N\varepsilon]$. We assume in the remainder of this paper that this constant is positive and we denote it by $\Omega^2(\varepsilon)$. Let us remark that if □ satisfies (3), then $\lim_{\varepsilon \to 0} \Omega^2(\varepsilon) = \omega^2$

locally uniformly in $]t_0, t_f[$, as the formula (22) in Proposition 1 shows, and the previous assumption is satisfied.

We observe that, in order that a configuration of n particles $\{(a_i, b_i)\}$ is a relative equilibrium solution of (24), it is necessary and sufficient that the set $\{(a_i, b_i)\}$ is solution of (31) with $\lambda = \omega^2$. Indeed, this is a simple consequence of plugging constant functions (a_i, b_i) in (24). In a similar way, we obtain the following

Proposition 3 – *In order that a configuration of n particles $\{(A_i, B_i)\}$ is a relative equilibrium solution of (25), it is necessary and sufficient that the set $\{(A_i, B_i)\}$ is solution of (31) with $\lambda = \Omega^2(\varepsilon)$*

Proof. We have in $[t_0 + 2N\varepsilon, t_f - 2N\varepsilon]$,

$$W_c(1) = \Omega^2(\varepsilon) = \frac{1}{\varepsilon^2} \sum_{\substack{|\ell| \leq 2N \\ |j| \leq N \\ |\ell + j| \leq N}} \gamma_{\ell+j} \gamma_j \cos(\ell \varepsilon \omega),$$

$$W_s(1) = \frac{1}{\varepsilon^2} \sum_{\substack{|\ell| \leq 2N \\ |j| \leq N \\ |\ell + j| \leq N}} \gamma_{\ell+j} \gamma_j \sin(\ell \varepsilon \omega).$$

By using an easy symmetry argument, we prove that, for all $t \in [t_0 + 2N\varepsilon, t_f - 2N\varepsilon]$, $W_s(1)(t) = 0$. So, when we suppose that the functions $\{(A_i, B_i)\}$ remain constant in $[t_0 + 2N\varepsilon, t_f - 2N\varepsilon]$, formula (25) gives rise to (31) with $\lambda = \Omega^2(\varepsilon)$. □

7.3 Expansion factor and constants of motion for generalized n-body problem with homogeneous potential functions

When the potentials are homogenous with exponent β, the solution sets of the two systems of algebraic equations (31), obtained when $\lambda = \omega^2$ and $\lambda = W_c(1)$, are the same up an homothety. The following result shows this claim.

Proposition 4 – *Let us suppose that $f_{ij}(r) = \mu_{ij} r^\beta$ with $\beta \neq 2$, then to each configuration of n bodies in relative equilibrium for (24) corresponds an homothetic configuration in relative equilibrium for (25) whose homothety ratio is the real number $\varphi(\varepsilon) = \left(\frac{\omega^2}{\Omega^2(\varepsilon)}\right)^{\frac{1}{2-\beta}}$. Furthermore, the kinetic energies T_C and T_D, the potential energies U_C and U_D and the angular momenta σ_C and σ_D of those two homothetic configurations are linked together as*

$$T_D = \varphi(\varepsilon)^2 T_C, \quad U_D = \varphi(\varepsilon)^\beta U_C \quad \textit{and} \quad \sigma_D = \varphi(\varepsilon)^2 \sigma_C.$$

Proof. Indeed, we see that the two systems of equations (31) obtained when $\lambda = \omega^2$ and $\lambda = \Omega^2(\varepsilon)$ may be rewritten respectively as

$$-\frac{\omega^2}{\beta} m_i a_i = \sum_{j\neq i} \mu_{ij} r_{ij}^{\beta-2}(a_i - a_j), \quad -\frac{\omega^2}{\beta} m_i b_i = \sum_{j\neq i} \mu_{ij} r_{ij}^{\beta-2}(b_i - b_j)$$

and

$$-\frac{\Omega^2(\varepsilon)}{\beta} m_i A_i = \sum_{j\neq i} \mu_{ij} R_{ij}^{\beta-2}(A_i - A_j), \quad -\frac{\Omega^2(\varepsilon)}{\beta} m_i B_i = \sum_{j\neq i} \mu_{ij} R_{ij}^{\beta-2}(B_i - B_j).$$

Searching for solutions of the second system of the shape $A_i = a_i\varphi$ and $B_i = b_i\varphi$, both systems agree if and only if $\Omega^2(\varepsilon)\varphi(\varepsilon)^{2-\beta} = \omega^2$ whence the value of φ.

Now, let us deal with the integrals of motion. Since U is homogeneous of degree β, we have $U_D = \varphi(\varepsilon)^\beta U_C$. Next, we use formulas (4) and (7) to compute $T_C = \omega^2 I_0$ and $T_D = \omega^2\varphi^2(\varepsilon) I_0$ where $I_0 = \frac{1}{2}\sum_i m_i(a_i^2 + b_i^2)$ is the moment of inertia. Lastly, the only nonzero component of the angular momentum tensor is equal to $\sigma_C = \sum_{i<j} \mu_{ij}(x_{i1}\dot{x}_{j2} - \dot{x}_{i1}x_{j2}) = \omega\sum_{i<j}\mu_{ij}(a_i a_j + b_i b_j)$ and obviously $\sigma_D = \varphi^2(\varepsilon)\sigma_C$. □

The homothety ratio $\varphi(\varepsilon)$ will be called the expansion factor. For arbitrary N, the condition (3) ensures that $\lim_{\varepsilon\to 0}\varphi(\varepsilon) = 1$ but the converse does not hold. For example, when $N = 1$, the expansion factor is equal to

$$\varphi(\varepsilon) = \left(\frac{\omega^2\varepsilon^2}{(\gamma_{-1}+\gamma_0+\gamma_1)^2 + 2\gamma_{-1}\gamma_1(\cos 2\omega\varepsilon - 1) + 2\gamma_0(\gamma_{-1}+\gamma_1)(\cos\omega\varepsilon - 1)}\right)^{\frac{1}{2-\beta}}.$$

The existence of a finite nonzero limit to $\varphi(\varepsilon)$ as ε tends to 0 is equivalent to the following two equations

$$\gamma_{-1} + \gamma_0 + \gamma_1 = 0 \quad \text{and} \quad \gamma_0\gamma_{-1} + \gamma_0\gamma_1 + 4\gamma_{-1}\gamma_1 = -1.$$

This system admits two families of solutions. The first one is a family of operators □ satisfying $\Box t = -1$ which do not check the condition (3). The second one is an affine straight line of operators □ satisfying (3) and given by

$$\Box^{[r,s]}\mathbf{x}(t) = -\frac{s}{\varepsilon}\mathbf{x}(t-\varepsilon)\chi(t-\varepsilon) + \frac{s-r}{\varepsilon}\mathbf{x}(t)\chi(t) + \frac{r}{\varepsilon}\mathbf{x}(t+\varepsilon)\chi(t+\varepsilon) \tag{33}$$

together with the condition $r + s = 1$, and that we have already encountered in Ryckelynck and Smoch (2013).

8 Numerical experiments

We present in the following the planar graphs associated to the restricted 3-body problem yielding a heavy, a light and a negligible bodies. The parameter μ stands for the normalized ratio between the lightest and the sum of the lightest and heaviest bodies. We choose to work only with the libration points L_4 and L_5 and not with the three unstable eulerian points L_1, L_2, L_3, see Boccaletti and Pucacco (1996). We consider an intermediate time $t_\nu \in [t_0 + 2N\varepsilon, t_f - 2N\varepsilon]$ at which the particle P_3 is located at the neighbourhood of L_4 (or L_5). We use some specific operators $\square$ of the shape (33) and especially $\square^{[1,0]}$, $\square^{[0,1]}$, $\square^{[\frac{1}{2},\frac{1}{2}]}$ and $\square^{[\frac{1-i}{2},\frac{1+i}{2}]}$.

Numerical experiments consist in solving (9) and (25). Although these equations are functional ones, we solve them numerically by computing $A_3(t)$ and $B_3(t)$ on the grid $\{t_\nu + k\varepsilon, k \in \mathbb{Z}\} \cap [t_0 + 2N\varepsilon, t_f - 2N\varepsilon]$. The convergence mode of operators $\square$ in the function space of continuously differentiable functions on $[t_0, t_f]$ is locally uniform in $]t_0, t_f[$ and this induces numerical difficulties relative to the stability of the Cauchy problem at $t = t_0$ or $t = t_f$. This is the reason why the intermediate time t_ν has been introduced.

In order to compare the performances of each operator $\square^{[r,s]}$ presented previously, we compute the error norm $err := \|\mathbf{x}(t_\nu + M\varepsilon) - \mathbf{x}(t_\nu)\|_2$ with $M = 5 \times m$ and $m \in \{0, \ldots, 100\}$. We use for this equations (9) and the system (12)-(15) which amounts to equations (25), abbreviated respectively as DEL (Delay Euler-Lagrange equations) and DHE (Delay Hamiltonian Equations).

Most numerical experiments use the value $\mu = 0.012$ associated to the system consisting of the Earth, the Moon and a rocket. The first one illustrates the fact that solving equations DHE gives more accurate results than solving equations DEL, see Figure 1. In addition, we note that whenever the operators $\square^{[1,0]}$ and $\square^{[0,1]}$ are not convenient to solve (9), they become the best choice for solving (25).

The second experiment uses the operator $\square_q := \square^{[\frac{1-i}{2},\frac{1+i}{2}]}$. The step number m per period which induces the smallness of ε is set to $m = 15, 30, 50$ with $[t_\nu, t_\nu + M\varepsilon] = [0, 5\pi]$. The six following figures 2 to 7 provide the trajectories of the three bodies when using equations DEL and DHE. The results with the three other operators $\square$ are quite similar. These figures illustrate the essential role played by M and the obvious performance of DHE versus DEL. As we can see, 30 steps per period are necessary to reach an acceptable solution of the restricted three-body problem.

The next experiment highlights the crucial role played by the perturbations of the position at time t_ν of the lightest body, particularly in the equations (9). We still use the operator $\square_q := \square^{[\frac{1-i}{2},\frac{1+i}{2}]}$ and the system Earth-Moon-rocket. Using the framework of Boccaletti and Pucacco (1996), we perturb $A_3(0, t_\nu) = \mu - \frac{1}{2}$ and $B_3(0, t_\nu) = \frac{\sqrt{3}}{2}$ respectively as $A_3(\varepsilon, t_\nu) = (\mu - \frac{1}{2})\varphi(\varepsilon) + \delta$ and $B_3(\varepsilon, t_\nu) = \frac{\sqrt{3}}{2}\varphi(\varepsilon) + \delta'$ and we choose $\delta = \delta' = 0.01$ and next, $\delta = \delta' = 0.05$. As we can see in Figures 8 and 9, the trajectory of the rocket becomes more unstable as δ increases. At last, we modify the ratio μ and give up the system earth-moon-rocket. The goal of this experiment is to illustrate the unstability of the system when $\mu > 0.0385$ (see Boccaletti and Pucacco (1996) for example). We use for this two values of μ which are greater than 0.04. As we can see in the two last figures 10 and 11, as soon as the ratio is greater

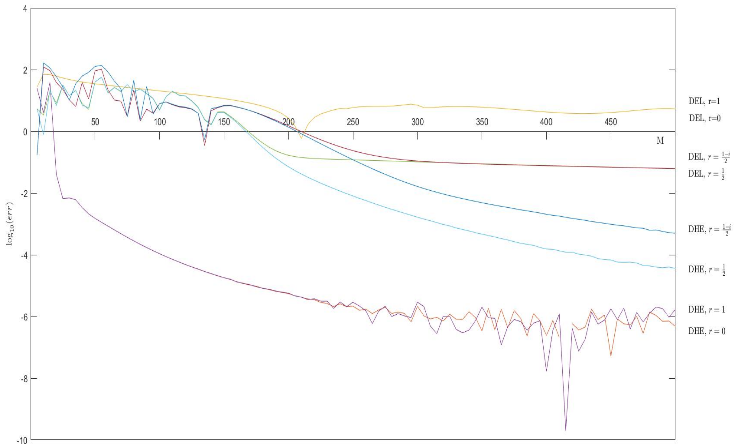

Figure 1 – 2-norm of the error $\mathbf{x}(t_\nu + M\varepsilon) - \mathbf{x}(t_\nu)$

than the limit value mentioned previously, the system becomes unstable.

9 Conclusion

The aim of this work was to apply the formalism of quantum calculus of variations to newtonian dynamics and especially celestial mechanics. As one knows, the search for particular solutions of the many-body problem has a particular importance in the historical development of celestial mechanics, probably because of the feeling that toy-models may be realistic and also that the simply-to-state but hard-to-prove questions in this domain are almost all linked with these particular solutions. In that sense, although we did not give the details of its application, the quantum calculus of variations equally applies to other generalized polygonal solutions, see for instance El Mabsout (1988, 1991). However, performing the effective experiments when applying quantum calculus of variations to choreographic solutions is much more complicated. Let us explain the difficulties in the case of a planar three-body choreographic solution such as the remarkable figure-eight solution found in 2000 by Chenciner and Montgomery (2000). Keeping the previous notations, and dealing with the grid $[t_0 + 2N\varepsilon, t_f - 2N\varepsilon] \cap (t_\nu + \varepsilon\mathbb{Z})$, the coordinates of three particles $x_{i,n} = x_i(t_0 + n\varepsilon)$, $y_{i,n} = y_i(t_0 + n\varepsilon)$ with $N \le n \le M - N$ and $M \le (t_f - t_0)/\varepsilon$, M being a multiple of 3, may be expressed as a system of algebraic equations with the additional constraints that $a_{i,n+M/3} = a_{i,n}$, $b_{i,n+M/3} = b_{i,n}$. The main task is to device an efficient method to solve the previous system which cannot be triangularized.

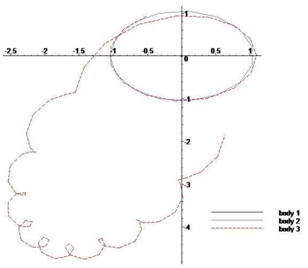

Figure 2 – DEL, $k = 5$, $m = 15$

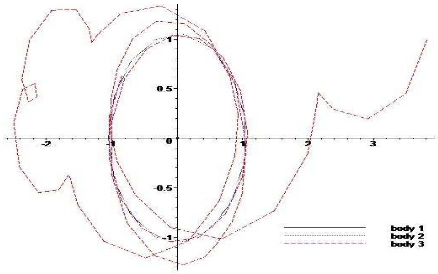

Figure 3 – DHE, $k = 5$, $m = 15$

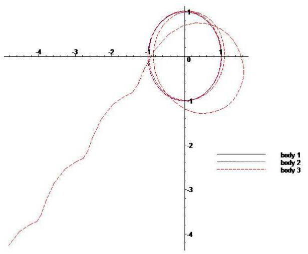

Figure 4 – DEL, $k = 5$, $m = 30$

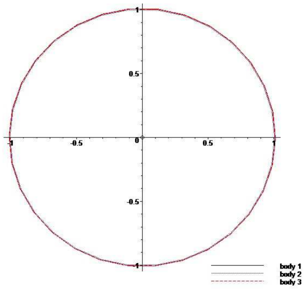

Figure 5 – DHE, $k = 5$, $m = 30$

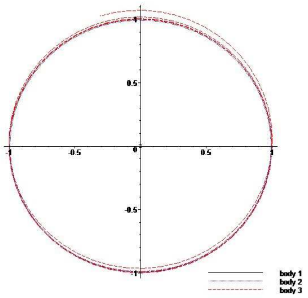

Figure 6 – DEL, $k = 5$, $m = 50$

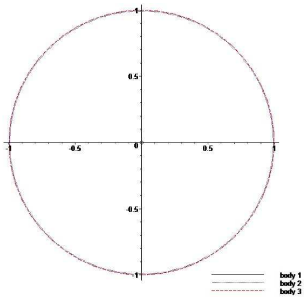

Figure 7 – DHE, $k = 5$, $m = 50$

We address this issue that is studied in a companion paper of the present one.

The link between the constants of motion of solutions of classical or quantum equations of motion has been established only in the case of relative equilibria. However, for arbitrary solutions, a phenomenon of diffusion of constants of motion appears due to the fact that the classical derivative and the generalized derivatives do not commute. Lastly, the application of Q.C.V. to systems of particles interacting

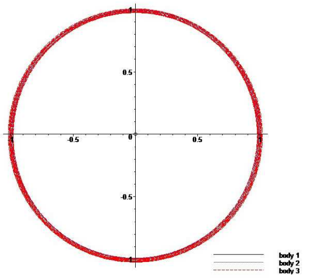

Figure 8 – DHE, $k = 20$, $m = 50$, $\delta = 0.01$

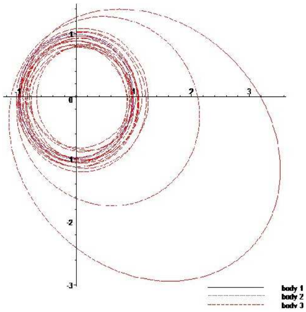

Figure 9 – DHE, $k = 20$, $m = 50$, $\delta = 0.05$

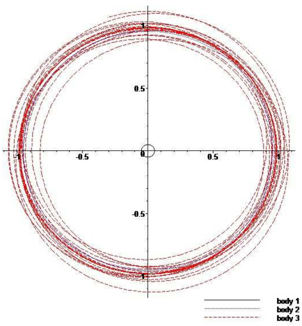

Figure 10 – DHE, $k = 20$, $m = 50$, $\mu = 0.05$, $\square = \square_q$

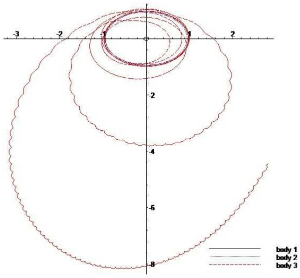

Figure 11 – DHE, $k = 20$, $m = 50$, $\mu = 0.06$, $\square = \square_q$

according to non-homogeneous potentials, for instance those of London and Laplace-Sellinger, is interesting and its treatment may be done through Puiseux series for the solutions of the many-body problem in a rotating frame. Indeed, in this general situation, we do not have an homothety anymore between the relative equilibria in the newtonian and in the QCV contexts.

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