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Extensions of Lipschitz functions and Grothendieck's bounded approximation property

Gilles Godefroy[1]

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Abstract

A metric compact space M is seen as the closure of the union of a sequence (M_n) of finite ϵ_n-dense subsets. Extending to M (up to a vanishing uniform distance) Banach-space valued Lipschitz functions defined on M_n, or defining linear continuous near-extension operators for real-valued Lipschitz functions on M_n, uniformly on n is shown to be equivalent to the bounded approximation property for the Lipschitz-free space $\mathcal{F}(M)$ over M. Several consequences are spelled out.

Keywords: extension of Lipschitz functions, free space over a metric space, bounded approximation property.

MSC: 46B20, 46B28.

1 Introduction

Let A be a metric space, and B be a non-empty subset of A. It is well-known that real-valued Lipschitz functions on B can be extended to Lipschitz functions on A with the same Lipschitz constant with an inf-convolution formula. Namely, if $f : B \to \mathbb{R}$ is L-Lipschitz, then the formula

$$\overline{f}(a) = \inf\{f(b) + Ld(a,b) \,/\, b \in B\}$$

which goes back to McShane[2] defines a L-Lipschitz function $\overline{f}$ on A which extends f. This formula relies of course on the order structure of the real line. Therefore it cannot be used for extending Banach-space valued Lipschitz functions - except in some specific cases. Another drawback of this formula is that it is not linear in f,

[1]Institut de Mathématiques de Jussieu, case 247, 4 place Jussieu, 75005 Paris, France

[2]McShane, 1934, "Extension of range of functions".

that is, the map $f \to \overline{f}$ that it defines is not a linear one. It is known that Banach-space valued Lipschitz functions do not generally admit Lipschitz extensions[3], and that generally no continuous linear extension operator exists for Lipschitz real-valued functions[4]. Canonical examples will be shown below (see remarks 1 and 2).

The purpose of this note is to relate these two conditions (extending Banach-space valued functions, finding linear extension operators for real-valued Lipschitz functions) and a combination of both with the validity of Grothendieck's bounded approximation property for Banach spaces which naturally show up in this context, namely the Lipschitz-free spaces. We will be dealing with finite subsets of a compact metric space M which approximate this space M, and these extension properties are easy for any given finite subset, but what matters is to find a uniform bound on the norm of the extension operators for this approximating sequence. Also, it turns out that the extension condition should be relaxed: what matters in this case is not an exact extension of a function f defined on a subset S of M, but a function on M whose restriction to S is uniformly close to f. Our proofs are simple, and rely on canonical constructions.

2 Results

We recall that a separable Banach space X has the bounded approximation property (BAP) if there exists a sequence of finite rank operators T_n such that $\lim \|T_n(x) - x\| = 0$ for every $x \in X$. It follows then from the uniform boundedness principle that $\sup \|T_n\| = \lambda < \infty$ and then we say that X has the λ-BAP. The BAP and the existence of Schauder bases for Lipschitz-free spaces has already been investigated in a number of articles[5]. It is shown in Godefroy and Ozawa (2014) that there exist compact metric spaces K such that $\mathcal{F}(K)$ fails the approximation property (AP): actually, if X

[3] Lindenstrauss, 1964, "On nonlinear projections in Banach spaces."; See corollary 1.29 in Benyamini and Lindenstrauss, 2000, *Geometric Nonlinear Functional Analysis*.

[4] Theorem 2.16 in A. Brudnyi and Y. Brudnyi, 2007, "Metric spaces with linear extensions preserving Lipschitz condition".

[5] Borel-Mathurin, 2012, "Approximation properties and nonlinear geometry of Banach spaces";
Cúth and Doucha, 2015, "Lipschitz-free spaces over ultrametric spaces";
Dalet, 2015, "Free spaces over countable compact metric spaces";
Dalet, 2014, "Free spaces over some proper metric spaces";
Godefroy and Kalton, 2003, "Lipschitz-free Banach spaces";
Hájek and Pernecká, 2014, "On Schauder bases in Lipschitz-free spaces";
Kalton, 2004, "Spaces of Lipschitz and Hölder functions and their applications";
Kalton, 2012, "The uniform structure of Banach spaces";
Kaufmann, 2014, "Products of Lipschitz-free spaces and applications";
Lancien and Pernecká, 2013, "Approximation properties and Schauder decompositions in Lipschitz-free spaces";
Pernecká and Smith, 2015, "The metric approximation property and Lipschitz-free spaces over subsets of $\mathbb{R}^n$".

is a separable Banach space failing the AP and C is a compact convex set containing 0 which spans a dense linear subspace of X, then $\mathcal{F}(C)$ fails the AP.

Let M be a metric space equipped with a distinguished point 0_M. The space $\mathrm{Lip}_0(M)$ of real-valued Lipschitz functions which vanish at 0_M is a Banach space for the Lipschitz norm, and its natural predual, i.e. the closed linear span of the Dirac measures, is denoted by $\mathcal{F}(M)$ and is called the Lipschitz-free space over M. The Dirac map $\delta : M \to \mathcal{F}(M)$ is an isometry. The distinguished point 0_M is a matter of convenience and changing it does not alter the isometric structure of the spaces we consider. Hence we will omit it and use the notation $\mathrm{Lip}(M)$ (resp. $\mathrm{Lip}(M, X)$) for real-valued (resp. X-valued with X a Banach space) Lipschitz functions on M, always assuming that these functions vanish at 0_M. The free spaces provide a canonical linearization procedure for Lipschitz maps between metric spaces[6] which will be used in this note.

If K is a compact metric space and $T : \mathrm{Lip}(K) \to \mathrm{Lip}(K)$ is a continuous linear operator, we denote by $\|T\|_L$ its operator norm when $\mathrm{Lip}(K)$ is equipped with the Lipschitz norm, and by $\|T\|_{L,\infty}$ its norm when the domain space is equipped with the Lipschitz norm and the range space with the uniform norm - alternatively, $\|T\|_{L,\infty}$ is the norm of T from $\mathrm{Lip}(K)$ to $C(K)$ when these spaces are equipped with their canonical norms. We use the same notation for X-valued Lipschitz functions. It should be noted that if M is a metric compact space, then the uniform norm induces on the unit ball of $\mathrm{Lip}(M)$ the weak* topology associated with the free space $\mathcal{F}(M)$.

Our main result states in particular that the uniform existence of near-extensions of Banach space valued Lipschitz maps from nearly dense subsets of a metric compact space M to the whole space M is equivalent to the existence of uniformly bounded linear near-extension operators for real-valued Lipschitz maps, to the bounded approximation property for the Lipschitz-free space over M, and to a combination of these two conditions, namely linear near-extension operators for Banach space valued functions. The terms "near-extension" means that in the notation used below, functions such as $E_n(F)$ or $G_n(F)$ will not necessarily be exact extensions of F, but their restriction to M_n will be uniformly close to F, with a uniform distance which decreases to 0 when n increases to infinity.

A subset S of a metric space M is said to be ϵ-dense if for all $m \in M$, one has $\inf\{d(m,s) / s \in S\} \leq \epsilon$. We denote by $\delta_n : M_n \to \mathcal{F}(M_n)$ the Dirac map relative to M_n. With this notation, the following holds.

Theorem 1 – *Let M be a compact metric space. Let $(M_n)_n$ be a sequence of finite ϵ_n-dense subsets of M, with $\lim(\epsilon_n) = 0$. We denote by $R_n(f)$ the restriction to M_n of a function f defined on M. Let $\lambda \geq 1$. The following assertions are equivalent:*

(A_1) *The free space $\mathcal{F}(M)$ over M has the λ-BAP.*

[6] See Weaver, 1999, *Lipschitz Algebras*;
Godefroy and Kalton, 2003, "Lipschitz-free Banach spaces".

(A_2) *There exist* $\alpha_n \geq 0$ *with* $\lim \alpha_n = 0$ *such that for every Banach space* X*, there exist linear operators* $E_n : \mathrm{Lip}(M_n, X) \to \mathrm{Lip}(M, X)$ *with* $\|E_n\|_L \leq \lambda$ *and*

$$\|R_n E_n - I\|_{L,\infty} \leq \alpha_n.$$

(A_3) *There exist linear operators* $G_n : \mathrm{Lip}(M_n) \to \mathrm{Lip}(M)$ *with* $\|G_n\|_L \leq \lambda$ *and*

$$\lim \|R_n G_n - I\|_{L,\infty} = 0.$$

(A_4) *For every Banach space* X*, there exist* $\beta_n \geq 0$ *with* $\lim \beta_n = 0$ *such that for every* 1*-Lipschitz function* $F : M_n \to X$*, there exists a* λ*-Lipschitz function* $H : M \to X$ *such that* $\|R_n(H) - F\|_{l_\infty(M_n,X)} \leq \beta_n$.

Proof.

- (A_1) $\Rightarrow$ (A_2): Let $Z = c((\mathcal{F}(M_n))$ be the Banach space of sequences (μ_n) with $\mu_n \in \mathcal{F}(M_n)$ for all n, such that (μ_n) is norm-converent in the Banach space $\mathcal{F}(M)$. We equip Z with the supremum norm, and we denote $Q : Z \to \mathcal{F}(M)$ the canonical quotient operator wich maps every sequence in Z to its limit.

  The kernel $Z_0 = c_0((\mathcal{F}(M_n))$ of Q is an M-ideal in Z, and the quotient space Z/Z_0 is isometric to $\mathcal{F}(M)$. It follows from (A_1) and the Ando-Choi-Effros theorem[7] that there exists a linear map $L : \mathcal{F}(M) \to Z$ such that $QL = \mathrm{Id}_{\mathcal{F}(M)}$ and $\|L\| \leq \lambda$.

  We let π_n be the canonical projection from Z onto $\mathcal{F}(M_n)$, and we define

  $$g_n = \pi_n L \delta : M \to \mathcal{F}(M_n).$$

  The maps g_n are λ-Lipschitz, and for every $m \in M$, we have

  $$\lim \|g_n(m) - \delta(m)\|_{\mathcal{F}(M)} = 0.$$

  Since M is compact, this implies by an equicontinuity argument that if we let

  $$\alpha_n = \sup_{m \in M} \|g_n(m) - \delta(m)\|_{\mathcal{F}(M)}$$

  then $\lim \alpha_n = 0$. Let now X be a Banach space, and $F : M_n \to X$ be a Lipschitz map. There exists a unique continuous linear map $\overline{F} : \mathcal{F}(M_n) \to X$ such that $\overline{F} \circ \delta_n = F$, and its norm is equal to the Lipschitz constant of F. In the notation of Godefroy and Kalton (2003), one has $\overline{F} = \beta_X \circ \hat{F}$ and in particular $\overline{F}$ depends linearly upon F. We let now

  $$E_n(F) = \overline{F} \circ g_n$$

  and it is easy to check that the sequence (E_n) satisfies the requirements of (A_2).

- $(A_2) \Rightarrow (A_3)$: it suffices to take $X = \mathbb{R}$ in (A_2).
- $(A_2) \Rightarrow (A_4)$: it suffices to take $H = E_n(F)$ and (A_4) follows with $\beta_n = \alpha_n$ (independent of X).
- $(A_3) \Rightarrow (A_1)$: We let $\|R_n G_n - I\|_{L,\infty} = \gamma_n$, with $\lim \gamma_n = 0$. If $H \in \mathrm{Lip}(M)$, then

  $$\|R_n G_n R_n(H) - R_n(H)\|_{l_\infty(M_n)} \leq \gamma_n \|H\|_L.$$

  In other words,

  $$\|R_n[G_n R_n(H) - H]\|_{l_\infty(M_n)} \leq \gamma_n \|H\|_L.$$

  If we let now $T_n = G_n R_n : \mathrm{Lip}(M) \to \mathrm{Lip}(M)$, we have $\|T_n\|_L \leq \lambda$ and since M_n is ϵ_n-dense in M with $\lim \epsilon_n = 0$, it follows from the above that for every $H \in \mathrm{Lip}(M)$, one has

  $$\lim \|T_n(H) - H\|_{l_\infty(M)} = 0.$$

  The operator R_n is a finite rank operator which is weak-star to norm-continuous, and so is T_n since $T_n = G_n R_n$. In particular, there exists $A_n : \mathcal{F}(M) \to \mathcal{F}(M)$ such that $A_n^* = T_n$. It is clear that $\|A_n\|_{\mathcal{F}(M)} \leq \lambda$ and that the sequence (A_n) converges to the identity for the weak operator topology, and this shows (A_1).
- $(A_4) \Rightarrow (A_1)$: It will be sufficient to apply condition (A_4) to a very natural sequence of 1-Lipschitz maps. We let $X = l_\infty(\mathcal{F}(M_n))$, and $j_n \circ \delta_n = \tilde{\delta_n} : M_n \to X$, where $j_n = \mathcal{F}(M_n) \to X$ is the obvious injection, such that $(j_n(\mu))_k = 0$ if $k \neq n$ and $(j_n(\mu))_n = \mu$. The map $\tilde{\delta_n}$ is an isometric injection from M_n into X.

  By (A_4), there exist λ-Lipschitz maps $H_n : M \to X$ such that

  $$\left\|R_n(H_n) - \tilde{\delta_n}\right\|_{l_\infty(M_n, X)} \leq \beta_n.$$

  We let $V_n = P_n H_n$, where $P_n : X \to \mathcal{F}(M_n)$ is the canonical projection. The maps V_n are λ-Lipschitz, and for every $m \in M_n$, one has since $P_n \tilde{\delta_n} = \delta_n$ that

  $$\|V_n(m) - \delta_n(m)\|_{\mathcal{F}(M_n)} \leq \beta_n.$$

  The Lipschitz map $V_n : M \to \mathcal{F}(M_n)$ extends to a linear map $\overline{V_n} : \mathcal{F}(M) \to \mathcal{F}(M_n)$ with $\left\|\overline{V_n}\right\| \leq \lambda$. By the above, the sequence $C_n = J_n \overline{V_n}$, where $J_n : \mathcal{F}(M_n) \to \mathcal{F}(M)$ is the canonical injection, converges to the identity of $\mathcal{F}(M)$ in the strong operator topology. This concludes the proof. □

In what follows, we will restrict our attention to actual extension operators, in other words to the case $\alpha_n = \beta_n = \gamma_n = 0$.

[7] See Harmand, D. Werner, and W. Werner, 1993, *M-ideals in Banach spaces and Banach algebras*, theorem II.2.1.

Remark 1 – Let M be a compact metric space with distinguished point 0_M, such that $\mathcal{F}(M)$ fails the BAP (such an M exists by Godefroy and Ozawa (2014)). We denote by M^∞ the Cartesian product of countably many copies of M equipped with $d^\infty(x_n, y_n) = \sup d(x_n, y_n)$, and by $P_n : M^\infty \to M$ the corresponding sequence of projections. We use the notation of the proof of $(\mathsf{A}_4) \Rightarrow (\mathsf{A}_1)$, and in particular we let $X = l_\infty(\mathcal{F}(M_n))$. We define a map Δ from the subset $L = \Pi_{n\geq 1} M_n$ of M^∞ to X by the formula

$$\Delta((m_n)) = (\tilde{\delta_n}(m_n))_n.$$

The map Δ is 1-Lipschitz. We denote by $i_n : M \to M^\infty$ the natural injection defined by $(i_n(m)_k = m$ if $k = n$ and 0_M otherwise. Assume that Δ admits a λ-Lipschitz extension $H : M^\infty \to X$. Then for every n, the map $H_n = P_n H i_n$ is a λ-Lipschitz extension of $\tilde{\delta_n}$. But then, the proof of $(\mathsf{A}_4) \Rightarrow (\mathsf{A}_1)$ shows that $\mathcal{F}(M)$ has the λ-BAP, contrarily to our assumption. Hence Δ cannot be extended to a Lipschitz map from M^∞ to X.

Remark 2 – In the notation of remark 1, assume that there exists a linear extension operator $E : \mathrm{Lip}(L) \to \mathrm{Lip}(M^\infty)$ with $\|E\|_L = \lambda < \infty$. If π_n denotes the canonical projection from L onto $i_n(M_n)$, then π_n is 1-Lipschitz and thus the map $E_n : \mathrm{Lip}(i_n(M_n)) \to \mathrm{Lip}(M^\infty)$ defined by $E_n(F) = E(F \circ \pi_n)$ satisfies $\|E_n\|_L \leq \lambda$. Composing E_n with the restriction to $i_n(M)$ shows the existence of a linear extension operator from $\mathrm{Lip}(M_n)$ to $\mathrm{Lip}(M)$ with norm at most λ for all n, and by $(\mathsf{A}_3) \Rightarrow (\mathsf{A}_1)$ this cannot be if $\mathcal{F}(M)$ fails BAP.

Remark 3 – The existence of linear extension operators for Lipschitz functions has already been investigated[8]. We recall the notation of A. Brudnyi and Y. Brudnyi (2007): if M is a metric space, then

$$\lambda(M) = \sup_{S\subset M} \inf\{\|E\|_L \,/\, E : \mathrm{Lip}(S) \to \mathrm{Lip}(M)\}$$

where E is assumed to be an extension operator. It is clear that if M is a compact metric space such that $\lambda(M) < \infty$, then $\mathcal{F}(M)$ has the λ-BAP with $\lambda \leq \lambda(M)$. It seems to be a natural question to decide whether a converse implication is valid. The article A. Brudnyi and Y. Brudnyi (2007) provides a wealth of metric spaces M such that $\lambda(M) < \infty$.

[8] For instance in A. Brudnyi and Y. Brudnyi, 2007, "Metric spaces with linear extensions preserving Lipschitz condition";

A. Brudnyi and Y. Brudnyi, 2008, "Linear and nonlinear extensions of Lipschitz functions from subsets of metric spaces";

Y. Brudnyi and Shvartsman, 2002, "Stability of the Lipschitz extension property under metric transforms";

Y. Brudnyi and Shvartsman, 1997, "The Whitney problem of existence of a linear extension operator", and more articles of the same authors.

Remark 4 – For some compact metric spaces M, the free space $\mathcal{F}(M)$ is isometric to the dual space of the little Lipschitz space lip(M) consisting of all Lipschitz functions f such that for any $\epsilon > 0$, there exists $\delta > 0$ such that if $d(x,y) < \delta$, then $|f(x) - f(y)| \leq \epsilon d(x,y)$. This happens when lip($M$) strongly separates M[9]. When it is so, $\mathcal{F}(M)$ is a separable dual space and it follows from a theorem of Grothendieck[10] that if $\mathcal{F}(M)$ satisfies the conditions of theorem 1 on page 31 for some $\lambda \in \mathbb{R}$, then in fact it satisfies them with $\lambda = 1$. Moreover, it follows easily from the local reflexivity principle that in this case, we may replace condition assertion (A_3) by the stronger requirement (A_3^*) that the operators (G_n) satisfy $\|G_n\|_L \leq 1$, $\lim\|R_n G_n - I\|_{L,\infty} = 0$ and $G_n(\mathrm{Lip}(M_n)) \subset \mathrm{lip}(M)$ for every n. This condition (A_3^*) is a linear version of Weaver's extension lemma[11].

Remark 5 – Let us observe that condition (A_1) is obviously independent of the particular approximating sequence (M_n) that we picked, therefore conditions (A_2), (A_3) and (A_4) are independent as well. Hence theorem 1 on page 31 is an invitation to consider geometrical conditions on a net N in M which would provide controlable extensions to M of Lipschitz functions defined on N, and to try to find N which such good properties. As an example of such a desirable behaviour, we mention the interpolation formula used in Hájek and Pernecká (2014), Lancien and Pernecká (2013), and Pernecká and Smith (2015) which allows to extend a function defined on the vertices of a cube without changing the Lipschitz constant relatively to the l_1-norm subordinated to the edges.

References

Benyamini, Y. and J. Lindenstrauss (2000). *Geometric Nonlinear Functional Analysis*. **48**. Colloquium Publications. American Mathematical Soc. 488 pp. (cit. on p. 30).

Borel-Mathurin, L. (2012). "Approximation properties and nonlinear geometry of Banach spaces". *Houst. J. Math.* **38** (4), pp. 1135–1148 (cit. on p. 30).

Brudnyi, A. and Y. Brudnyi (2007). "Metric spaces with linear extensions preserving Lipschitz condition". *Amer. J. Math.* **129** (1), pp. 217–314 (cit. on pp. 30, 34).

Brudnyi, A. and Y. Brudnyi (2008). "Linear and nonlinear extensions of Lipschitz functions from subsets of metric spaces". *St Petersburg Math. J.* **19** (3), pp. 397–407. ISSN: 1061-0022. DOI: 10.1090/S1061-0022-08-01003-0 (cit. on p. 34).

Brudnyi, Y. and P. Shvartsman (1997). "The Whitney problem of existence of a linear extension operator". *J. Geom. Anal.* **7** (4), pp. 515–574 (cit. on p. 34).

Brudnyi, Y. and P. Shvartsman (2002). "Stability of the Lipschitz extension property under metric transforms". *Geom. Funct. Anal.* **12** (1), pp. 73–79 (cit. on p. 34).

[9] See chapter 3 in Weaver, 1999, *Lipschitz Algebras*.

[10] Grothendieck, 1955, "Produits tensoriels topologiques et espaces nucléaires".

[11] Lemma 3.2.3 in Weaver, 1999, *Lipschitz Algebras*.

Cúth, M. and M. Doucha (2015). "Lipschitz-free spaces over ultrametric spaces". *Mediterr. J. Math.* Pp. 1–14. DOI: 10.1007/s00009-015-0566-7 (cit. on p. 30).
Dalet, A. (2014). "Free spaces over some proper metric spaces". *Mediterr. J. Math.* **12** (3), pp. 973–986 (cit. on p. 30).
Dalet, A. (2015). "Free spaces over countable compact metric spaces". *Proc. Amer. Math. Soc.* **143** (8), pp. 3537–3546 (cit. on p. 30).
Godefroy, G. and N. J. Kalton (2003). "Lipschitz-free Banach spaces". *Stud. Math.* **159** (1), pp. 121–141 (cit. on pp. 30–32).
Godefroy, G. and N. Ozawa (2014). "Free Banach spaces and the approximation properties". *Proc. Am. Math. Soc.* **142** (5), pp. 1681–1687 (cit. on pp. 30, 34).
Grothendieck, A. (1955). "Produits tensoriels topologiques et espaces nucléaires". French. *Mem. Amer. Math. Soc.* **16** (cit. on p. 35).
Hájek, P. and E. Pernecká (2014). "On Schauder bases in Lipschitz-free spaces". *J. Math. Anal. Appl.* **416** (2), pp. 629–646 (cit. on pp. 30, 35).
Harmand, P., D. Werner, and W. Werner (1993). *M-ideals in Banach spaces and Banach algebras*. **1547**. Lecture Notes in Maths. Springer-Verlag (cit. on p. 33).
Kalton, N. J. (2004). "Spaces of Lipschitz and Hölder functions and their applications". *Collect. Math.* **55** (2), pp. 171–217 (cit. on p. 30).
Kalton, N. J. (2012). "The uniform structure of Banach spaces". *Math. Ann.* **354** (4), pp. 1247–1288 (cit. on p. 30).
Kaufmann, P. L. (2014). "Products of Lipschitz-free spaces and applications". *Stud. Math.* **226** (3), pp. 213–227 (cit. on p. 30).
Lancien, G. and E. Pernecká (2013). "Approximation properties and Schauder decompositions in Lipschitz-free spaces". *J. of Funct. Anal.* **264** (10), pp. 2323–2334 (cit. on pp. 30, 35).
Lindenstrauss, J. (1964). "On nonlinear projections in Banach spaces." *Michigan Math. J.* **11** (3), pp. 263–287 (cit. on p. 30).
McShane, E. J. (1934). "Extension of range of functions". *Bull. Amer. Math. Soc.* **40** (12), pp. 837–842 (cit. on p. 29).
Pernecká, E. and R. J. Smith (2015). "The metric approximation property and Lipschitz-free spaces over subsets of $\mathbb{R}^n$". *J. of Appr. Theory* **199**, pp. 29–44 (cit. on pp. 30, 35).
Weaver, N. (1999). *Lipschitz Algebras*. World Scientific. 240 pp. (cit. on pp. 31, 35).

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