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A short proof of the boundedness of the composition operators on the Hardy space $\mathcal{H}^2$

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Abstract

Following an idea of J. Shapiro, we give a simple proof of the fact that an element of the Gordon Hedenmalm class Φ such that $\Phi(\infty) = \infty$ defines a contractive composition operator C_Φ on the space $\mathcal{H}^2$ of Dirichlet series.

Keywords: Hardy space - composition operator - Dirichlet series.

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1 Introduction

In this note, we deal with the space $\mathcal{H}^2$ of Dirichlet series which has been first introduced in Gordon and Hedenmalm (1999). Recall that a Dirichlet series:

$$f(s) = \sum_{n=1}^{+\infty} a_n n^{-s} \qquad (a_n, s \in \mathbb{C})$$

belongs to $\mathcal{H}^2$ if and only if the complex sequence $(a_n)_{n\geqslant 1}$ satisfies the following growth condition:

$$\sum_{n=1}^{+\infty} |a_n|^2 < \infty$$

The space $\mathcal{H}^2$ is a complex Hilbert space when it is equipped with the norm:

$$\|f\| = \left(\sum_{n=1}^{+\infty} |a_n|^2\right)^{1/2}$$

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Such a Dirichlet series is absolutely convergent on the half-plane:

$$\mathbb{C}_{1/2} := \{s \in \mathbb{C}; \mathfrak{Re}(s) > 1/2\}$$

and, for every $s \in \mathbb{C}_{1/2}$, the point evaluation function:

$$\delta_s : \begin{cases} \mathcal{H}^2 & \longrightarrow & \mathbb{C} \\ f & \longmapsto & f(s) \end{cases}$$

is bounded and $\|\delta_s\| = \sqrt{\zeta(2\mathfrak{Re}(s))}$.

For every non-negative real number θ, we denote by $\mathbb{C}_\theta$ the half space $\mathfrak{Re}(s) > \theta$ (and for the special case $\theta = 0$, we put $\mathbb{C}_0 = \mathbb{C}_+$).

The paper Gordon and Hedenmalm (1999) initiated the study of various spaces of Dirichlet series. The reader can also see for instance Bayart (2002a) for the Hardy spaces $\mathcal{H}^p$ or Bailleul and Lefèvre (2013) for two classes of Bergman spaces of Dirichlet series.

Let $\mathcal{D}$ be the space of Dirichlet series which admit a representation by a convergent Dirichlet series on some half-plane. It is well-known from Gordon and Hedenmalm (1999, Theorem B) that an analytic function $\Phi : \mathbb{C}_{1/2} \longrightarrow \mathbb{C}_{1/2}$ defines a bounded composition operator on $\mathcal{H}^2$ if and only if the two following conditions are fullfilled:

(a) Φ is of the form:

$$\Phi(s) = c_0 s + \varphi(s)$$

where c_0 is a non-negative integer and $\varphi \in \mathcal{D}$;

(b) Φ has an analytic extension to $\mathbb{C}_+$ (also denoted by Φ) such that φ converges uniformly in $\mathbb{C}_\varepsilon$ for every $\varepsilon > 0$ and has the following properties:

- (i) if $c_0 \geqslant 1$, then either $\varphi(\mathbb{C}_+) \subset \mathbb{C}_+$ or $\varphi(s) = i\tau$ for some $\tau \in \mathbb{R}$;
- (ii) if $c_0 = 0$, $\varphi(\mathbb{C}_+) \subset \mathbb{C}_{1/2}$.

The Gordon-Hedenmalm class consists in the class of functions satisfying both (a) and (b).

The aim of this paper is to give a new proof of the boundedness of the composition operator C_Φ when the integer c_0 is positive. The proof is inspired from that of Littlewood's Subordination Principle Shapiro (1993) in the context of power series. This is the result we are going to prove in section 3 (Gordon and Hedenmalm (1999, Theorem B) when $\Phi(\infty) = \infty$).

Theorem 1 – *Let Φ be an analytic function such that $\varphi(\mathbb{C}_+) \subset \mathbb{C}_+$ of the form:*

$$\Phi(s) = c_0 s + \varphi(s)$$

where c_0 is a positive integer and $\varphi \in \mathcal{D}$. Then the composition operator C_Φ defines a bounded linear operator on $\mathcal{H}^2$ such that $\|C_\Phi\| = 1$.

The boundedness of multiplication operators on $\mathcal{H}^2$ plays a key role in the proof of the above theorem. To define such an operator:

$$M_\psi : \begin{cases} \mathcal{H}^2 & \longrightarrow & \mathcal{H}^2 \\ f & \longmapsto & f\psi \end{cases}$$

the symbol ψ must belong to the set of multipliers of $\mathcal{H}^2$, that is:

$$\mathcal{M} = \left\{\psi \in \mathcal{D}; \forall f \in \mathcal{H}^2,\ \psi f \in \mathcal{H}^2\right\}$$

It is well-know from Hedenmalm, Lindqvist, and Seip (1997) that the set of multipliers $\mathcal{M}$ is equal to the space $\mathcal{H}^\infty$ of bounded Dirichlet series on $\mathbb{C}_+$ (and that the multiplication operators are bounded on $\mathcal{H}^2$). It is the aim of section 2 to give a shortest proof of this result by using the boundedness of the point evaluation on $\mathcal{H}^2$.

Theorem 2 (Hedenmalm, Lindqvist, and Seip (1997), Theorem 3.1) – *Let $\mathcal{M}$ be the set of multipliers of $\mathcal{H}^2$. Then $\mathcal{M} = \mathcal{H}^\infty$ and for every $\phi \in \mathcal{H}^\infty$, $\|M_\phi\| = \|\phi\|_\infty$.*

Note that this theorem has been generalized for a large scale of spaces of Dirichlet series: see Bayart (2002b) for the Hardy spaces $\mathcal{H}^p$ and Bailleul and Brevig (2016), Bailleul (2015) for the two classicals families of Bergman spaces.

2 Boundedness of multipliers

In the next proof, an important argument is the following:

Theorem 3 (Bohr's lemma) – *If a function $f \in \mathcal{D}$ has an analytic extension to a bounded function on some half space $\mathbb{C}_\theta$ then the Dirichlet series associated to f converges uniformly on every half space $\mathbb{C}_{\theta'}$ with $\theta' > \theta$.*

By using the closed graph theorem, we can prove that the operator of multiplication by ϕ, denoted by M_ϕ, is bounded on $\mathcal{H}^2$.

Proof (Proof of Theorem 2). We only give a new proof of the *hard part* of the theorem, that is if $m \in \mathcal{M}$ then $m \in \mathcal{H}^\infty$. The idea is to use the point evaluation on $\mathcal{H}^2$. Let $f \in \mathcal{H}^2$ such that:

$$f(s) = \sum_{n=1}^{+\infty} a_n n^{-s} \qquad (a_n, s \in \mathbb{C})$$

we denote by f_ℓ the following partial Dirichlet series:

$$f_\ell(s) = \sum_{p_+(n)\leqslant p_\ell} a_n n^{-s}$$

for every non-negative integer ℓ, where $p_+(n)$ denotes the largest prime factor of n and $(p_\ell)_{\ell\geqslant 1}$ is the increasing sequence of prime numbers. Then by Cauchy-Schwarz inequality, we have for every $\sigma > 0$:

$$\sum_{p_+(n)\leqslant p_\ell} |a_n| n^{-\sigma} \leqslant \left(\sum_{p_+(n)\leqslant p_\ell} n^{-2\sigma} \right)^{1/2} \|f_\ell\|_2$$

With a classical Euler product argument and by using the fact that $\|f_\ell\|_2 \leqslant \|f\|_2$, we obtain:

$$\sum_{p_+(n)\leqslant p_\ell} |a_n| n^{-\sigma} \leqslant \left(\prod_{i=1}^{\ell} \frac{1}{1-p_i^{-2\sigma}} \right)^{1/2} \|f\|_2 \tag{1}$$

It follows that f_ℓ converges on $\mathbb{C}_+$ and that:

$$\forall f \in \mathcal{H}^2,\ \forall s \in \mathbb{C}_+, \qquad |f_\ell(s)| \leqslant \left(\prod_{i=1}^{\ell} \frac{1}{1-p_i^{-2\mathfrak{Re}(s)}} \right)^{1/2} \|f\|_2$$

Let $s \in \mathbb{C}_+$ and $\phi \in \mathcal{M}$. Then $\phi \in \mathcal{H}^2$ since $1 \in \mathcal{H}^2$ and $\|\phi\|_2 \leqslant \|M_\phi\|$. By induction, we obtain:

$$\forall k \geqslant 1, \qquad \left\|\phi^k\right\|_2 \leqslant \|M_\phi\|^k$$

and so:

$$\forall k \geqslant 1, \qquad \left\|\phi^k\right\|_2^{1/k} \leqslant \|M_\phi\|$$

Let ℓ be a positive integer. We point out that for every $k \geqslant 1$, the Dirichlet series ϕ_ℓ^k has only coefficients b_n with $p_+(n) \leqslant \ell$. According to (1), there exists $C_{s,\ell} > 0$ such that:

$$\forall k \geqslant 1, \qquad |\phi_\ell^k(s)| \leqslant C_{s,\ell} \|\phi^k\|_2$$

and then:

$$\forall k \geqslant 1, \qquad |\phi_\ell(s)| \leqslant C_{s,\ell}^{1/k} \|M_\phi\|$$

Letting k going to $+\infty$, we get:

$$|\phi_\ell(s)| \leqslant \|M_\phi\|$$

Hence $(\phi_\ell)_{\ell\geqslant 1}$ is a uniformly bounded sequence of $\mathcal{H}^\infty$ so by Montel's lemma in this context (Bayart (2002b, Lemma 5.2)), there exists $\tilde{\phi}\in\mathcal{H}^\infty$ and a subsequence $(\phi_{\ell_k})_{k\geqslant 1}$ which converges uniformly to $\tilde{\phi}$ on every half space $\mathbb{C}_\varepsilon$ with $\varepsilon>0$. We know that $\phi\in\mathcal{H}^2$ so $(\phi_{\ell_k})_{k\geqslant 1}$ also converges uniformly to ϕ on every half space $\mathbb{C}_{1/2+\varepsilon}$ with $\varepsilon>0$. By uniqueness, we conclude that $\tilde{\phi}$ is a bounded analytic extension of ϕ on $\mathbb{C}_+$ by Bohr's lemma, *i.e.* $\phi\in\mathcal{H}^\infty$. □

3 A Shapiro type result

The proof of Theorem 1 is also based on the following easy lemma.

Lemma 1 – *Let $\varphi\in\mathcal{D}$ and $\alpha>0$. Then $\alpha^{-\varphi}\in\mathcal{D}$. If in addition $\varphi(\mathbb{C}_+)\subset\mathbb{C}_+$ then $\alpha^{-\varphi}\in\mathcal{H}^\infty$ and $\|\alpha^{-\varphi}\|_\infty\leqslant 1$.*

Proof. There exists $\theta>0$ such that we have the following Dirichlet series expansion of φ in the half space $\mathbb{C}_\theta$:

$$\varphi(s)=\sum_{n=1}^{+\infty}\frac{a_n}{n^s}\qquad\qquad (s\in\mathbb{C}_\theta)$$

For every $s\in\mathbb{C}_\theta$:

$$\begin{aligned}\alpha^{-\varphi}=e^{-\varphi\ln(\alpha)}&=\exp\left(-\ln(\alpha)\sum_{n=1}^{+\infty}a_n n^{-s}\right)\\&=\sum_{k=0}^{+\infty}\frac{(-\ln(\alpha))^k}{k!}\left(\sum_{n=1}^{+\infty}\frac{a_n}{n^s}\right)^k\end{aligned}$$

There exists $\sigma>\theta$ such that $\lambda:=\sum_{n=1}^{+\infty}\frac{|a_n|}{n^\sigma}<+\infty$ hence:

$$\sum_{k=1}^{+\infty}\frac{|\ln(\alpha)|^k}{k!}\left(\sum_{n=1}^{+\infty}\frac{|a_n|}{n^\sigma}\right)^k=e^{|\ln(\alpha)|\lambda}<+\infty$$

It is then possible to apply Fubini's theorem in $\mathbb{C}_\sigma$. If we denote by $\sum_{n\geqslant 1}\frac{a_{n,k}}{n^s}$ the Dirichlet series $\left(\sum_{n=1}^{+\infty}\frac{a_n}{n^s}\right)^k$ then:

$$\alpha^{-\varphi}=\sum_{n=1}^{+\infty}\left(\sum_{k=0}^{+\infty}\frac{(-\ln(\alpha))^k}{k!}a_{n,k}\right)n^{-s}$$

which means that $\alpha^{-\varphi}$ belongs to $\mathcal{D}$. Moreover if $\varphi(\mathbb{C}_+)\subset\mathbb{C}_+$ then, for every $s\in\mathbb{C}_+$, we have $|\alpha^{-\varphi(s)}|=\alpha^{-\mathfrak{Re}(\varphi(s))}\leqslant 1$. It follows that $\alpha^{-\varphi}$ is bounded on $\mathbb{C}_+$ and that $\|\alpha^{-\varphi}\|_\infty\leqslant 1$. According to Bohr's lemma we can conclude that $\alpha^{-\varphi}$ belongs to $\mathcal{H}^\infty$.□

We are now able to prove Theorem 1.

Proof (of Theorem 1). Let us first consider a Dirichlet polynomial $f(s)=\sum_{n=1}^{N}a_n n^{-s}$ where N is a positive integer and $s,a_1,\dots,a_N\in\mathbb{C}$. Since $C_\Phi f$ is a bounded Dirichlet series in $\mathbb{C}_+$, it belongs to $\mathcal{H}^2$. Furthermore, for every integer $n\geqslant 2$, we have:

$$a_n n^{-\Phi(s)}=\frac{a_n}{n^{c_0 s}}n^{-\varphi(s)}$$

and according to Lemma 1, we know that $n^{-\varphi}\in\mathcal{D}$. Since $c_0\geqslant 1$, it is clear from the above identity that the constant coefficient of $C_\Phi f$ is a_1. Hence:

$$\begin{aligned}\|C_\Phi f\|_2^2=\left\|\sum_{n=1}^{N}a_n n^{-\Phi(s)}\right\|_2^2&=\left\|a_1+\sum_{n=2}^{N}a_n n^{-\Phi(s)}\right\|_2^2\\&=|a_1|^2+\left\|\sum_{n=2}^{N}a_n n^{-\Phi(s)}\right\|_2^2\end{aligned}$$

The following identity allows us to get the $|a_2|^2$ term:

$$\sum_{n=2}^{N}a_n n^{-\Phi(s)}=2^{-\varphi(s)}\left(a_2 2^{-c_0 s}+a_3 3^{-c_0 s}\left(\frac{3}{2}\right)^{-\varphi(s)}+\cdots+a_N N^{-c_0 s}\left(\frac{N}{2}\right)^{-\varphi(s)}\right)$$

We know from Lemma 1 that $2^{-\varphi}\in\mathcal{H}^\infty$ and that $\|2^{-\varphi}\|_\infty\leqslant 1$ (since $\varphi(\mathbb{C}_+)\subset\mathbb{C}_+$). Theorem 2 implies that:

$$\begin{aligned}\left\|\sum_{n=2}^{N}a_n n^{-\Phi(s)}\right\|_2^2&\leqslant\left\|2^{-\varphi}\right\|_\infty^2\left\|a_2 2^{-c_0 s}+a_3 3^{-c_0 s}\left(\frac{3}{2}\right)^{-\varphi(s)}+\cdots+a_N N^{-c_0 s}\left(\frac{N}{2}\right)^{-\varphi(s)}\right\|_2^2\\&\leqslant\left\|a_2 2^{-c_0 s}+a_3 3^{-c_0 s}\left(\frac{3}{2}\right)^{-\varphi(s)}+\cdots+a_N N^{-c_0 s}\left(\frac{N}{2}\right)^{-\varphi(s)}\right\|_2^2\end{aligned}$$

For any integer $k\in\{3,\dots,N\}$, $k^{-c_0 s}\left(\frac{k}{2}\right)^{-\varphi(s)}$ is a Dirichlet series of the form $\sum_{j=1}^{+\infty}b_j(k^{c_0}j)^{-s}$. In particular, we see that 2^{c_0} can not be express of the form $k^{c_0}j$ for some positive integer j (recall that c_0 is a positive integer). By orthogonality, it follows that:

$$\|C_\Phi f\|_2^2\leqslant|a_1|^2+|a_2|^2+\left\|a_3 3^{-c_0 s}\left(\frac{3}{2}\right)^{-\varphi(s)}+\cdots+a_N N^{-c_0 s}\left(\frac{N}{2}\right)^{-\varphi(s)}\right\|_2^2$$

Factorizing by $\left(\frac{3}{2}\right)^{-\varphi(s)}$ which is a Dirichlet series whose infinite norm is less than 1 (according to Lemma 1 and using Theorem 2) we get the $|a_3|^2$ term:

$$\|C_\Phi f\|_2^2 \leqslant |a_1|^2 + |a_2|^2 + |a_3|^2 + \left\| a_4 4^{-c_0 s}\left(\frac{4}{3}\right)^{-\varphi(s)} + \cdots + a_N N^{-c_0 s}\left(\frac{N}{3}\right)^{-\varphi(s)} \right\|_2^2$$

An obvious induction gives us:

$$\|C_\Phi f\|^2 \leqslant \sum_{n=1}^{N} |a_n|^2$$

This proves that $\|C_\Phi f\|_2 \leqslant \|f\|_2$ when f belongs to the space $\mathcal{P}$ of Dirichlet polynomials. We conclude the proof with a classical density argument which relies on the continuity of the point evaluation map. Indeed, since $\mathcal{P}$ is dense in $\mathcal{H}^2$, the operator $C_\Phi : \mathcal{P} \longrightarrow \mathcal{H}^2$ extends uniquely to an operator $T \in \mathcal{B}(\mathcal{H}^2)$. Let $f \in \mathcal{H}^2$, $(f_n)_{n\geqslant 0}$ a sequence of elements of $\mathcal{P}$ which converges to f in $\mathcal{H}^2$ and $s \in \mathbb{C}_{1/2}$. Since the evaluation map:

$$\delta_s : \left\{ \begin{array}{ccc} \mathcal{H}^2 & \longrightarrow & \mathbb{C} \\ g & \longmapsto & g(s) \end{array} \right.$$

is bounded and $\Phi(s) \in \mathbb{C}_{1/2}$, we get:

$$\begin{aligned} Tf(s) = \lim_{n\to+\infty} Tf_n(s) = \lim_{n\to+\infty} f_n(\Phi(s)) &= \lim_{n\to+\infty} \delta_{\Phi(s)}(f_n) \\ &= \delta_{\Phi(s)}(f) \\ &= f(\Phi(s)) \\ &= C_\Phi(s) \end{aligned}$$

Then $C_\Phi = T$ is bounded on the space $\mathcal{H}^2$ and $\|C_\Phi\| \leqslant 1$. Since C_Φ fixes the constants, we eventually have $\|C_\Phi\| = 1$. □

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Contents