North-Western European Journal of Mathematics

Composition operators between $\mathcal{N}_p$-spaces in the ball

Bingyang Hu[1] Le Hai Khoi[2] Trieu Le[3]

Received: January 28, 2016/Accepted: June 6, 2016/Online: July 22, 2016

Abstract

We study composition operators acting between $\mathcal{N}_p$-spaces in the unit ball in $\mathbb{C}^n$. We obtain characterizations of the boundedness and compactness of $C_\varphi : \mathcal{N}_p \longrightarrow \mathcal{N}_q$ for $p, q > 0$.

Keywords: $\mathcal{N}_p$-space, composition operator, boundedness, compactness.

MSC: 32A36, 47B33.

1 Introduction

Let $\mathbb{B}$ be the open unit ball in $\mathbb{C}^n$. The space $\mathcal{O}(\mathbb{B})$ consists of all holomorphic functions in $\mathbb{B}$. For any holomorphic self-mapping φ of the unit ball $\mathbb{B}$, the linear operator $C_\varphi : \mathcal{O}(\mathbb{B}) \longrightarrow \mathcal{O}(\mathbb{B})$ defined by

$$C_\varphi(f) = f \circ \varphi, \quad f \in \mathcal{O}(\mathbb{B}),$$

is called the *composition operator* with symbol φ. We are often interested in the study of C_φ acting between Banach spaces contained in $\mathcal{O}(\mathbb{B})$.

Composition operators on spaces of holomorphic functions in the unit disk $\mathbb{D}$ and the unit ball $\mathbb{B}$ such as the Hardy, Bergman, Bloch spaces, just to name a few, have been studied intensively in many settings. We refer the reader to the monographs of Cowen and MacCluer[4] and Shapiro[5] for detailed information. In this paper we would like to investigate composition operators acting on a different class of Banach spaces, the $\mathcal{N}_p$-spaces.

[1]Department of Mathematics, University of Wisconsin, Madison, WI 53706-1388, USA

[2]Division of Mathematical Sciences, School of Physical and Mathematical Sciences, Nanyang Technological University (NTU), 637371 Singapore

[3]Department of Mathematics and Statistics, Mail Stop 942, University of Toledo, Toledo, OH 43606, USA

[4]Cowen and MacCluer, 1995, *Composition operators on spaces of analytic functions.*

[5]Shapiro, 1993, *Composition operators and classical function theory.*

The $\mathcal{N}_p$-spaces on the unit disk were introduced and studied by Palmberg[6]. For $p > 0$, the space $\mathcal{N}_p(\mathbb{D})$ consists of functions in $\mathcal{O}(\mathbb{D})$ such that

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f(z)|^2 (1 - |\sigma_a(z)|^2)^p \,\mathrm{d}A(z) < \infty.$$

The study of such spaces was motivated by Q_p-spaces, which have been of interests by many researchers. The book of Xiao[7] provides an excellent source of information about Q_p-spaces.

Some important properties of $\mathcal{N}_p$-spaces are: for $p > 1$, $\mathcal{N}_p$-spaces all coincide with $A^{-1}(\mathbb{D})$ and for $p \in (0,1]$, the $\mathcal{N}_p$-spaces are all different. Here $A^{-p}(\mathbb{D})$ denotes the Bergman-type space that consists of functions in $\mathcal{O}(\mathbb{D})$ such that $\sup_{z\in\mathbb{D}} |f(z)|(1-|z|^2)^p < \infty$.

In Palmberg (2007), Palmberg studied the boundedness and compactness of composition operators acting between $\mathcal{N}_p$-spaces and from an $\mathcal{N}_p$-space into $A^{-q}(\mathbb{D})$. Certain characterizations of boundedness and compactness were obtained. In Ueki (2012), Ueki investigated weighted composition operators between $\mathcal{N}_p$-spaces and A^{-q}-spaces.

The first two authors[8] introduced and studied properties of $\mathcal{N}_p$-spaces in higher dimensions. For $p > 0$, the $\mathcal{N}_p$-space of $\mathbb{B}$ is defined as follows:

$$\begin{aligned}\mathcal{N}_p &= \mathcal{N}_p(\mathbb{B}) \\ &:= \left\{ f \in \mathcal{O}(\mathbb{B}) : \|f\|_p = \sup_{a\in\mathbb{B}} \left(\int_{\mathbb{B}} |f(z)|^2 (1 - |\Phi_a(z)|^2)^p \,\mathrm{d}V(z) \right)^{1/2} < \infty \right\},\end{aligned}$$

where $\mathrm{d}V$ is the normalized Lebesgue volume measure over $\mathbb{B}$ and Φ_a is the involutive automorphism of $\mathbb{B}$ that interchanges 0 and a. We also defined the little space of $\mathcal{N}_p$ as

$$\mathcal{N}_p^0 = \mathcal{N}_p^0(\mathbb{B}) := \left\{ f \in \mathcal{N}_p : \lim_{|a|\to 1^-} \int_{\mathbb{B}} |f(z)|^2 (1 - |\Phi_a(z)|^2)^p \,\mathrm{d}V(z) = 0 \right\},$$

which is a closed subspace of $\mathcal{N}_p$.

Several basic properties of $\mathcal{N}_p$-spaces are proved, in connection with the Bergman-type spaces A^{-q}, which, similar to the one dimensional cases, consists of functions $f \in \mathcal{O}(\mathbb{B})$ for which

$$|f|_p = \sup_{z\in\mathbb{B}} |f(z)|(1 - |z|^2)^p < \infty.$$

Recall that H^∞ denotes the Banach space of all bounded functions in $\mathcal{O}(\mathbb{B})$ with the norm $\|f\|_\infty = \sup_{z\in\mathbb{B}} |f(z)|$.

[6]Palmberg, 2007, "Composition operators acting on $\mathcal{N}_p$-spaces".

[7]Xiao, 2001, *Holomorphic Q classes*.

[8]Hu and Khoi, 2013, "Weighted composition operators on $\mathcal{N}_p$-spaces in the ball".

Theorem 1 (Hu and Khoi[9]) – *The following statements hold:*

1. *For $p > q > 0$, we have $H^\infty \hookrightarrow \mathcal{N}_q \hookrightarrow \mathcal{N}_p \hookrightarrow A^{-\frac{n+1}{2}}$.*
2. *For $p > 0$, if $p > 2k-1, k \in (0, \frac{n+1}{2}]$, then $A^{-k} \hookrightarrow \mathcal{N}_p$. In particular, when $p > n$, $\mathcal{N}_p = A^{-\frac{n+1}{2}}$.*
3. *$\mathcal{N}_p$ is a functional Banach space with the norm $\|\cdot\|_p$, and moreover, its norm topology is stronger than the compact-open topology.*
4. *For $0 < p < \infty$, $\mathcal{B} \hookrightarrow \mathcal{N}_p$, where $\mathcal{B}$ is the Bloch space in $\mathbb{B}$.*

Considering weighted composition operators between $\mathcal{N}_p$ and Bergman-type spaces A^{-q}

$$W_{u,\varphi}(f) = u \cdot (f \circ \varphi), f \in \mathcal{O}(\mathbb{B}),$$

where $u\colon \mathbb{B} \to \mathbb{C}$ is a holomorphic function, the properties above allow us to prove criteria for boundedness and compactness of these operators[10]. In Hu and Khoi (2015) and Hu, Khoi, and Le (2016a), compact differences and the estimate for essential norm of weighted composition operators acting from an $\mathcal{N}_p$-space to an A^{-q}-space were investigated. In these works, various properties stated in Theorem 1 were used.

All of the aforementioned results concern weighted composition operators $W_{u,\varphi}$ acting between the spaces $\mathcal{N}_p$ and A^{-q}. A natural question one may ask is: what is the situation like when we consider composition operators acting between $\mathcal{N}_p$-spaces themselves? The aim of the present paper is to investigate this question. More precisely, we study the boundedness and compactness of composition operators C_φ acting between $\mathcal{N}_p$ and $\mathcal{N}_q$ for $p, q > 0$. To our knowledge, this problem has not been treated before, neither in one variable nor several variables.

The structure of this paper is as follows. Section 2 on the next page is devoted to the boundedness of C_φ. We obtain sufficient conditions and necessary conditions for C_φ to be bounded from $\mathcal{N}_p$ into $\mathcal{N}_q$. In Section 3 on p. 118, we investigate the compactness of C_φ. Different characterizations for compactness are provided. We also show in this section that the range of any compact composition operator $C_\varphi : \mathcal{N}_p \longrightarrow \mathcal{N}_q$ must be contained in $\mathcal{N}_q^0$. Consequently, the compactness of C_φ from $\mathcal{N}_p$ into $\mathcal{N}_q$ and from $\mathcal{N}_p$ into $\mathcal{N}_q^0$ are equivalent.

Throughout the paper, $d\sigma$ denotes the normalized surface measure on the sphere $\mathbb{S}$, the boundary of $\mathbb{B}$. For two quantities a and b of a certain variable, we write $a \lesssim b$ (respectively, $a \gtrsim b$) if there exists a positive number C independent of the variable under consideration such that $a \le Cb$ (respectively, $a \ge Cb$). Moreover, if both $a \lesssim b$ and $a \gtrsim b$ hold, then we write $a \simeq b$.

[9] Hu and Khoi, 2013, "Weighted composition operators on $\mathcal{N}_p$-spaces in the ball".

[10] Ibid., Theorems 3.2 and 3.4.

2 Boundedness of composition operators from $\mathcal{N}_p$ to $\mathcal{N}_q$

In this section we investigate the boundedness of composition operators between $\mathcal{N}_p$-spaces. The one dimensional case is considered by Palmberg[11]. A sufficient condition and a necessary condition for C_φ to be bounded on $\mathcal{N}_p$ of the unit disk were given there. These conditions involve the generalized Nevanlinna counting function introduced by Shapiro.

We begin with the case φ is a univalent holomorphic self-mapping of the unit ball $\mathbb{B}$.

Theorem 2 – *Let φ be a univalent holomorphic self-mapping of $\mathbb{B}$ and p be any positive real number. Suppose that*

$$\delta = \inf\{|J\varphi(z)| : z \in \mathbb{B}\} > 0,$$

where $J\varphi$ the complex Jacobian of φ. Then $C_\varphi : \mathcal{N}_p \longrightarrow \mathcal{N}_p$ is a bounded operator with $\|C_\varphi\| \le \delta^{-1}$ for all $p > 0$.

Proof. Let f be any function in $\mathcal{N}_p$. For $a \in \mathbb{B}$, and $z \in \mathbb{B}$, the Schwarz-Pick lemma[12] gives $|\Phi_{\varphi(a)}(\varphi(z))| \le |\Phi_a(z)|$. Since φ is univalent, it is biholomorphic from $\mathbb{B}$ onto $\varphi(\mathbb{B})$[13]. This makes the change of variables possible in the following estimates:

$$\begin{aligned}
&\delta^2 \int_{\mathbb{B}} |f(\varphi(z))|^2 (1 - |\Phi_a(z)|^2)^p \,\mathrm{d}V(z) \\
&\le \int_{\mathbb{B}} |f(\varphi(z))|^2 (1 - |\Phi_{\varphi(a)}(\varphi(z))|^2)^p |J\varphi(z)|^2 \,\mathrm{d}V(z) \\
&= \int_{\varphi(\mathbb{B})} |f(w)|^2 (1 - |\Phi_{\varphi(a)}(z)|^2)^p \,\mathrm{d}V(w) \quad \text{(by the change-of-variables } w = \varphi(z)) \\
&\le \int_{\mathbb{B}} |f(w)|^2 (1 - |\Phi_{\varphi(a)}(z)|^2)^p \,\mathrm{d}V(w) \\
&\le \|f\|_p^2.
\end{aligned}$$

Taking supremum over $a \in \mathbb{B}$ gives $\delta^2 \|C_\varphi f\|_p^2 \le \|f\|_p^2$, which implies $\|C_\varphi f\|_p \le \delta^{-1}\|f\|_p$. Since f was an arbitrary element in $\mathcal{N}_p$, we conclude that C_φ is a bounded operator on $\mathcal{N}_p$ with $\|C_\varphi\| \le \delta^{-1}$ as desired. □

[11] Palmberg, 2007, "Composition operators acting on $\mathcal{N}_p$-spaces", Theorem 4.2.
[12] Rudin, 1980, *Function theory in the unit ball of* $\mathbb{C}^n$, Theorem 8.1.4.
[13] See e.g. ibid., Theorem 15.1.8.

Corollary 1 – *Suppose $A : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ is an invertible linear operator and b is a vector in $\mathbb{C}^n$ such that $\varphi(z) = Az + b$ is a self-mapping of $\mathbb{B}$. Then $C_\varphi : \mathcal{N}_p \longrightarrow \mathcal{N}_p$ is bounded for all $p > 0$.*

Proof. Since $|J\varphi(z)| = |\det(A)| > 0$, Theorem 2 on the preceding page provides the desired conclusion. □

For general self-mappings φ of the unit ball, we offer a necessary condition and a sufficient condition for C_φ to be bounded from $\mathcal{N}_p$ into $\mathcal{N}_q$. We shall make use of a sequence of homogeneous polynomials $\{P_m\}$ which satisfy $\deg(P_m) = m$,

$$\|P_m\|_\infty = \sup_{|\zeta|=1} |P_m(\zeta)| = 1, \qquad \text{and} \int_{\mathbb{S}} |P_m(\zeta)|^2 \mathrm{d}\sigma(\zeta) \geq \frac{\pi}{4^n}. \tag{1}$$

The existence of such a sequence was proved in Ryll and Wojtaszczyk (1983). We shall also need the inequality

$$1 + \sum_{k=0}^{\infty} 2^{k\gamma} x^{2^k} \gtrsim (1-x)^{-\gamma} \qquad \text{for } 0 \leq x < 1, \tag{2}$$

where γ is a positive number. Indeed, it was showed[14] that

$$\sum_{k=0}^{\infty} 2^{k\gamma} x^{2^k} \gtrsim (1-x)^{-\gamma} \qquad \text{for } x > \frac{1}{2}.$$

On the other hand, it is clear that $2^\gamma \geq (1-x)^{-\gamma}$ for $0 \leq x \leq \frac{1}{2}$. Therefore, (2) holds.

The following preparatory result will be needed in obtaining the necessary condition for the boundedness of C_φ.

Proposition 1 – *Let p and q be positive numbers and φ a holomorphic self-mapping of $\mathbb{B}$. Let $\mathbb{B}_{\mathcal{N}_p}$ denotes the unit ball of $\mathcal{N}_p$. Suppose μ is a positive number and E is a measurable subset of $\mathbb{B}$ such that*

$$\sup_{a\in\mathbb{B}, f\in\mathbb{B}_{\mathcal{N}_p}} \int_E |f(\varphi(z))|^2 (1-|\Phi_a(z)|^2)^q \mathrm{d}V(z) \leq \mu.$$

Then for any $0 < \varepsilon \leq p+1$, we have

$$\sup_{a\in\mathbb{B}} \int_E \frac{(1-|\Phi_a(z)|^2)^q}{(1-|\varphi(z)|^2)^{p+1-\varepsilon}} \mathrm{d}V(z) \lesssim \mu.$$

[14] Ueki, 2012, “Weighted composition operators acting between the $\mathcal{N}_p$-space and the weighted-type space H_α^∞”, p. 247.

Proof. The hypothesis implies that for any $a \in \mathbb{B}$ and any $f \in \mathcal{N}_p$, we have

$$\int_E |f(\varphi(z))|^2(1-|\Phi_a(z)|^2)^q \mathrm{d}V(z) \le \mu\|f\|_p^2. \tag{3}$$

Put

$$F(z) = 1 + \sum_{k=0}^{\infty} 2^{k(p+1-\varepsilon)/2} P_{2^k}(z), \qquad z \in \mathbb{B}.$$

By Hu, Khoi, and Le (2016b, Theorem 5.1),

$$\|F\|_p^2 \approx \sum_{k=0}^{\infty} \frac{2^{k(p+1-\varepsilon)}}{2^{k(p+1)}} = \sum_{k=0}^{\infty} 2^{-k\varepsilon} < \infty,$$

which shows that F belongs to $\mathcal{N}_p$. Let $\mathfrak{U}_n$ denote the group of all unitary operators on the Hilbert space $\mathbb{C}^n$. For any $U \in \mathfrak{U}_n$, the unitary invariant property of the volume measure shows that $F \circ U$ also belongs to $\mathcal{N}_p$ and $\|F \circ U\|_p = \|F\|_p$. Setting $f = F \circ U$ in (3) gives

$$\int_E |F(U\varphi(z))|^2(1-|\Phi_a(z)|^2)^q \mathrm{d}V(z) \le \mu\|F \circ U\|_p^2 = \mu\|F\|_p^2,$$

for each $a \in \mathbb{B}$ and $U \in \mathfrak{U}_n$.

Now fix $a \in \mathbb{B}$. Integration with respect to the Haar measure $\mathrm{d}U$ on $\mathfrak{U}_n$ and Fubini's Theorem yield

$$\int_E \Big(\int_{\mathfrak{U}_n} |F(U\varphi(z))|^2 \mathrm{d}U\Big)\Big(1-|\Phi_a(z)|^2\Big)^q \mathrm{d}V(z) \le \mu\|F\|_p^2.$$

For any $z \in \mathbb{B}$, we compute

$$\begin{aligned}
\int_{\mathfrak{U}_n} |F(U\varphi(z))|^2 \mathrm{d}U &= \int_{\mathbb{S}} \Big|F\Big(|\varphi(z)|\zeta\Big)\Big|^2 \mathrm{d}\sigma(\zeta) \\
&\qquad \text{(by Rudin (1980, Proposition 1.4.7))} \\
&= \int_{\mathbb{S}} \Big|1 + \sum_{k=0}^{\infty} 2^{k(p+1-\varepsilon)/2} P_{2^k}(|\varphi(z)|\zeta)\Big|^2 \mathrm{d}\sigma(\zeta) \\
&= 1 + \sum_{k=0}^{\infty} 2^{k(p+1-\varepsilon)}(|\varphi(z)|^2)^{2^k} \int_{\mathbb{S}} |P_{2^k}(\zeta)|^2 \mathrm{d}\sigma(\zeta) \\
&\qquad \text{(by the orthogonality and the homogeneity of } \{P_{2^k}\}) \\
&\ge \frac{\pi}{4^n}\Big(1 + \sum_{k=0}^{\infty} 2^{k(p+1-\varepsilon)}(|\varphi(z)|^2)^{2^k}\Big) \qquad \text{(by (1))}
\end{aligned}$$

(Cont. next page)

$$\gtrsim \left(1-|\varphi(z)|^2\right)^{-(p+1-\varepsilon)} \qquad \text{(by (2))}.$$

Consequently,

$$\begin{aligned}\int_E & \frac{(1-|\Phi_a(z)|^2)^q}{(1-|\varphi(z)|^2)^{p+1-\varepsilon}} \mathrm{d}V(z) \\ &\lesssim \int_E \left(\int_{\mathfrak{U}} |F(U\varphi(z))|^2 \mathrm{d}U\right)(1-|\Phi_a(z)|^2)^q \mathrm{d}V(z) \le \mu \|F\|_p^2\end{aligned}$$

as desired. □

We are now ready to prove a necessary condition and a sufficient condition for the boundedness of $C_\varphi : \mathcal{N}_p \longrightarrow \mathcal{N}_q$.

Theorem 3 – *Let p and q be two positive numbers and φ a holomorphic self-mapping of $\mathbb{B}$. If*

$$\sup_{a\in\mathbb{B}} \int_{\mathbb{B}} \frac{(1-|\Phi_a(z)|^2)^q}{(1-|\varphi(z)|^2)^{n+1}} \mathrm{d}V(z) < \infty, \tag{4}$$

then $C_\varphi : \mathcal{N}_p \longrightarrow \mathcal{N}_q$ is bounded.

Conversely, if $C_\varphi : \mathcal{N}_p \longrightarrow \mathcal{N}_q$ is bounded, then for any $0 < \varepsilon \le p+1$,

$$\sup_{a\in\mathbb{B}} \int_{\mathbb{B}} \frac{(1-|\Phi_a(z)|^2)^q}{(1-|\varphi(z)|^2)^{p+1-\varepsilon}} \mathrm{d}V(z) < \infty.$$

Proof. Suppose (4) holds. By Item 1 of Theorem 1 on p. 113, there is a constant $C > 0$ such that for each $f \in \mathcal{N}_p$, we have

$$|f(z)|(1-|z|^2)^{\frac{n+1}{2}} \le C\|f\|_p, \quad \forall z \in \mathbb{B}.$$

Hence,

$$\sup_{a\in\mathbb{B}} \int_{\mathbb{B}} |f(\varphi(z))|^2 (1-|\Phi_a(z)|^2)^q \mathrm{d}V(z) \le (C\|f\|_p)^2 \sup_{a\in\mathbb{B}} \int_{\mathbb{B}} \frac{(1-|\Phi_a(z)|^2)^q}{(1-|\varphi(z)|^2)^{n+1}} \mathrm{d}V(z),$$

which shows that C_φ is bounded from $\mathcal{N}_p$ into $\mathcal{N}_q$.

Conversely, suppose $C_\varphi : \mathcal{N}_p \longrightarrow \mathcal{N}_q$ is bounded. Then

$$\sup_{a\in\mathbb{B}, f\in\mathbb{B}_{\mathcal{N}_p}} \int_{\mathbb{B}} |f(\varphi(z))|^2 (1-|\Phi_a(z)|^2)^q \mathrm{d}V(z) = \sup_{f\in\mathbb{B}_{\mathcal{N}_p}} \|C_\varphi f\|_q^2 \le \|C_\varphi\|^2.$$

The desired inequality now follows from Proposition 1 on p. 115. □

An application of Theorem 3 immediately gives the following result. In fact, the operator C_φ in the corollary is compact, as we shall see in the next section.

Corollary 2 – *Let φ be a holomorphic self-mapping of $\mathbb{B}$ such that $\|\varphi\|_\infty < 1$. Then $C_\varphi : \mathcal{N}_p \longrightarrow \mathcal{N}_q$ is bounded for all $p, q > 0$.*

3 Compactness of composition operators from $\mathcal{N}_p$ to $\mathcal{N}_q$

3.1 General characterizations

In this section we study the compactness of composition operators between $\mathcal{N}_p$-spaces. By standard argument[15], we have the following criterion for compactness.

Lemma 1 – *A bounded composition operator $C_\varphi\colon \mathcal{N}_p \to \mathcal{N}_q$ is compact if and only if for any bounded sequence $\{f_m\} \subset \mathcal{N}_p$ converging to zero uniformly on compact subsets of $\mathbb{B}$, the sequence $\left\{\|f_m \circ \varphi\|_q\right\}$ converges to zero as $m \to \infty$.*

It turns out, as we shall see below, that the range of any compact composition operator $C_\varphi : \mathcal{N}_p \longrightarrow \mathcal{N}_q$ must be contained in the little space $\mathcal{N}_q^0$. We first prove an important property of elements in $\mathcal{N}_q^0$.

Lemma 2 – *Let h be an element in the space $\mathcal{N}_q^0$. Suppose $\{A_k\}_{k\geq 1}$ is a decreasing sequence of measurable subsets of $\mathbb{B}$ whose intersection is empty. Then*

$$\lim_{k\to\infty}\left[\sup_{a\in\mathbb{B}}\int_{A_k}|h(z)|^2(1-|\Phi_a(z)|^2)^q \mathrm{d}V(z)\right] = 0. \tag{5}$$

Proof. Let $\varepsilon > 0$ be given. Since $h \in \mathcal{N}_q^0$, there exists a positive number $0 < \delta < 1$ such that

$$\sup_{\delta<|a|<1}\int_{\mathbb{B}}|h(z)|^2(1-|\Phi_a(z)|^2)^q \mathrm{d}V(z) < \varepsilon. \tag{6}$$

On the other hand, if $a \in \mathbb{B}$ with $|a| \leq \delta$, then for $z \in \mathbb{B}$,

$$1-|\Phi_a(z)|^2 = \frac{(1-|a|^2)(1-|z|^2)}{|1-\langle z,a\rangle|^2} \leq \frac{1+|a|}{1-|a|}(1-|z|^2) \leq \frac{1+\delta}{1-\delta}(1-|z|^2).$$

Consequently, for any integer $k \geq 1$, we have

$$\sup_{|a|\leq\delta}\int_{A_k}|h(z)|^2(1-|\Phi_a(z)|^2)^q \mathrm{d}V(z) \leq \left(\frac{1+\delta}{1-\delta}\right)^q \int_{A_k}|h(z)|^2(1-|z|^2)^q \mathrm{d}V(z).$$

Since $h \in \mathcal{N}_q^0 \subset A_q^2$, the function $|h(z)|^2(1-|z|^2)^q$ belongs to $L^1(\mathbb{B}, dV)$. Because $\{A_{k\geq 1}\}$ is a decreasing sequence of subsets whose intersection is empty, there exists a positive integer k_ε such that for all $k \geq k_\varepsilon$,

$$\left(\frac{1+\delta}{1-\delta}\right)^q \int_{A_k}|h(z)|^2(1-|z|^2)^q \mathrm{d}V(z) < \varepsilon.$$

[15] As in Cowen and MacCluer, 1995, *Composition operators on spaces of analytic functions*, Proposition 3.11.

This implies that for such k,

$$\sup_{|a|\le\delta}\int_{A_k}|h(z)|^2(1-|\Phi_a(z)|^2)^q\,\mathrm{d}V(z)<\varepsilon. \tag{7}$$

Combining (6) and (7) yields

$$\sup_{a\in\mathbb{B}}\int_{A_k}|h(z)|^2(1-|\Phi_a(z)|^2)^q\,\mathrm{d}V(z)<\varepsilon$$

for all $k\ge k_\varepsilon$. Since ε was arbitrary, (5) follows. □

The following result provides a necessary condition for C_φ to be a compact operator.

Proposition 2 – *Suppose $C_\varphi:\mathcal{N}_p\longrightarrow\mathcal{N}_q$ is a compact composition operator. Let $\mathbb{B}_{\mathcal{N}_p}=\{f\in\mathcal{N}_p:\|f\|_p\le 1\}$. Then the following statements are true.*

1. *$C_\varphi(\mathcal{N}_p)\subset\mathcal{N}_q^0$ and*

$$\lim_{|a|\to 1^-}\sup_{f\in\mathbb{B}_{\mathcal{N}_p}}\int_{\mathbb{B}}|f(\varphi(z))|^2(1-|\Phi_a(z)|^2)^q\,\mathrm{d}V(z)=0. \tag{8}$$

2. *For any decreasing sequence $\{A_k\}_{k\ge1}$ of measurable subsets of the unit ball whose intersection is empty, we have*

$$\lim_{k\to\infty}\sup_{f\in\mathbb{B}_{\mathcal{N}_p}}\left[\sup_{a\in\mathbb{B}}\int_{A_k}|f(\varphi(z))|^2(1-|\Phi_a(z)|^2)^q\,\mathrm{d}V(z)\right]=0. \tag{9}$$

Proof. We first prove that $C_\varphi(\mathcal{N}_p)$ is a subset of $\mathcal{N}_q^0$. Let f be in $\mathcal{N}_p$. For any integer $m\ge1$, put $f_m(z)=f(\frac{m}{m+1}z)$. Then each f_m belongs to H^∞ and the sequence $\{f_m\}$ is bounded on $\mathcal{N}_p$ and converges to f uniformly on compact subsets of $\mathbb{B}$. By Lemma 1 on the preceding page, the sequence $\{C_\varphi f_m\}$ converges to $C_\varphi f$ in $\mathcal{N}_q$ as $m\to\infty$. Since each function $C_\varphi f_m$ belongs to $H^\infty\subset\mathcal{N}_q^0$ and $\mathcal{N}_q^0$ is a closed subspace of $\mathcal{N}_p$, we conclude that $C_\varphi f$ is an element in $\mathcal{N}_q^0$. Since f was arbitrary, it follows that the image of $\mathcal{N}_p$ under C_φ is contained in $\mathcal{N}_q^0$.

Let $\varepsilon>0$ be given. Since C_φ is compact and its range is contained in $\mathcal{N}_q^0$, the image $C_\varphi(\mathbb{B}_{\mathcal{N}_p})$ is pre-compact in $\mathcal{N}_q^0$. Therefore, $C_\varphi(\mathbb{B}_{\mathcal{N}_p})$ can be covered by finitely many $\frac{\sqrt{\varepsilon}}{2}$-balls. That is, there exists a finite set $\{f_1,\dots,f_M\}\subset\mathbb{B}_{\mathcal{N}_p}$ such that for any $f\in\mathbb{B}_{\mathcal{N}_p}$, there is a number $j\in\{1,2,\dots,M\}$ for which

$$\|C_\varphi(f)-C_\varphi(f_j)\|_q^2<\frac{\varepsilon}{4}. \tag{10}$$

On other hand, since $\{f_1 \circ \varphi, \dots, f_M \circ \varphi\}$ is contained in $\mathcal{N}_q^0$, there exists $0 < r < 1$ such that for all $1 \le j \le M$ and $|a| > r$,

$$\int_{\mathbb{B}} |f_j(\varphi(z))|^2 (1 - |\Phi_a(z)|^2)^q \mathrm{d}V(z) < \frac{\varepsilon}{4}. \tag{11}$$

For each $a \in \mathbb{B}$ with $|a| > r$ and $f \in \mathcal{N}_p$ with $\|f\|_p < 1$, choose $1 \le j \le M$ such that (10) holds. Combining with (11), we have

$$\begin{aligned}
&\int_{\mathbb{B}} |f(\varphi(z))|^2 (1 - |\Phi_a(z)|^2)^q \mathrm{d}V(z) \\
&\qquad \le 2 \int_{\mathbb{B}} \Big(|f(\varphi(z)) - f_j(\varphi(z))|^2 + |f_j(\varphi(z))|^2 \Big) (1 - |\Phi_a(z)|^2)^q \mathrm{d}V(z) \\
&\qquad = 2\|C_\varphi(f) - C_\varphi(f_j)\|_q^2 + 2 \int_{\mathbb{B}} |f_j(\varphi(z))|^2 (1 - |\Phi_a(z)|^2)^q \mathrm{d}V(z) \\
&\qquad < 2\left(\frac{\varepsilon}{4} + \frac{\varepsilon}{4} \right) = \varepsilon.
\end{aligned}$$

This shows that for all $r < |a| < 1$,

$$\sup_{f \in \mathbb{B}_{\mathcal{N}_p}} \int_{\mathbb{B}} |f(\varphi(z))|^2 (1 - |\Phi_a(z)|^2)^q \mathrm{d}V(z) \le \varepsilon,$$

which implies (8).

Now let $\{A_k\}$ be a decreasing sequence of measurable subsets of $\mathbb{B}$ whose intersection is empty. Since $\{f_1 \circ \varphi, \dots, f_M \circ \varphi\}$ is contained in $\mathcal{N}_q^0$, Lemma 2 on p. 118 shows that there exists an integer k_ε such that for any $k \ge k_\varepsilon$ and any $1 \le j \le M$,

$$\sup_{a \in \mathbb{B}} \int_{A_k} |f_j(\varphi(z))|^2 (1 - |\Phi_a(z)|^2)^q \mathrm{d}V(z) < \frac{\varepsilon}{4}. \tag{12}$$

Inequalities (10) and (12) together give

$$\begin{aligned}
&\sup_{a \in \mathbb{B}} \int_{A_k} |f(\varphi(z))|^2 (1 - |\Phi_a(z)|^2)^q \mathrm{d}V(z) \\
&\quad \le 2 \sup_{a \in \mathbb{B}} \int_{A_k} \Big(|f(\varphi(z)) - f_j(\varphi(z))|^2 + |f_j(\varphi(z))|^2 \Big) (1 - |\Phi_a(z)|^2)^q \mathrm{d}V(z) \\
&\quad \le 2\|C_\varphi(f) - C_\varphi(f_j)\|_q^2 + 2 \sup_{a \in \mathbb{B}} \int_{A_k} |f_j(\varphi(z))|^2 (1 - |\Phi_a(z)|^2)^q \mathrm{d}V(z) \\
&\quad < 2\left(\frac{\varepsilon}{4} + \frac{\varepsilon}{4} \right) = \varepsilon,
\end{aligned}$$

for any $k \ge k_\varepsilon$ and $f \in \mathbb{B}_{\mathcal{N}_p}$. The limit (9) then follows. □

Now we give the following characterization of the compactness of C_φ acting from $\mathcal{N}_p$ into $\mathcal{N}_q$.

Theorem 4 – *Let p, q be positive numbers and φ a holomorphic self-mapping of $\mathbb{B}$ such that the composition operator $C_\varphi : \mathcal{N}_p \longrightarrow \mathcal{N}_q$ is bounded. Let $E_1 \subset E_2 \subset \cdots \subset \mathbb{B}$ be an increasing sequence of measurable sets such that $\cup_{k\geq 1} E_k = \mathbb{B}$ and for each k, the closure $\overline{\varphi(E_k)}$ is compact in $\mathbb{B}$. Then $C_\varphi : \mathcal{N}_p \longrightarrow \mathcal{N}_q$ is compact if and only if*

$$\lim_{k\to\infty} \sup_{f\in\mathbb{B}_{\mathcal{N}_p}} \left[\sup_{a\in\mathbb{B}} \int_{\mathbb{B}\setminus E_k} |f(\varphi(z))|^2 (1-|\Phi_a(z)|^2)^q \mathrm{d}V(z)\right] = 0. \tag{13}$$

We recall here that $\mathbb{B}_{\mathcal{N}_p} = \{f \in \mathcal{N}_p : \|f\|_p \leq 1\}$ is the unit ball of $\mathcal{N}_p$.

Proof. **Necessity.** Set $A_k = \mathbb{B}\setminus E_k$ for all integers $k \geq 1$. Then $\{A_k\}_{k\geq 1}$ is a decreasing sequence of measurable subsets of $\mathbb{B}$ and

$$\bigcap_{k\geq 1} A_k = \mathbb{B} \setminus \Big(\bigcup_{k\geq 1} E_k\Big) = \varnothing.$$

If C_φ is a compact operator from $\mathcal{N}_p$ into $\mathcal{N}_q$, then (13) follows from Item 2 of Proposition 2 on p. 119.

Sufficiency. Suppose (13) holds. Take any bounded sequence $\{f_m\} \subset \mathcal{N}_p$ converging to zero uniformly on every compact subset of $\mathbb{B}$. By Lemma 1 on p. 118, it suffices to show that the sequence $\{\|f_m \circ \varphi\|_q\}$ converges to zero as $m \to \infty$.

Let $\varepsilon > 0$ be given. By (13), there exists a positive integer k such that for any $m \in \mathbb{N}$,

$$\sup_{a\in\mathbb{B}} \int_{\mathbb{B}\setminus E_k} |f_m(\varphi(z))|^2 (1-|\Phi_a(z)|^2)^q \mathrm{d}V(z) < \frac{\varepsilon}{2}. \tag{14}$$

On the other hand, since $\{f_m\}$ converges to zero uniformly on the compact set $\overline{\varphi(E_k)}$, for sufficiently large m, we have

$$\begin{aligned}\sup_{a\in\mathbb{B}} \int_{E_k} |f_m(\varphi(z))|^2 (1-|\Phi_a(z)|^2)^q \mathrm{d}V(z) &\leq \int_{E_k} |f_m(\varphi(z))|^2 \mathrm{d}V(z)\\ &\leq \sup_{z\in E_k} |f_m(\varphi(z))|^2\\ &= \sup_{w\in\varphi(E_k)} |f_m(w)|^2 < \frac{\varepsilon}{2}.\end{aligned}$$

This estimate together with (14) immediately yields

$$\|C_\varphi f_m\|_p^2 = \sup_{a\in\mathbb{B}} \int_{\mathbb{B}} |f_m(\varphi(z))|^2 (1-|\Phi_a(z)|^2)^q \mathrm{d}V(z) < \varepsilon,$$

for sufficiently large integers m. Consequently, $\|C_\varphi f_m\|_q \to 0$ as desired. □

By weakening condition (13) and adding an extra condition, we obtain an equivalent characterization for the compactness of C_φ as follows.

Theorem 5 – *Let p, q be positive numbers and φ a holomorphic self-mapping of $\mathbb{B}$ such that the composition operator $C_\varphi : \mathcal{N}_p \longrightarrow \mathcal{N}_q$ is bounded. Let $E_1 \subset E_2 \subset \cdots \subset \mathbb{B}$ be an increasing sequence of measurable sets such that $\cup_{k\geq 1} E_k = \mathbb{B}$ and for each k, the closure $\overline{\varphi(E_k)}$ is compact in $\mathbb{B}$. Then the following statements are equivalent.*

1. *C_φ is compact from $\mathcal{N}_p$ into $\mathcal{N}_q$.*
2. *C_φ is compact from $\mathcal{N}_p$ into $\mathcal{N}_q^0$.*
3. *The following two conditions are satisfied*

$$\lim_{k\to\infty}\left(\sup_{f\in\mathbb{B}_{\mathcal{N}_p}}\left[\int_{\mathbb{B}\setminus E_k}|f(\varphi(z))|^2(1-|z|^2)^q \mathrm{d}V(z)\right]\right)=0; \tag{15}$$

and

$$\lim_{|a|\to 1^-}\sup_{f\in\mathbb{B}_{\mathcal{N}_p}}\int_{\mathbb{B}}|f(\varphi(z))|^2(1-|\Phi_a(z)|^2)^q \mathrm{d}V(z)=0. \tag{16}$$

Proof. The equivalence of 1 and 2 comes from the fact that $\mathcal{N}_q^0$ is a subspace of $\mathcal{N}_q$ and Item 1 of Proposition 2 on p. 119, which shows that whenever $C_\varphi : \mathcal{N}_p \longrightarrow \mathcal{N}_q$ is compact, the range of C_φ is actually contained in $\mathcal{N}_q^0$. We only need to prove the equivalence of 1 and 3.

Suppose 1 holds, that is, $C_\varphi : \mathcal{N}_p \to \mathcal{N}_q$ is compact. Then (15) follows from Item 2 of Proposition 2 on p. 119 with $A_k = \mathbb{B}\setminus E_k$ and $a = 0$. In addition, (16) follows from Item 1 of Proposition 2 on p. 119.

Now suppose 3 hold, that is, both (15) and (16) are satisfied. Take any bounded sequence $\{f_m\} \subset \mathcal{N}_p$ converging to zero uniformly on every compact subset of $\mathbb{B}$. We may assume that $\|f_m\|_p \leq 1$ for each $m \in \mathbb{N}$. To show C_φ is compact, it suffices to show that

$$\lim_{m\to\infty}\|C_\varphi(f_m)\|_q = 0.$$

Let $\varepsilon > 0$ be given. By (16), there exists $0 < \delta < 1$ such that

$$\sup_{\delta<|a|<1}\left(\sup_{f\in\mathbb{B}_{\mathcal{N}_p}}\int_{\mathbb{B}}|f(\varphi(z))|^2(1-|\Phi_a(z)|^2)^q dV(z)\right)<\frac{\varepsilon}{3}. \tag{17}$$

By (15), there exists an integer $k \geq 1$ such that

$$\sup_{f\in\mathbb{B}_{\mathcal{N}_p}}\left[\int_{\mathbb{B}\setminus E_k}|f(\varphi(z))|^2(1-|z|^2)^q \mathrm{d}V(z)\right]<\frac{(1-\delta)^{2q}\cdot\varepsilon}{3}. \tag{18}$$

If $a \in \mathbb{B}$ with $|a| \le \delta$, then

$$1 - |\Phi_a(z)|^2 = \frac{(1-|a|^2)(1-|z|^2)}{|1-\langle z, a\rangle|^2} \le \frac{1-|z|^2}{(1-\delta)^2}.$$

This together with (18) implies

$$\begin{aligned}\sup_{f \in \mathbb{B}_{\mathcal{N}_p}} &\left(\sup_{|a|\le\delta} \int_{\mathbb{B}\setminus E_k} |f(\varphi(z))|^2 (1-|\Phi_a(z)|^2)^q \mathrm{d}V(z) \right) \\ &\le \frac{1}{(1-\delta)^{2q}} \sup_{f\in\mathbb{B}_{\mathcal{N}_p}} \int_{\mathbb{B}\setminus E_k} |f(\varphi(z))|^2 (1-|z|^2)^q \mathrm{d}V(z) < \frac{\varepsilon}{3}.\end{aligned}$$

Since $\{f_m\}$ converges to zero uniformly on the compact set $\overline{\varphi(E_k)}$, there exists $m_0 \in \mathbb{N}$ such that whenever $m > m_0$,

$$\begin{aligned}\sup_{a\in\mathbb{B}} \left(\int_{E_k} |f_m(\varphi(z))|^2 (1-|\Phi_a(z)|^2)^q \mathrm{d}V(z) \right) &\le \int_{E_k} |f_m(\varphi(z))|^2 \mathrm{d}V(z) \\ &\le \sup_{w\in\varphi(E_k)} |f_m(w)| < \frac{\varepsilon}{3}. \qquad (19)\end{aligned}$$

For $m > m_0$, by (17), (18) and (19), we have

$$\begin{aligned}\|C_\varphi(f_m)\|_q^2 &= \sup_{a\in\mathbb{B}} \int_{\mathbb{B}} |f_m(\varphi(z))|^2 (1-|\Phi_a(z)|^2)^q \mathrm{d}V(z) \\ &\le \sup_{\delta<|a|<1} \int_{\mathbb{B}} |f_m(\varphi(z))|^2 (1-|\Phi_a(z)|^2)^q \mathrm{d}V(z) \\ &\quad + \sup_{|a|\le\delta} \int_{\mathbb{B}} |f_m(\varphi(z))|^2 (1-|\Phi_a(z)|^2)^q \mathrm{d}V(z) \\ &\le \sup_{\delta<|a|<1} \int_{\mathbb{B}} |f_m(\varphi(z))|^2 (1-|\Phi_a(z)|^2)^q \mathrm{d}V(z) \\ &\quad + \sup_{|a|\le\delta} \int_{\mathbb{B}\setminus E_k} |f_m(\varphi(z))|^2 (1-|\Phi_a(z)|^2)^q \mathrm{d}V(z) \\ &\quad + \sup_{|a|\le\delta} \int_{E_k} |f_m(\varphi(z))|^2 (1-|\Phi_a(z)|^2)^q \mathrm{d}V(z) \\ &< \varepsilon.\end{aligned}$$

It follows that $\|C_\varphi f_m\|_q \to 0$ as required. □

By using Theorem 4 on p. 121 with certain choices of the sets $\{E_k\}_{k\ge1}$, we obtain somewhat more concrete criteria for the compactness of C_φ.

Corollary 3 – *Let p, q be positive numbers and φ a holomorphic self-mapping of $\mathbb{B}$ such that the composition operator $C_\varphi : \mathcal{N}_p \longrightarrow \mathcal{N}_q$ is bounded. Then the following statements are equivalent.*

1. *C_φ is a compact operator from $\mathcal{N}_p$ into $\mathcal{N}_q$.*
2. $\lim_{t\to 1^-}\Big(\sup_{a\in\mathbb{B},\ f\in\mathbb{B}_{\mathcal{N}_p}}\Big[\int_{|z|>t}|f(\varphi(z))|^2(1-|\Phi_a(z)|^2)^q \mathrm{d}V(z)\Big]\Big) = 0.$
3. $\lim_{t\to 1^-}\Big(\sup_{a\in\mathbb{B},\ f\in\mathbb{B}_{\mathcal{N}_p}}\Big[\int_{|\varphi(z)|>t}|f(\varphi(z))|^2(1-|\Phi_a(z)|^2)^q \mathrm{d}V(z)\Big]\Big) = 0.$

Proof. Observe that statement 2 and respectively, statement 3, is equivalent to the statement that for any sequence $\{t_k\}_{k\geq 1}$ of positive numbers increasing to 1, we have

$$\lim_{k\to\infty}\Big(\sup_{a\in\mathbb{B}, f\in\mathbb{B}_{\mathcal{N}_p}}\Big[\int_{|z|>t_k}|f(z)|^2(1-|\Phi_a(z)|^2)^q \mathrm{d}V(z)\Big]\Big) = 0,$$

and respectively,

$$\lim_{k\to\infty}\Big(\sup_{a\in\mathbb{B}, f\in\mathbb{B}_{\mathcal{N}_p}}\Big[\int_{|\varphi(z)|>t_k}|f(z)|^2(1-|\Phi_a(z)|^2)^q \mathrm{d}V(z)\Big]\Big) = 0.$$

For each integer $k \geq 1$, define $E_k = \{z : |z| \leq t_k\}$ in the case of statement 2 and $E_k = \{z : |\varphi(z)| \leq t_k\}$ in the case of statement 3. Since $\{t_k\}_{k\geq 1}$ is increasing to 1, we see that $\{E_k\}_{k\geq 1}$ is an increasing sequence of measurable sets and $\cup_{k=1}^{\infty} E_k = \mathbb{B}$. Furthermore, it is clear that the set $\overline{\varphi(E_k)}$ is compact for each k. The equivalence of 1 and 2 and the equivalence of 1 and 3 now follow from Theorem 4 on p. 121.□

Corollary 4 – *Let φ be a holomorphic self-mapping of $\mathbb{B}$ such that $\|\varphi\|_\infty < 1$. Then $C_\varphi : \mathcal{N}_p \longrightarrow \mathcal{N}_q$ is compact for all $p, q > 0$.*

Proof. By Corollary 2 on p. 117, the operator C_φ is bounded from $\mathcal{N}_p$ into $\mathcal{N}_q$. In addition, condition (d) in Corollary 3 is clearly satisfied since the set $\{|\varphi(z)| > t\}$ is empty for all $\|\varphi\|_\infty < t < 1$. Consequently, C_φ is compact. □

By changing the role of $\mathcal{N}_p$ to $\mathcal{N}_p^0$ in the proofs of Theorems 4 and 5 on p. 121 and on p. 122, we immediately obtain the following result describing the compactness of composition operators acting between $\mathcal{N}_p^0$ and $\mathcal{N}_q$.

Theorem 6 – *Let p, q be positive numbers and φ a holomorphic self-mapping of $\mathbb{B}$ such that the composition operator $C_\varphi : \mathcal{N}_p^0 \longrightarrow \mathcal{N}_q$ is bounded. Let $E_1 \subset E_2 \subset \cdots \subset \mathbb{B}$ be an increasing sequence of measurable sets such that $\cup_{k\geq 1} E_k = \mathbb{B}$ and for each k, the closure $\overline{\varphi(E_k)}$ is compact in $\mathbb{B}$. Then the following statements are equivalent.*

1. *C_φ is compact from $\mathcal{N}_p^0$ into $\mathcal{N}_q$.*

2. *C_φ is compact from $\mathcal{N}_p^0$ into $\mathcal{N}_q^0$.*

3. $$\lim_{k\to\infty} \sup_{f\in \mathbb{B}_{\mathcal{N}_p^0}} \left[\sup_{a\in\mathbb{B}} \int_{\mathbb{B}\setminus E_k} |f(\varphi(z))|^2(1-|\Phi_a(z)|^2)^q \mathrm{d}V(z)\right] = 0.$$

4. *The following two conditions are satisfied*

$$\lim_{k\to\infty}\left(\sup_{f\in \mathbb{B}_{\mathcal{N}_p^0}} \left[\int_{\mathbb{B}\setminus E_k} |f(\varphi(z))|^2(1-|z|^2)^q \mathrm{d}V(z)\right]\right) = 0;$$

and

$$\lim_{|a|\to 1^-} \sup_{f\in \mathbb{B}_{\mathcal{N}_p^0}} \int_{\mathbb{B}} |f(\varphi(z))|^2(1-|\Phi_a(z)|^2)^q \mathrm{d}V(z) = 0.$$

Here $\mathbb{B}_{\mathcal{N}_p^0} = \{f \in \mathcal{N}_p^0 : \|f\|_p \le 1\}$ is the unit ball of $\mathcal{N}_p^0$.

3.2 Some simplifications

Although Theorems 4 to 6 on p. 121, on p. 122 and on the preceding page offer several characterizations of the compactness of composition operators $C_\varphi : \mathcal{N}_p \longrightarrow \mathcal{N}_q$, the conditions are rather abstract and difficult to check. We shall provide in this section a necessary condition and a sufficient condition for the compactness of C_φ directly in terms of φ. These conditions seem to be more useful in applications.

Theorem 7 – *Let $p, q \in (0, n]$ be two positive numbers and φ a holomorphic self-mapping of $\mathbb{B}$ such that $C_\varphi : \mathcal{N}_p \longrightarrow \mathcal{N}_q$ is bounded. If*

$$\lim_{t\to 1^-} \sup_{a\in\mathbb{B}} \int_{|\varphi(z)|>t} \frac{(1-|\Phi_a(z)|^2)^q}{(1-|\varphi(z)|^2)^{n+1}} \mathrm{d}V(z) = 0, \tag{20}$$

then C_φ is compact.

Conversely, if $C_\varphi : \mathcal{N}_p \to \mathcal{N}_q$ is compact, then for $0 < \varepsilon \le p+1$,

$$\lim_{t\to 1^-} \sup_{a\in\mathbb{B}} \int_{|\varphi(z)|>t} \frac{(1-|\Phi_a(z)|^2)^q}{(1-|\varphi(z)|^2)^{p+1-\varepsilon}} \mathrm{d}V(z) = 0. \tag{21}$$

Proof. Suppose (20) holds. As in the proof of Theorem 3 on p. 117, there is a constant $C > 0$ such that for any $f \in \mathcal{N}_p$ with $\|f\|_p \le 1$,

$$|f(z)| \le C\|f\|_p (1-|z|^2)^{-(n+1)/2} \le C(1-|z|^2)^{-(n+1)/2}.$$

It implies that for any $0 < t < 1$ and any $a \in \mathbb{B}$,

$$\int_{|\varphi(z)|>t} |f(\varphi(z))|^2(1-|\Phi_a(z)|^2)^q \mathrm{d}V(z) \le C^2 \int_{|\varphi(z)|>t} \frac{(1-|\Phi_a(z)|^2)^q}{(1-|z|^2)^{n+1}} \mathrm{d}V(z).$$

Now (20) shows that statement 3 in Corollary 3 on p. 124 is satisfied. As a result, C_φ is a compact operator from $\mathcal{N}_p$ into $\mathcal{N}_q$.

Now suppose $C_\varphi : \mathcal{N}_p \longrightarrow \mathcal{N}_q$ is compact. Let $\mu > 0$ be given. By Corollary 3 on p. 124, there is a number $t_\mu \in (0,1)$ such that for all $t_\mu < t < 1$,

$$\sup_{a\in\mathbb{B}, f\in\mathbb{B}_{\mathcal{N}_p}} \int_{|\varphi(z)|>t} |f(\varphi(z))|^2(1-|\Phi_a(z)|^2)^q \mathrm{d}V(z) \leq \mu.$$

An application of Proposition 1 on p. 115 with $E = \{z \in \mathbb{B} : |\varphi(z)| > t\}$ yields

$$\sup_{a\in\mathbb{B}} \int_{|\varphi(z)|>t} \frac{(1-|\Phi_a(z)|^2)^q}{(1-|z|^2)^{p+1-\varepsilon}} \mathrm{d}V(z) \lesssim \mu,$$

for all $t_\mu < t < 1$. Since μ is arbitrary, (21) follows. □

As applications of Theorems 3 and 7 on p. 117 and on the previous page, we have the following result.

Corollary 5 – *Suppose $k > 0, p, q, r \in (0, n), r \geq q, \varepsilon \in (0, q+1)$ and φ is a holomorphic self-mapping of $\mathbb{B}$. The following statements hold.*

1. *If $C_\varphi : A^{-k(q+1-\varepsilon)} \longrightarrow A^{-k(n+1)}$ is a bounded operator, then $C_\varphi : \mathcal{N}_p \longrightarrow \mathcal{N}_r$ is a bounded operator;*
2. *If $C_\varphi : A^{-k(q+1-\varepsilon)} \longrightarrow A^{-k(n+1)}$ is a compact operator, then $C_\varphi : \mathcal{N}_p \longrightarrow \mathcal{N}_r$ is a compact operator.*

Proof. The boundedness of $C_\varphi : A^{-k(q+1-\varepsilon)} \longrightarrow A^{-k(n+1)}$ is equivalent to[16]

$$M := \sup_{z\in\mathbb{B}} \frac{(1-|z|^2)^{q+1-\varepsilon}}{(1-|\varphi(z)|^2)^{n+1}} < \infty.$$

By Theorem 3 on p. 117, it suffices to show that

$$\sup_{a\in\mathbb{B}} \int_{\mathbb{B}} \frac{(1-|\Phi_a(z)|^2)^r}{(1-|\varphi(z)|^2)^{n+1}} \mathrm{d}V(z) < \infty.$$

Indeed, we have

$$\begin{aligned}
&\sup_{a\in\mathbb{B}} \int_{\mathbb{B}} \frac{(1-|\Phi_a(z)|^2)^r}{(1-|\varphi(z)|^2)^{n+1}} \mathrm{d}V(z) \\
&\quad\leq M \sup_{a\in\mathbb{B}} \int_{\mathbb{B}} \frac{(1-|\Phi_a(z)|^2)^r}{(1-|z|^2)^{q+1-\varepsilon}} \mathrm{d}V(z) \\
&\quad= M \sup_{a\in\mathbb{B}} (1-|a|^2)^r \int_{\mathbb{B}} \frac{(1-|z|^2)^{r-q-1+\varepsilon}}{|1-\langle z,a\rangle|^{2r}} \mathrm{d}V(z)
\end{aligned}$$

(Cont. next page)

$$= M \sup_{a \in \mathbb{B}} (1 - |a|^2)^r \int_{\mathbb{B}} \frac{(1 - |z|^2)^{r-q-1+\varepsilon}}{|1 - \langle z, a \rangle|^{n+1+(r-q-1+\varepsilon)+(r+q-n-\varepsilon)}} \mathrm{d}V(z).$$

We have[17]

$$\sup_{a \in \mathbb{B}} (1 - |a|^2)^r \int_{\mathbb{B}} \frac{(1 - |z|^2)^{r-q-1+\varepsilon}}{|1 - \langle z, a \rangle|^{n+1+(r-q-1+\varepsilon)+(r+q-n-\varepsilon)}} \mathrm{d}V(z) < \infty,$$

which implies the desired result.

The proof of statement 2 is similar to that of 1 and we use the fact[18] that $C_\varphi : A^{-k(q+1-\varepsilon)} \longrightarrow A^{-k(n+1)}$ is compact if and only if

$$\lim_{t \to 1^-} \sup_{|\varphi(z)| > t} \frac{(1 - |z|^2)^{q+1-\epsilon}}{(1 - |\varphi(z)|^2)^{n+1}} = 0.$$

We leave the details for the interested reader. □

Acknowledgments

Supported in part by MOE's AcRF Tier 1 grant M4011166.110 (RG24/13).

References

Contreras, M. D. and A. G. Hernandez-Diaz (2000). "Weighted composition operators in weighted Banach spaces of analytic functions". *J. Austral. Math. Soc. Ser. A* **69** (1), pp. 41–60. ISSN: 0263-6115 (cit. on p. 127).

Cowen, C. C. and B. D. MacCluer (1995). *Composition operators on spaces of analytic functions*. Studies in Advanced Mathematics. CRC Press, Boca Raton, FL. 388 pp. ISBN: 0-8493-8492-3 (cit. on pp. 111, 118).

Hu, B. and L. H. Khoi (2013). "Weighted composition operators on $\mathcal{N}_p$-spaces in the ball". *C. R. Math. Acad. Sci. Paris* **351** (19-20), pp. 719–723. ISSN: 1631-073X. DOI: `10.1016/j.crma.2013.10.002` (cit. on pp. 112, 113).

Hu, B. and L. H. Khoi (2015). "Compact difference of weighted composition operators on $\mathcal{N}_p$-spaces in the ball". *Rev. Roumaine Math. Pures Appl.* **60** (2), pp. 101–116. ISSN: 0035-3965 (cit. on p. 113).

[16] By Contreras and Hernandez-Diaz, 2000, "Weighted composition operators in weighted Banach spaces of analytic functions", Proposition 3.1.

[17] Applying Rudin, 1980, *Function theory in the unit ball of* $\mathbb{C}^n$, Proposition 1.4.10.

[18] Contreras and Hernandez-Diaz, 2000, "Weighted composition operators in weighted Banach spaces of analytic functions", Corollary 4.3.

Hu, B., L. H. Khoi, and T. Le (2016a). "Essential norms of weighted composition operators on $\mathcal{N}_p$-spaces in the ball". *Vietnam J. Math.* **44** (2), pp. 431–439. ISSN: 2305-221X. DOI: 10.1007/s10013-015-0177-4 (cit. on p. 113).

Hu, B., L. H. Khoi, and T. Le (2016b). "On the structure of $\mathcal{N}_p$-spaces in the ball". arXiv: 1607.07415 [math.FA] (cit. on p. 116).

Palmberg, N. (2007). "Composition operators acting on $\mathcal{N}_p$-spaces". *Bull. Belg. Math. Soc. Simon Stevin* **14** (3), pp. 545–554. ISSN: 1370-1444 (cit. on pp. 112, 114).

Rudin, W. (1980). *Function theory in the unit ball of* $\mathbb{C}^n$. **241**. Springer-Verlag, New York-Berlin. 436 pp. ISBN: 0-387-90514-6. DOI: 10.1007/978-1-4613-8098-6 (cit. on pp. 114, 116, 127).

Ryll, J. and P. Wojtaszczyk (1983). "On homogeneous polynomials on a complex ball". *Trans. Amer. Math. Soc.* **276** (1), pp. 107–116. ISSN: 0002-9947. DOI: 10.2307/1999419 (cit. on p. 115).

Shapiro, J. H. (1993). *Composition operators and classical function theory*. Universitext: Tracts in Mathematics. Springer-Verlag, New York. 223 pp. ISBN: 0-387-94067-7. DOI: 10.1007/978-1-4612-0887-7 (cit. on p. 111).

Ueki, S.-i. (2012). "Weighted composition operators acting between the $\mathcal{N}_p$-space and the weighted-type space H^∞_α". *Indag. Math. (N.S.)* **23** (3), pp. 243–255. ISSN: 0019-3577. DOI: 10.1016/j.indag.2011.11.006 (cit. on pp. 112, 115).

Xiao, J. (2001). *Holomorphic Q classes*. **1767**. Lecture Notes in Mathematics. Springer-Verlag, Berlin. 112 pp. ISBN: 3-540-42625-6. DOI: 10.1007/b87877 (cit. on p. 112).

Contents