North-Western European Journal of Mathematics

Sum of dilates in vector spaces

Antal Balog[1] George Shakan[2]

Received: September 30, 2014/Accepted: April 23, 2015/Online: June 12, 2015

Abstract

Let $d \geq 2$, $A \subset \mathbb{Z}^d$ be finite and not contained in a translate of any hyperplane, and $q \in \mathbb{Z}$ such that $|q| \geq 2$. We show

$$|A + q \cdot A| \geq (|q| + d + 1)|A| - O(1).$$

Keywords: higher dimensional sumsets, dilations.

MSC: 11B13, 11B30, 11P70.

1 Introduction

Let A and B be finite sets of real numbers. The sumset and the productset of A and B are defined by

$$A + B = \{a + b \,/\, a \in A,\ b \in B\},$$
$$A \cdot B = \{ab \,/\, a \in A,\ b \in B\}.$$

For a real number $d \neq 0$ the dilation of A by d is defined by

$$d \cdot A = \{d\} \cdot A = \{da \,/\, a \in A\},$$

while for any real number x, the translation of A by x is defined by

$$x + A = \{x\} + A = \{x + a \,/\, a \in A\}.$$

The following (actually more) was shown in Balog and Shakan (n.d.[a]).

Theorem 1 (Balog and Shakan[3]) – *Let $q \in \mathbb{Z}$. Then there is a constant C_q such that every finite $A \subset \mathbb{Z}$ satisfies*

$$|A + q \cdot A| \geq (|q| + 1)|A| - C_q. \tag{1}$$

[1] Alfréd Rényi Institute of Mathematics, Budapest, P.O.Box 127, 1364–Hungary
[2] University of Wyoming Department of Mathematics, Laramie, Wyoming 82072, USA

This was obtained after the works of Bukh; Cilleruelo, Hamidoune, and Serra; Du S., Cao H., and Sun Z.; Hamidoune and Rué; Ljujic[4]. The reader is invited to see the introductions of Balog and Shakan (n.d.[a]) and Bukh (2008) for a more detailed introduction to this problem.

For a finite $A \subset \mathbb{Z}^d$, we say the *rank* of A is the smallest dimension of an affine space that contains A. When A is a set of high rank, one might expect to be able to improve the lower bound in equation (1) on page 57, which is the goal of our current note. Ruzsa proved the following in Ruzsa (1994).

Theorem 2 (Ruzsa[5]) – *Let $A, B \subset \mathbb{Z}^d$ be finite such that $A + B$ has rank d and $|A| \geq |B|$. Then*

$$|A+B| \geq |A| + d|B| - \frac{d(d+1)}{2}$$

Let $A \subset \mathbb{Z}^d$ be finite of rank d and q be an integer. The main objective here is to improve upon equation (1) on page 57 and theorem 2 in the case $B = q \cdot A$. In this note $O(1)$ will always depend on the relevant d and q. Our main theorem is the following.

Theorem 3 – *Let $A \subset \mathbb{Z}^d$ of rank $d \geq 2$ and $|q| \geq 2$ be an integer. Then*

$$|A + q \cdot A| \geq (|q| + d + 1)|A| - O(1).$$

The authors would like to thank Imre Ruzsa for drawing our attention to the problem considered here. We remark that we do not believe even the multiplicative constant of $(|q|+d+1)$ is the best possible, and we now present our best construction. For $1 \leq i \leq d$, let e_i be the standard basis vectors of $\mathbb{Z}^d$. For $N \in \mathbb{Z}$, consider

$$A_N = \{e_1, \ldots, e_d\} \cup \{ne_1 \,/\, 0 < n < N\ ,\ n \in \mathbb{Z}\}.$$

An easy induction on d reveals

$$|A_N + q \cdot A_N| \leq (q + 2d - 1)|A_N| - (d-1)(|q| - 2(d-1) + 1) \tag{2}$$

This shows that theorem 3 is the best possible up to the additive constant for $d = 2$. We are also able to handle the case $d = 3$.

Theorem 4 – *Let $A \subset \mathbb{Z}^3$ be finite of rank 3 and $|q| \geq 2$. Then*

$$|A + q \cdot A| \geq (|q| + 5)|A| - O(1).$$

[3] Balog and Shakan, n.d.(a), *On the sum of dilations of a set*.

[4] Bukh, 2008, “Sums of Dilates”;
Cilleruelo, Hamidoune, and Serra, 2009, “On sums of dilates”;
Du S., Cao H., and Sun Z., 2015, “On a sumset problem for integers”;
Hamidoune and Rué, 2011, “A lower bound for the size of a Minkowski sum of dilates”;
Ljujic, 2013, “A lower bound for the size of a sum of dilates”.

[5] Ruzsa, 1994, “Sum of sets in several dimensions”.

Furthermore, we can prove the following bound for all q, and this is best possible, up to the additive constant, when $|q| = 2$. One can check the example for equation (2) on page 58 to see that

$$|A_N \pm 2 \cdot A_N| = (2d+1)|A_N| - d(d+1).$$

Theorem 5 – *Let $A \subset \mathbb{Z}^d$ be finite of rank d and $|q| > 1$. Then*

$$|A + q \cdot A| \geq (2d+1)|A| - d(d+1)^2/2.$$

Our basic intuition is that to minimize $|A + q \cdot A|$ one should choose A to be as close to a one dimensional set as possible. One should proceed with caution with this intuition because when $q = -1$, a clever construction in Stanchescu (2001) shows that this is not the best strategy. Nevertheless, given the evidence of theorem 4 on page 58 and theorem 5 we present the following conjecture.

Conjecture 1 – *Suppose $A \subset Z^d$ is finite of rank d and q is an integer with $|q| \geq 2$. Then*

$$|A + q \cdot A| \geq (|q| + 2d - 1)|A| - O(1).$$

We remark that the cases $A + A$ and $A - A$ have different behavior. Theorem 2 on page 58, which in the case $B = A$ was proved by Freiman[6], says that $|A + A| \geq (d+1)|A| - d(d+1)/2$. This is the best possible due to equation (2) on page 58, which shows theorem 3 on page 58 is false with $q = 1$. The reason that one can improve when $q \neq 1$ is simply that in $A + A$, the roles of the summands are interchangeable, while in the case $A + q \cdot A$, the roles of A and $q \cdot A$ are not interchangeable. We have already mentioned that there is a tricky construction in Stanchescu (2001), which shows $|A - A|$ can be as small as $(2d - 2 + \frac{1}{d-1})|A| - (2d^2 - 4d + 3)$. In the same paper, the author conjectures that this is the best possible. It is curious that best known lower bound is $|A - A| \geq (d+1)|A| - d(d+1)/2$. The case $q = -1$ is also different in the sense that it is important that when $|q| > 1$, we can split A into cosets modulo $q \cdot \mathbb{Z}^d$. This will be seen in our argument below.

Let $L : \mathbb{Z}^d \to \mathbb{Z}^d$ be a linear transformation. In this note we are primarily concerned with $|A + LA|$ where L is a scalar multiple of the identity. The study of other choices of L would be natural, but we do not do it here.

2 Proof of theorems 3 and 5

Fix $A \subset \mathbb{Z}^d$ of rank $d \geq 2$ and an integer q with $|q| \geq 2$ Since the rank of A is d, we must have that A contains at least $(d+1)$ elements. We first partition A into its intersections with cosets of the lattice $q \cdot \mathbb{Z}^d$. Note there are $|q|^d$ such cosets. Let

$$A = \bigcup_{i=1}^{r} A_i, \;\; A_i = a_i + q \cdot A_i', \;\; a_i \in \{0, \ldots, |q| - 1\}^d, \;\; A_i' \neq \varnothing,$$

[6] Freiman, 1973, *Foundations of structural theory of set addition*.

where the unions are disjoint. We obtain the preliminary estimate

Lemma 1 – *Let $A \subset \mathbb{Z}^d$ and $q \in \mathbb{Z}$ such that $|q| \geq 2$. Suppose that A intersects r cosets of the lattice $q \cdot \mathbb{Z}^d$. Then*

$$|A + q \cdot A| \geq (d + r)|A| - rd(d + 1)/2.$$

Proof. Using theorem 2 on page 58, we obtain

$$\begin{aligned} |A + q \cdot A| &= \sum_{i=1}^{r} |A_i + q \cdot A| \\ &\geq \sum_{i=1}^{r} \left(d|A_i| + |A| - \frac{d(d+1)}{2} \right) \\ &= (d + r)|A| - rd(d + 1)/2. \end{aligned}$$

□

We say that A is fully distributed (FD) modulo $q \cdot \mathbb{Z}^d$ if A intersects every coset of $q \cdot \mathbb{Z}^d$. Note that for a FD modulo $q \cdot \mathbb{Z}^d$ set, theorem 3 on page 58 and conjecture 1 on page 59 are far from optimal.

We now describe the process of reducing A. Applying an invertible linear transformation to A does not change $|A + q \cdot A|$. Suppose there is some $a \in A$ such that the lattice $\langle A - a \rangle_{\mathbb{Z}} = \Gamma$ is a non-trivial sublattice of $\mathbb{Z}^d$. Let $L : \mathbb{Z}^d \to \mathbb{Z}^d$ be a linear transformation that maps the standard basis vectors to the basis vectors of Γ, that is $\Gamma = L\mathbb{Z}^d$. Since A has rank d, L is invertible. Then we may replace A with $L^{-1}(A - a)$. Note that $L^{-1}(A - a) \subset \mathbb{Z}^d$ since $A \subset a + L\mathbb{Z}^d$. Since $1 < \det(L) \in \mathbb{Z}$, each reduction reduces the volume of the convex hull of A by at least $\frac{1}{2}$. The volume of the convex hull of A is always bounded from below by the volume of the d-dimensional simplex so eventually this process must stop. Thus we may assume $\langle A - a \rangle_{\mathbb{Z}} = \mathbb{Z}^d$ for all $a \in A$. Then it follows that we have for all $1 \leq i \leq r$,

$$\mathbb{Z}^d = \langle A - a \rangle_{\mathbb{Z}} \subset \langle a_1 - a_i, \ldots, a_r - a_i, qe_1, \ldots, qe_d \rangle_{\mathbb{Z}} \subset \mathbb{Z}^d. \tag{3}$$

Here we used that if $x \in A - a$ and $a \in A_i$, then for some $1 \leq j \leq r$ we have $x \in a_j - a + q \cdot A'_j \subset \langle a_j - a_i, qe_1, \ldots, qe_d \rangle_{\mathbb{Z}}$. We say A is *reduced* if equation (3) holds.

Proof (of theorem 5 on page 59). By the discussion above, we may assume A is reduced. We first aim to show that a reduced set must intersect at least $d + 1$ cosets of $q \cdot \mathbb{Z}^d$, and then we will appeal to the argument of lemma 1.

Observe that the linear combinations of $a_1 - a_1, \ldots, a_r - a_1$ can only take at most $|q|^{r-1}$ different vectors mod $q \cdot \mathbb{Z}^d$. Since A is reduced, by equation (3), these vectors must intersect every coset modulo $q \cdot \mathbb{Z}^d$. Thus we have that $|q|^{r-1} \geq |q|^d$, and so $r - 1 \geq d$.

Then by theorem 2 on page 58, we find

$$\begin{aligned}|A+q\cdot A| &\geq \left(\sum_{i=1}^{d}|A_i+q\cdot A|\right)+\left|A\setminus\left(\bigcup_{i=1}^{d}A_i\right)+q\cdot A\right|\\ &\geq \left(\sum_{i=1}^{d}(d|A_i|+|A|-d(d+1)/2)\right)+d\left|A\setminus\left(\bigcup_{i=1}^{d}A_i\right)\right|+|A|-d(d+1)/2\\ &=(2d+1)|A|-d(d+1)^2/2.\end{aligned}$$ □

We now focus our attention to the proof of theorem 3 on page 58. We start with a special case. Recall that we assume $d \geq 2$.

Lemma 2 – *Suppose A is contained in d parallel lines. Then $|A+q\cdot A| \geq (|q|+2d-1)|A|-O(1)$.*

Proof. Suppose A is contained in $x_1+\ell,\dots,x_d+\ell$ for some 1 dimensional subspace ℓ. After a translation of A by $-a$ for an element $a \in A$ we can suppose $x_1 = 0$ and without loss of generality, we may suppose $x_2,\dots,x_d$ are elements of $\ell^{\perp} \cong \mathbb{R}^{d-1}$. Moreover, we have that $x_2,\dots,x_d$ are linearly independent over $\mathbb{R}$ since A has rank d. This implies that for all $1 \leq i,j \leq d$, the lines $(x_i+\ell)+q\cdot(x_j+\ell)$ are pairwise disjoint. For $1 \leq i \leq d$, let $B_i := A\cap(x_i+\ell)$. It follows, using equation (1) on page 57 that

$$\begin{aligned}|A+q\cdot A| &= \sum_{i=1}^{d}\sum_{j=1}^{d}|B_i+q\cdot B_j|\\ &=\sum_{i=1}^{d}\left(|B_i+q\cdot B_i|+\sum_{j\neq i}|B_i+q\cdot B_j|\right)\\ &\geq \sum_{i=1}^{d}\left(\Big((|q|+1)|B_i|-O(1)\Big)+\sum_{j\neq i}(|B_i|+|B_j|-1)\right)\\ &=(|q|+2d-1)|A|-O(1).\end{aligned}$$ □

We remark that the lack of a satisfactory higher dimensional analog of lemma 2 is essentially what blocks us from improving the multiplicative constant in m theorem 3 on page 58.

We prove theorem 3 on page 58 by induction on d starting from $d = 2$ (the statement is not true for $d = 1$). Note that the proof of the next lemma does not use the induction hypothesis for $d = 2$ but only for $d \geq 3$.

Lemma 3 – *Let $B \subset A$ and suppose that the rank of B is $1 \leq f < d$. Then*

$$|B+q\cdot A| \geq (|q|+d+1)|B|-O(1),$$

or A is contained in d parallel lines.

Proof. Note that the rank of $B+q\cdot B$ is also f. Since $B+q\cdot A$ is of rank d, we may find an $x\in A$ such that $B+qx$ is not in the affine span of $B+q\cdot B$. Thus $B+q\cdot B$ and $B+qx$ are disjoint. The rank of $B\cup\{x\}+q\cdot(B\cup\{x\})$ is $f+1$. We may repeat this process with $B\cup\{x\}+q\cdot(B\cup\{x\})$ in the place of $B+q\cdot B$, and so on, a total of $(d-f)$ times. Thus we find $x_1,\dots,x_{(d-f)}\in A$ such that $B+q\cdot B, B+qx_1,\dots,B+qx_{(d-f)}$ are pairwise disjoint. When $f\geq 2$ (so $d\geq 3$) we use the induction hypothesis, that is theorem 3 on page 58 for the sum $B+q\cdot B$ where B is of rank $2\leq f<d$ to get

$$|B+q\cdot A|\geq|B+q\cdot B|+\sum_{j=1}^{d-f}|B+qx_j|\geq(|q|+d+1)|B|-O(1).$$

Now we handle the case $f=1$ (this is the only possibility when $d=2$), in this case we do not use the induction hypothesis. B is contained in a line. We may suppose A is not contained in d parallel lines. We proceed as above to find $x_1,\dots,x_{d-1}$ such that $B+q\cdot B, B+qx_1,\dots,B+qx_{(d-1)}$ are pairwise disjoint. Since A is not contained in d parallel lines, we may find an $x_d\in A$ such that $B+qx_d$ is disjoint from all $B+q\cdot B, B+qx_1,\dots,B+qx_{(d-1)}$. It follows from theorem 1 on page 57 applied to the sum $B+q\cdot B$ that

$$|B+q\cdot A|\geq|B+q\cdot B|+\sum_{j=1}^{d}|B+qx_j|\geq(|q|+d+1)|B|-O(1). \qquad \square$$

The next lemma is a higher dimensional analog of Lemma 3.1 in Balog and Shakan (n.d.[a]).

Lemma 4 – *Let $1\leq i\leq r$. Then either A_i' is* FD *modulo $q\cdot\mathbb{Z}^d$ or*

$$|A_i+q\cdot A|\geq|A_i+q\cdot A_i|+\min_{1\leq w\leq r}|A_w|.$$

Proof. Suppose

$$|A_i+q\cdot A|<|A_i+q\cdot A_i|+\min_{1\leq w\leq r}|A_w|.$$

Fix $1\leq w\leq r$. Since $A_w\subset A$, we find that

$$|(A_i+q\cdot A_w)\setminus(A_i+q\cdot A_i)|<|A_w|.$$

Translation by $-a_i$ and dilation by $\frac{1}{q}$ reveals that

$$|(a_w-a_i+A_i'+q\cdot A_w')\setminus(A_i'+q\cdot A_i')|<|A_w'|.$$

Thus for any $x\in A_i'$ there is a $y\in A_w'$ such that $a_w-a_i+x+qy\in A_i'+q\cdot A_i'$. It follows that there is a $x'\in A_i'$ such that $a_w-a_i+x\equiv x' \mod q\cdot\mathbb{Z}^d$. We may repeat this argument with x' in the place of x, and so on, and for each $1 \le w \le r$ to obtain that for any $u_1, \dots, u_r \in \mathbb{Z}$ there is a $x'' \in A_i'$ such that

$$u_1(a_1 - a_i) + \cdots + u_r(a_r - a_i) + x \equiv x'' \mod q \cdot \mathbb{Z}^d.$$

Since A is reduced, this describes all of the cosets modulo $q \cdot \mathbb{Z}^d$ and it follows that A_i' is FD mod $q \cdot \mathbb{Z}^d$. □

We are now ready to prove theorem 3 on page 58. We start with $|A + q \cdot A| \ge |A|$ and improve upon the multiplicative constant iteratively.

Proposition 1 – *Suppose $A \subset \mathbb{Z}^d$ such that A has rank d. Let $q \in \mathbb{Z}$ such that $|q| \ge 2$. Then for every $|q| + d + 1 \le m \le (|q| + d + 1)^2$, one has*

$$A + q \cdot A| \ge \frac{m}{|q| + d + 1}|A| - O(1),$$

where $O(1)$ also depends on m.

Proof. Observe that $m = (|q| + d + 1)^2$ is precisely theorem 3 on page 58. For convenience, set $S = |q| + d + 1$. We prove by induction on m, where $|A + q \cdot A| \ge |A|$ trivially starts the induction. Suppose now that proposition 1 is true for a fixed $S \le m < S^2$, and we prove it for $m + 1$.

If A is contained in d parallel lines, then lemma 2 on page 61 immediately implies theorem 3 on page 58, and so proposition 1 is especially true for $m + 1$ as well. Thus we may assume A is not contained in d parallel lines.

Consider a set $B \subset A$. If it is $1 \le f < d$ dimensional, then lemma 3 on page 61 shows that $|B + q \cdot A| \ge S|B| - O(1)$. If B is d dimensional, then by the induction hypothesis on m, we have

$|B + q \cdot A| \ge |B + q \cdot B| \ge \frac{m}{S}|B| - O(1)$. In either case, using that $m < S^2$, we have

$$|B + q \cdot A| \ge \frac{m}{S}|B| - O(1). \tag{4}$$

First, assume there is an $1 \le i \le r$ such that $|A_i| \le \frac{1}{S}|A|$. We have by equation (4) and theorem 2 on page 58, that

$$\begin{aligned}|A + q \cdot A| &\ge |A_i + q \cdot A| + |(A \setminus A_i) + q \cdot A| \ge \\ &\ge |A_i| + |A| - 1 + \frac{m}{S}(|A| - |A_i|) - O(1) \ge \frac{m+1}{S}|A| - O(1).\end{aligned}$$

Thus we may assume that every A_i has more than $\frac{1}{S}|A|$ elements.

Suppose now that every A_i is strictly less than d dimensional. Then lemma 3 on page 61 shows that

$$\begin{aligned}|A + q \cdot A| &= \sum_{i=1}^{r} |A_i + q \cdot A| \ge \sum_{i=1}^{r} ((|q| + d + 1)|A_i| - O(1)) \\ &= (|q| + d + 1)|A| - O(1) \ge \frac{m+1}{S}|A| - O(1).\end{aligned}$$

Thus we may assume that there is an A_i that is d dimensional.

If the corresponding A_i' is not FD modulo $q\cdot\mathbb{Z}^d$, then by lemma 4 on page 62, equation (4) on page 63, and by the induction hypothesis for A_i we have

$$\begin{aligned}|A+q\cdot A| &\geq |A_i+q\cdot A|+|(A\setminus A_i)+q\cdot A|\\ &\geq |A_i+q\cdot A_i|+\min_{1\leq w\leq r}|A_w|+\frac{m}{S}(|A|-|A_i|)-O(1)\\ &\geq \frac{m}{S}|A_i|-O(1)+\frac{1}{S}|A|+\frac{m}{S}(|A|-|A_i|)-O(1)=\frac{m+1}{S}|A|-O(1).\end{aligned}$$

Similarly if A_i' is FD mod $q\cdot\mathbb{Z}^d$ (and A_i is d dimensional) then by lemma 1 on page 60 and equation (4) on page 63 we have

$$\begin{aligned}|A+q\cdot A| &= |A_i+q\cdot A|+|A\setminus A_i+q\cdot A|\geq |A_i'+q\cdot A_i'|+|A\setminus A_i+q\cdot A|\\ &\geq (|q|^d+d)|A_i'|-O(1)+\frac{m}{S}(|A|-|A_i|)-O(1)\geq \frac{m+1}{S}|A|-O(1).\end{aligned}$$

Note that the only place where we have used the hypothesis of the induction on d is the $f\geq 2$ case of the proof of lemma 3 on page 61, what we do not use when $d=2$ thus this argument also proves theorem 3 on page 58 in that case. □

3 Proof of theorem 4

Let $A\subset\mathbb{Z}^3$ of rank 3 and q be a positive integer such that $|q|\geq 2$.

The proof of theorem 4 on page 58 is almost identical to that of theorem 3 on page 58. The only difference is that we have to strengthen lemma 2 on page 61. The reader is invited to check that it is enough to prove lemma 2 on page 61 in the case where $d=3$ and A is contained in two parallel planes or 4 parallel lines and then the proof of theorem 3 on page 58 goes through in an identical manner. Indeed, if one was able to prove conjecture 1 on page 59 in the special cases for each $1\leq f\leq d-1$, and A is contained in $2(d-f)$ translates of a f-dimensional subspace, then this along with the proof of theorem 3 on page 58 would imply conjecture 1 on page 59 in general.

Lemma 5 – *Suppose A is contained in two parallel hyperplanes. Then*

$$|A+q\cdot A|\geq (|q|+5)|A|-O(1).$$

Proof. Suppose $A\subset H\cup(H+x)$ for some hyperplane H and some $x\in\mathbb{Z}^3$. Since $|q|>1$, we have that

$$(H+q\cdot H),(H+x+q\cdot H),(H+q\cdot(H+x)),((H+x)+q\cdot(H+x)),$$

are disjoint Let $B_1 = H \cap A$ and $B_2 = (H + x) \cap A$. Then we have that

$$|A + q \cdot A| \geq |B_1 + q \cdot B_1| + |B_1 + q \cdot B_2| + |B_2 + q \cdot B_1| + |B_2 + q \cdot B_2|. \tag{5}$$

Suppose, without loss of generality, that $|B_1| \geq |B_2|$. We separately consider several cases.

1. Suppose B_1 has rank 2. Then by theorem 3 on page 58, we have $|B_1 + q \cdot B_1| \geq (|q| + 3)|B_1| - O(1)$. Furthermore by theorem 2 on page 58, we have $|B_1 + q \cdot B_2| + |B_2 + q \cdot B_1| \geq 2(|B_1| + 2|B_2| - 3)$. Lastly, by equation (1) on page 57, we have $|B_2 + q \cdot B_2| \geq (|q| + 1)|B_2| - O(1)$. Combining this three inequalities with equation (5) yields $|A + q \cdot A| \geq (|q| + 5)|A| - O(1)$. Note that this case applies when B_2 consists of a single point.

2. Suppose B_1 has rank 1 and B_2 has rank 2. By equation (1) on page 57, $|B_1 + q \cdot B_1| \geq (|q| + 1)|B_1| - O(1)$ and by theorem 3 on page 58, $|B_2 + q \cdot B_2| \geq (|q| + 3)|B_2| - O(1)$. We have that B_1 lies in a translate of some line, say ℓ. Suppose B_2 lies in some distinct lines $x_1 + \ell, \ldots, x_m + \ell$ such that each $x_j + \ell$ intersects B_2 in at least one point. Note that $m \geq 2$ since A has rank 3. For each $1 \leq j \leq m$, let $B_2^j = B_2 \cap (x_j + \ell)$. Then by the one dimensional theorem 2 on page 58, we have

$$|B_1 + q \cdot B_2| \geq \sum_{j=1}^{m} |B_1 + q \cdot B_2^j| \geq m|B_1| + \sum_{j=1}^{m} (|B_2^j| - 1) \geq 2|B_1| + |B_2| - 2.$$

   Similarly, $|B_2 + q \cdot B_1| \geq 2|B_1| + |B_2| - 2$. Combining these four inequalities with equation (5), we obtain $|A + q \cdot A| \geq (|q| + 5)|A| - O(1)$.

3. Suppose B_1 and B_2 are both rank 1. Then the sets $x + q \cdot B_1$ and $B_1 + q \cdot x$ where $x \in B_2$ are all disjoint. Using equation (1) on page 57, we obtain (the extremal case being $|B_2| = 2$)

$$|A + q \cdot A| \geq (|q| + 1)|A| - O(1) + 2|B_1||B_2| \geq (|q| + 5)|A| - O(1). \qquad \square$$

We now have to consider the case where A is contained in four parallel lines.

Lemma 6 – *Suppose A is contained in four parallel lines. Then*

$$|A + q \cdot A| \geq (|q| + 5)|A| - O(1).$$

Proof. Suppose A is contained in four parallel lines, all parallel to some line through the origin ℓ. Then $\mathbb{Z}^3/\ell \cong \mathbb{Z}^2$ and say $A' = \{x_1, x_2, x_3, x_4\} \subset \mathbb{Z}^3/\ell$ are the 4 cosets that intersect A. Note that A' must be a 2 dimensional set since A is 3 dimensional. We want to show $|A' + q \cdot A'| \geq 14$. By the argument of lemma 1 on page 60, we may assume that A' intersects at least 3 residue classes modulo $q \cdot (\mathbb{Z}^3/\ell)$. If A'

intersects four residue classes, then $|A' + q \cdot A'| = 16$. Otherwise let A'_1 be the intersection of A' with the residue class that contains 2 elements of A'. Since A' is 2 dimensional, it is not an arithmetic progression, so $|A'_1 + q \cdot A'| \geq |A'_1| + |A'| = 6$. Then $|A' + q \cdot A'| = 8 + |A'_1 + q \cdot A'| \geq 14$.

Let $A = B_1 \cup \cdots \cup B_4$ where $B_i = (\ell + x_i) \cap A$. Then $B_i + q \cdot B_j$ are all disjoint, if we drop at most two pairs $\{i, j\}$. We do not need to drop a pair in the form $\{i, i\}$ because an equation in the form $x_i + qx_i = x_j + qx_j$ is not possible in A'. That means, any set B_i can appear in a dropped pair at most twice. Then

$$\begin{aligned}|A + q \cdot A| &\geq \sum_{i=1}^{4} |B_i + q \cdot B_i| + \sum_{\substack{i \neq j \\ \text{not dropped}}} |B_i + q \cdot B_j| \\ &\geq \sum_{i=1}^{4} ((|q| + 1)|B_i| - O(1)) \\ &\quad + \sum_{i \neq j} (|B_i| + |B_j| - 1) - 2|A| = (|q| + 5)|A| - O(1). \qquad \square\end{aligned}$$

Finally we can express the analog of lemma 3 on page 61. Note that the proof uses theorem 3 on page 58 and theorem 1 on page 57 rather than any induction, otherwise identical to the proof of lemma 3 on page 61.

Lemma 7 – *Let $A \subset \mathbb{Z}^3$ of rank 3, $B \subset A$ and suppose that the rank of B is $1 \leq f < 3$. Then*

$$|B + q \cdot A| \geq (|q| + 5)|B| - O(1),$$

or A is contained in two parallel hyperplanes or in four parallel lines.

Acknowledgments

The first author's research was supported by the Hungarian National Science Foundation Grants K81658 and K104183. Research conducted while the second author enjoyed the hospitality of the Alfréd Rényi Institute of Mathematics, and benefited from the OTKA grant K109789.

References

Balog, A. and G. Shakan. *On the sum of dilations of a set*. to appear: Acta Arithmetica (cit. on pp. 57, 58, 62).

Bukh, B. (2008). "Sums of Dilates". *Combinatorics, Probability and Computing* **17**, pp. 627–639 (cit. on p. 58).

Cilleruelo, J., Y. Hamidoune, and O. Serra (2009). "On sums of dilates". *Combinatorics, Probability and Computing* **18**, pp. 871–880 (cit. on p. 58).

Du S., S., Q. Cao H., and W. Sun Z. (2015). "On a sumset problem for integers". *Electronic Journal of Combinatorics* **21**, pp. 1–25 (cit. on p. 58).

Freiman, G. (1973). *Foundations of structural theory of set addition*. American Math Society (cit. on p. 59).

Hamidoune, Y. and J. Rué (2011). "A lower bound for the size of a Minkowski sum of dilates". *Combinatorics, Probability and Computing* **20**, pp. 249–256 (cit. on p. 58).

Ljujic, Z. (2013). "A lower bound for the size of a sum of dilates". *Journal of Combinatorial Number Theory* **5**, pp. 31–51 (cit. on p. 58).

Ruzsa, I. (1994). "Sum of sets in several dimensions". *Combinatorica* **14**, pp. 48–490 (cit. on p. 58).

Stanchescu, Y. (2001). "An upper bound for d-dimensional difference sets". *Combinatorica* **21**, pp. 591–595 (cit. on p. 59).

Contents