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A combinatorial definition of the Θ-invariant from Heegaard diagrams

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Abstract

The invariant Θ is the simplest 3-manifold invariant defined by counting graph configurations. It is actually an invariant of rational homology 3-spheres M equipped with a combing X over the complement of a point, where a combing is a homotopy class of nowhere vanishing vector fields. The invariant $\Theta(M,X)$ is the sum of $6\lambda(M)$ and $p_1(X)/4$, where λ denotes the Casson-Walker invariant, and p_1 is an invariant of combings, which is an extension of a first relative Pontrjagin class, and which is simply related to a Gompf invariant θ_G. In Lescop (2015a), we proved a combinatorial formula for the Θ-invariant in terms of decorated Heegaard diagrams. In this article, we study the variations of the invariants p_1 or θ_G when the decorations of the Heegaard diagrams that define the combings change, independently. Then we prove that the formula of Lescop (2015a) defines an invariant of combed once punctured rational homology 3-spheres without referring to configuration spaces. Finally, we prove that this invariant is the sum of $6\lambda(M)$ and $p_1(X)/4$ for integer homology 3-spheres, by proving surgery formulae both for the combinatorial invariant and for p_1.

Keywords: Θ-invariant, Heegaard splittings, Heegaard diagrams, combings, Gompf invariant, Casson-Walker invariant, finite type invariants of 3-manifolds, homology spheres, configuration space integrals, perturbative expansion of Chern-Simons theory.

MSC: 57M27, 57N10, 57M25, 55R80.

1 Introduction

In this article, a $\mathbb{Q}$-*sphere* or *rational homology* 3-*sphere* (resp. a $\mathbb{Z}$-*sphere* or *integer homology* 3-*sphere*) is a smooth closed oriented 3-manifold that has the same rational (resp. integral) homology as S^3.

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1.1 General introduction

The work of Witten[2] pioneered the introduction of many $\mathbb{Q}$-sphere invariants, among which the Le-Murakami-Ohtsuki universal finite type invariant[3] and the Kontsevich configuration space invariant[4], which was proved to be equivalent to the LMO invariant for integer homology 3-spheres by Kuperberg and Thurston[5]. The construction of the Kontsevich configuration space invariant for a $\mathbb{Q}$-sphere M involves a point ∞ in M, an identification of a neighborhood of ∞ with a neighborhood $S^3 \setminus B(1)$ of ∞ in $S^3 = \mathbb{R}^3 \cup \{\infty\}$, and a parallelization τ of $(\check{M} = M \setminus \{\infty\})$ that coincides with the standard parallelization of $\mathbb{R}^3$ on $\mathbb{R}^3 \setminus B(1)$, where $B(r)$ denotes the ball centered at 0 with radius r in $\mathbb{R}^3$. The Kontsevich configuration space invariant is in fact an invariant of (M,τ). Its degree one part $\Theta(M,\tau)$ is the sum of $6\lambda(M)$ and $p_1(\tau)/4$, where λ is the Casson-Walker invariant and p_1 is a Pontrjagin number associated with τ, according to a theorem of Kuperberg and Thurston[6] generalized to rational homology 3-spheres in Lescop (2004b). Here, the Casson-Walker invariant λ is normalized like in Akbulut and McCarthy (1990), Guillou and Marin (1992), and Marin (1988) for integer homology 3-spheres, and like $\frac{1}{2}\lambda_W$ for rational homology 3-spheres where λ_W is the Walker normalisation in Walker (1992).

Let B_M denote the complement in M of the neighborhood of ∞ identified with $S^3 \setminus B(1)$, B_M is a rational homology ball. An *∞-combing* of such a rational homology 3-sphere M is a section of the unit tangent bundle $U\check{M}$ of $\check{M}$ that is constant on $\check{M} \setminus B_M$ (via the identifications above with $\mathbb{R}^3 \setminus B(1)$ near ∞), up to homotopies through this kind of sections. As it is shown in Lescop (2015a), $\Theta(M,.)$ is actually an invariant of rational homology 3-spheres equipped with such ∞-combings.

In this article, a *genus g handlebody* is the 3-manifold bounded by the genus g surface embedded in a standard way in $\mathbb{R}^3$ as in Figure 1 on the next page. Every closed oriented 3-manifold M can be written as the union of two handlebodies $H_{\mathcal{A}}$ and $H_{\mathcal{B}}$ glued along their common boundary, which is a genus g surface, as

$$M = H_{\mathcal{A}} \cup_{\partial H_{\mathcal{A}}} H_{\mathcal{B}}$$

where $\partial H_{\mathcal{A}} = -\partial H_{\mathcal{B}}$. Such a decomposition is called a *Heegaard decomposition* of M. A *system of meridians* for $H_{\mathcal{A}}$ is a system of g disjoint curves α_i of $\partial H_{\mathcal{A}}$ that bound disjoint disks $D(\alpha_i)$ properly embedded in $H_{\mathcal{A}}$ such that the union of the α_i does not separate $\partial H_{\mathcal{A}}$. For a positive integer g, we will denote the set $\{1,2,\dots,g\}$ by $\underline{g}$. Let $(\alpha_i)_{i\in\underline{g}}$ be a system of meridians for $H_{\mathcal{A}}$ and let $(\beta_j)_{j\in\underline{g}}$ be such a system for $H_{\mathcal{B}}$. Then the surface equipped with the collections of the curves α_i and the curves

[2] Witten, 1989, "Quantum field theory and the Jones polynomial".
[3] Le, Murakami, and Ohtsuki, 1998, "On a universal perturbative invariant of 3-manifolds".
[4] Kontsevich, 1994, "Feynman diagrams and low-dimensional topology".
[5] Kuperberg and Thurston, 1999, "Perturbative 3-manifold invariants by cut-and-paste topology".
[6] Ibid.

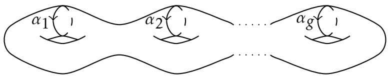

Figure 1 – A genus g handlebody equipped with a system $\{\alpha_i\}_{i\in\underline{g}}$ of meridians

$\beta_j = \partial D(\beta_j)$ determines M. When the collections $(\alpha_i)_{i\in\underline{g}}$ and $(\beta_j)_{j\in\underline{g}}$ are transverse, the data $\mathcal{D} = (\partial H_{\mathcal{A}}, (\alpha_i)_{i\in\underline{g}}, (\beta_j)_{j\in\underline{g}})$ is called a *Heegaard diagram*.

Such a Heegaard diagram may be obtained from a Morse function f_M of M that has one minimum mapped to (-3), one maximum mapped to 9, g index one points a_i and g index 2 points b_j, such that f_M maps index 1 points to 1 and index 2 points to 5, and f_M satisfies generic Morse-Smale conditions ensuring transversality of descending and ascending manifolds of critical points, with respect to a Euclidean metric $\mathfrak{g}$ of M. Thus the surface $\partial H_{\mathcal{A}}$ is $f_M^{-1}(3)$, the ascending manifolds of the a_i intersect $H_{\mathcal{A}}$ as disks $D(\alpha_i)$ bounded by the α_i and the descending manifolds of the b_j intersect $H_{\mathcal{B}}$ as disks $D(\beta_j)$ bounded by the β_j, and the flow line closures from a_i to b_j are in natural one-to-one correspondence with the crossings of $\alpha_i \cap \beta_j$. Conversely, for any Heegaard diagram, there exists a Morse function f_M with the properties above.

A *matching* in a genus g Heegaard diagram

$$\mathcal{D} = (\partial H_{\mathcal{A}}, \{\alpha_i\}_{i=1,\dots,g}, \{\beta_j\}_{j=1,\dots,g})$$

is a set $\mathfrak{m}$ of g crossings such that every curve of the diagram contains one crossing of $\mathfrak{m}$. An *exterior point* in such a diagram $\mathcal{D}$ is a point of $\partial H_{\mathcal{A}} \setminus \left(\coprod_{i=1}^{g} \alpha_i \cup \coprod_{j=1}^{g} \beta_j\right)$. The choice of a matching $\mathfrak{m}$ and of an exterior point w in a diagram $\mathcal{D}$ of M equips M with the following ∞-combing $X(w,\mathfrak{m}) = X(\mathcal{D}, w, \mathfrak{m})$.

Remove an open ball around the flow line from the minimum to the maximum that goes through w, so that we are left with a rational homology ball

$$B_M(2) = B_M \cup_{\partial B(1) = \partial B_M} B(2) \setminus \mathring{B}(1)$$

where the gradient field of f_M is vertical near the boundary. Reversing the gradient field along the flow lines $\gamma(c)$ through the crossings c of $\mathfrak{m}$ as in Section 3.1 on p. 31 produces the ∞-combing $X(w,\mathfrak{m})$ of M.

Let θ_G denote the invariant of combings of rational homology 3-spheres introduced by Gompf in Gompf (1998, Section 4). A choice of a standard modification described in Section 4.2 on p. 37 of $X(w,\mathfrak{m})$ in the fixed neighborhood of ∞ identified with $S^3 \setminus B(2)$ transforms $X(w,\mathfrak{m})$ into a combing $X(M,w,\mathfrak{m})$ such that $p_1(X(w,\mathfrak{m})) - \theta_G(X(M,w,\mathfrak{m}))$ is independent of $(M,w,\mathfrak{m})$.

In Lescop (2015a, Theorem 1.5), we express $\Theta(M, X(w,\mathfrak{m}))$ as a combination

$$\tilde{\Theta}(\mathcal{D}, w, \mathfrak{m}) = \ell_2(\mathcal{D}) + s_\ell(\mathcal{D}, \mathfrak{m}) - e(\mathcal{D}, w, \mathfrak{m})$$

of invariants of Heegaard diagrams $\mathcal{D}$ equipped with a matching $\mathfrak{m}$ and an exterior point w. First combinatorial expressions of the ingredients $\ell_2(\mathcal{D})$, $s_\ell(\mathcal{D},\mathfrak{m})$, and $e(\mathcal{D},w,\mathfrak{m})$ are given in the end of this introduction section whereas Section 2 on p. 24 provides alternative expressions and properties of these quantities.

In this article, we give several expressions of the variations of $p_1(X(w,\mathfrak{m}))$, or, equivalently of $\theta_G(X(M,w,\mathfrak{m}))$, when w and $\mathfrak{m}$ vary, for a fixed Heegaard diagram. Expressions in terms of linking numbers are given in Section 3.2 on p. 33 and derived combinatorial expressions can be found in Section 4 on p. 36.

The latter ones allow us to give combinatorial proofs that

$$\left(4\tilde{\Theta}(\mathcal{D},w,\mathfrak{m})-p_1(X(w,\mathfrak{m}))\right)$$

is independent of $(w,\mathfrak{m})$ in Section 5 on p. 47. We prove that

$$\tilde{\lambda}(\mathcal{D})=\frac{1}{24}\left(4\tilde{\Theta}(\mathcal{D},w,\mathfrak{m})-p_1(X(w,\mathfrak{m}))\right)$$

only depends on the presented rational homology 3-sphere M, combinatorially, in Section 6 on p. 52. We set $\tilde{\lambda}(M)=\tilde{\lambda}(\mathcal{D})$ so that $\tilde{\lambda}$ is a topological invariant of rational homology 3-spheres.

Then we give a direct combinatorial proof that $\tilde{\lambda}$ satisfies the Casson surgery formula for $\frac{1}{n}$-Dehn surgeries along null-homologous knots in Section 7 on p. 68. This implies that $\tilde{\lambda}$ coincides with the Casson invariant for integer homology 3-spheres. Our proof also yields a surgery formula for p_1, which is stated in Theorem 6 on p. 70.

Thus this article contains an independent construction of the Casson invariant, which includes a direct proof of the Casson surgery formula, and an independent combinatorial proof of the formula of Lescop (2015a, Theorem 3.8) for the Θ-invariant in terms of Heegaard diagrams in the case of $\mathbb{Z}$-spheres. It also describes the behaviour of the four quantities $\ell_2(\mathcal{D})$, $s_\ell(\mathcal{D},\mathfrak{m})$, $e(\mathcal{D},w,\mathfrak{m})$ and $p_1(X(\mathcal{D},w,\mathfrak{m}))$ (or equivalently $\theta_G(X(M,w,\mathfrak{m}))$) associated with Heegaard diagrams $\mathcal{D}$ decorated with $(w,\mathfrak{m})$ under standard modifications of Heegaard diagrams, and Dehn surgeries. These quantities might show up in combinatorial definitions of other invariants from Heegaard diagrams, as θ_G, which Gripp and Huang use to define the Heegaard Floer homology $\widehat{HF}$ grading in Ramos and Huang (2017).

The definitions introduced in Lescop (2015a) are recalled here for the reader's convenience.

I thank Jean-Mathieu Magot for useful conversations.

1.2 Conventions and notations

Unless otherwise mentioned, all manifolds are oriented. Boundaries are oriented by the outward normal first convention. Products are oriented by the order of the factors. More generally, unless otherwise mentioned, the order of appearance of coordinates or parameters orients chains or manifolds. For a manifold M, $(-M)$ denotes the manifold obtained from M by reversing its orientation. The normal bundle $\mathfrak{N}(A)$ of an oriented submanifold A is oriented so that the normal bundle followed by the tangent bundle of the submanifold induce the orientation of the ambient manifold, fiberwise. The transverse intersection of two submanifolds A and B of a manifold C is oriented so that the normal bundle $\mathfrak{N}_x(A \cap B)$ of $A \cap B$ at x is oriented as $(\mathfrak{N}_x(A) \oplus \mathfrak{N}_x(B))$. When the dimensions of two such submanifolds add up to the dimension of C, each intersection point is equipped with a sign ± 1, which is 1 if and only if $(\mathfrak{N}_x(A) \oplus \mathfrak{N}_x(B))$ (or equivalently $(T_x(A) \oplus T_x(B))$) induces the orientation of C. When A is compact, the sum of the signs of the intersection points is the *algebraic intersection number* $\langle A, B\rangle_C$. The *linking number* $lk(L_1, L_2) = lk_C(L_1, L_2)$ of two disjoint null-homologous cycles L_1 and L_2 of respective dimensions d_1 and d_2 in an oriented $(d_1 + d_2 + 1)$-manifold C is the algebraic intersection $\langle L_1, W_2\rangle_C$ of L_1 with a chain W_2 bounded by L_2 in C. This definition extends to rationally null-homologous cycles by bilinearity.

1.3 Introduction to the combinatorial definition of $\tilde{\Theta}$

In the end of this section, we give explicit formulas for the ingredients $\ell_2(\mathcal{D})$, $s_\ell(\mathcal{D}, \mathfrak{m})$ and $e(\mathcal{D}, w, \mathfrak{m})$ in the formula

$$\tilde{\Theta}(\mathcal{D}, w, \mathfrak{m}) = \ell_2(\mathcal{D}) + s_\ell(\mathcal{D}, \mathfrak{m}) - e(\mathcal{D}, w, \mathfrak{m})$$

for a Heegaard diagram $\mathcal{D}$ equipped with a matching $\mathfrak{m}$ and an exterior point w. These ingredients will be studied in more details in Section 2 on p. 24.

Let $\mathcal{D} = (\partial H_{\mathcal{A}}, (\alpha_i)_{i \in \underline{g}}, (\beta_j)_{j \in \underline{g}})$ be a Heegaard diagram of a rational homology 3-sphere. A *crossing* c of $\mathcal{D}$ is an intersection point of a curve $\alpha_{i(c)} = \alpha(c)$ and a curve $\beta_{j(c)} = \beta(c)$. Its sign $\sigma(c)$ is 1 if $\partial H_{\mathcal{A}}$ is oriented by the oriented tangent vector of $\alpha(c)$ followed by the oriented tangent vector of $\beta(c)$ at c as above. It is (-1) otherwise. The set of crossings of $\mathcal{D}$ is denoted by $\mathcal{C}$.

Let

$$[\mathcal{J}_{ji}]_{(j,i) \in \underline{g}^2} = [\langle \alpha_i, \beta_j \rangle_{\partial H_{\mathcal{A}}}]^{-1}$$

denote the inverse matrix of the intersection matrix.

$$\sum_{i=1}^{g} \mathcal{J}_{ji} \langle \alpha_i, \beta_k \rangle_{\partial H_{\mathcal{A}}} = \delta_{jk} = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{otherwise.} \end{cases}$$

When d and e are two crossings of α_i, $[d, e]_{\alpha_i} = [d, e]_\alpha$ denotes the set of crossings from d to e (including them) along α_i, or the closed arc from d to e in α_i depending on the context. Then $[d, e[_\alpha = [d, e]_\alpha \setminus \{e\}$, $]d, e]_\alpha = [d, e]_\alpha \setminus \{d\}$ and $]d, e[_\alpha = [d, e[_\alpha \setminus \{d\}$.

Now, for such a part I of α_i,

$$\langle I, \beta_j \rangle = \langle I, \beta_j \rangle_{\partial H_{\mathcal{A}}} = \sum_{c \in I \cap \beta_j} \sigma(c).$$

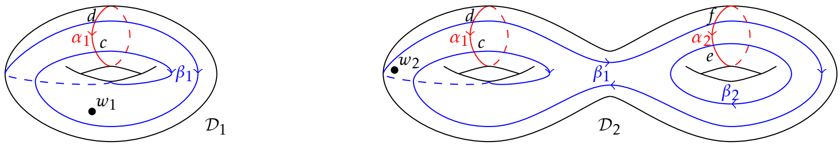

Figure 2 – Two Heegaard diagrams of $\mathbb{RP}^3$

We use the notation | for ends of arcs to say that an end is half-contained in an arc, and that it must be counted with coefficient 1/2. ("$[d,e|_\alpha = [d,e]_\alpha \setminus \{e\}/2$"). We agree that $|d,d|_\alpha = \emptyset$.

We use the same notation for arcs $[d,e|_{\beta_j} = [d,e|_\beta$ of β_j. For example, if d is a crossing of $\alpha_i \cap \beta_j$, then

$$\langle [d,d|_\alpha, \beta_j\rangle = \frac{\sigma(d)}{2}$$

and

$$\langle [c,d|_\alpha, [e,d|_\beta\rangle = \frac{\sigma(d)}{4} + \sum_{c \in [c,d[_\alpha \cap [e,d[_\beta} \sigma(c).$$

Example 1 – In the Heegaard diagrams of $\mathbb{RP}^3$ in Figure 2, $\langle [c,c|_\alpha,\ [c,c|_\beta\rangle = \frac{1}{4}$, $\langle [c,c|_\alpha,\ [c,d|_\beta\rangle = \langle [c,d|_\alpha,\ [c,c|_\beta\rangle = \frac{1}{2}$, $\langle [c,d|_\alpha,\ [c,d|_\beta\rangle = \frac{5}{4}$, $\langle [c,c|_\alpha, \beta_1\rangle = \frac{1}{2}$, $\langle [c,d|_\alpha, \beta_1\rangle = \frac{3}{2}$.

1.4 First combinatorial definitions of ℓ_2 and $s_\ell(\mathcal{D}, \mathfrak{m})$

Choose a matching $\mathfrak{m} = \{m_i; i \in \underline{g}\}$ where $m_i \in \alpha_{\rho^{-1}(i)} \cap \beta_i$, for a permutation ρ of $\underline{g}$. For two crossings c and d of $\mathcal{C}$, set

$$\tilde{\ell}_{\mathfrak{m}}(c,d) = \langle |m_{\rho(i(c))}, c|_\alpha, |m_{j(d)}, d|_\beta\rangle - \sum_{(i,j)\in \underline{g}^2} \mathcal{J}_{ji} \langle |m_{\rho(i(c))}, c|_\alpha, \beta_j\rangle \langle \alpha_i, |m_{j(d)}, d|_\beta\rangle.$$

Then

$$\ell_2(\mathcal{D}) = \sum_{(c,d)\in\mathcal{C}^2} \mathcal{J}_{j(c)i(d)} \mathcal{J}_{j(d)i(c)} \sigma(c)\sigma(d)\tilde{\ell}_{\mathfrak{m}}(c,d) - \sum_{c\in\mathcal{C}} \mathcal{J}_{j(c)i(c)}\sigma(c)\tilde{\ell}_{\mathfrak{m}}(c,c)$$

and

$$s_\ell(\mathcal{D}, \mathfrak{m}) = \sum_{(c,d)\in\mathcal{C}^2} \mathcal{J}_{j(c)i(c)} \mathcal{J}_{j(d)i(d)} \sigma(c)\sigma(d)\tilde{\ell}_{\mathfrak{m}}(c,d).$$

Example 2 – For the genus one Heegaard diagram $\mathcal{D}_1$ of Figure 2 on the preceding page, we have $\sigma(c) = 1$, $\langle\alpha_1, \beta_1\rangle_{\partial H_{\mathcal{A}}} = 2$, $\mathcal{J}_{11} = \frac{1}{2}$, choose $\{c\}$ as a matching, $\tilde{\ell}_{\{c\}}(c,c) = \tilde{\ell}_{\{c\}}(c,d) = \tilde{\ell}_{\{c\}}(d,c) = 0$, $\tilde{\ell}_{\{c\}}(d,d) = \frac{1}{2} - \mathcal{J}_{11} = 0$ so that $\ell_2(\mathcal{D}_1) = s_\ell(\mathcal{D}_1, \{c\}) = 0$.

For the genus two Heegaard diagram $\mathcal{D}_2$ of Figure 2 on the preceding page, $\mathcal{J}_{11} = \frac{1}{2} = -\mathcal{J}_{21}$, $\mathcal{J}_{22} = 1$ and $\mathcal{J}_{12} = 0$. Choose $\{c,e\}$ as a matching. For any crossing x of $\mathcal{D}_2$,

$$0 = \tilde{\ell}_{\{c,e\}}(c,x) = \tilde{\ell}_{\{c,e\}}(x,c) = \tilde{\ell}_{\{c,e\}}(e,x) = \tilde{\ell}_{\{c,e\}}(x,e) = \tilde{\ell}_{\{c,e\}}(d,d),$$

and

$$\begin{aligned}
\tilde{\ell}_{\{c,e\}}(f,f) &= \frac{1}{4} - \frac{3}{4}\mathcal{J}_{11} - \frac{1}{4}\mathcal{J}_{12} - \frac{3}{4}\mathcal{J}_{21} - \frac{1}{4}\mathcal{J}_{22} = 0\\
\tilde{\ell}_{\{c,e\}}(d,f) &= \frac{3}{4} - \frac{3}{2}\mathcal{J}_{11} - \frac{1}{2}\mathcal{J}_{12} = 0\\
\tilde{\ell}_{\{c,e\}}(f,d) &= -\frac{1}{2}\mathcal{J}_{11} - \frac{1}{2}\mathcal{J}_{21} = 0
\end{aligned}$$

so that $\ell_2(\mathcal{D}_2) = s_\ell(\mathcal{D}_2, \{c,e\}) = 0$.

1.5 Combinatorial definition of $e(\mathcal{D}, w, \mathfrak{m})$

Let w be an exterior point of $\mathcal{D}$. The choice of $\mathfrak{m}$ being fixed, represent the Heegaard diagrams in a plane by removing from $\partial H_{\mathcal{A}}$ a disk around w that does not intersect the diagram curves, and by cutting the surface $\partial H_{\mathcal{A}}$ along the α_i. Each α_i gives rise to two copies α_i' and α_i'' of α_i, which are represented as the boundaries of two disjoint disks with opposite orientations in the plane. Locate the crossing m_i at the points with upward tangent vectors of α_i' and α_i'', and locate the other crossings near the points with downward tangent vectors as in Figure 3. Draw the arcs of the curves β_j so that they have horizontal tangent vectors near the crossings.

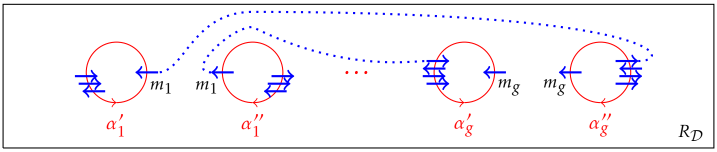

Figure 3 – The Heegaard surface cut along the α_i and deprived of a neighborhood of w

The rectangle has the standard parallelization of the plane. Then there is a map "unit tangent vector" from each partial projection of a beta curve β_j in the plane to S^1. The total degree of this map for the curve β_j is denoted by $d_e(\beta_j)$. For a crossing $c \in \beta_j$, $d_e(|m_j, c|_\beta) \in \frac{1}{2}\mathbb{Z}$ denotes the degree of the restriction of this map to the arc

$|m_j, c|_\beta$. This degree is the average of the degrees of this map at the upward vertical vector and at the downward one. For any $c \in \mathcal{C}$, define

$$d_e(c) = d_e(|m_{j(c)}, c|_\beta) - \sum_{(r,s)\in \underline{g}^2} \mathcal{J}_{sr}\langle \alpha_r, |m_{j(c)}, c|_\beta \rangle d_e(\beta_s).$$

Set

$$e(\mathcal{D}, w, \mathfrak{m}) = \sum_{c\in\mathcal{C}} \mathcal{J}_{j(c)i(c)}\sigma(c) d_e(c)$$

so that the combinatorial expression

$$\tilde{\Theta}(\mathcal{D}, w, \mathfrak{m}) = \ell_2(\mathcal{D}) + s_\ell(\mathcal{D}, \mathfrak{m}) - e(\mathcal{D}, w, \mathfrak{m}),$$

which is studied in this article, is completely defined.

Example 3 – For the rectangular diagram of $(\mathcal{D}_1, \{c\}, w_1)$ of Figure 4, $d_e(|c,c|_\beta) = 0$ and $d_e(c) = 0$, $d_e(|c,d|_\beta) = \frac{1}{2}$, $d_e(\beta_1) = 2$ so that $d_e(d) = -\frac{1}{2}$, $e(\mathcal{D}_1, w_1, \{c\}) = -\frac{1}{4}$ and $\tilde{\Theta}(\mathcal{D}_1, w_1, \{c\}) = \frac{1}{4}$.

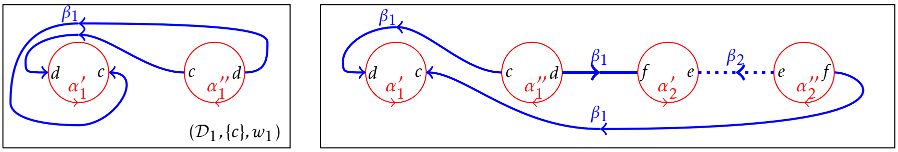

Figure 4 – Rectangular diagrams of $(\mathcal{D}_1, \{c\}, w_1)$ and $(\mathcal{D}_2, \{c,e\}, w_2)$

For the rectangular diagram of $(\mathcal{D}_2, \{c,e\}, w_2)$ of Figure 4, $d_e(c) = d_e(e) = d_e(\beta_1) = d_e(\beta_2) = 0$, $d_e(d) = d_e(f) = \frac{1}{2}$, $e(\mathcal{D}_2, w_2, \{c,e\}) = \frac{1}{4}$ and $\tilde{\Theta}(\mathcal{D}_2, w_2, \{c,e\}) = -\frac{1}{4}$.

2 More on the combinatorial definition of $\tilde{\Theta}$

In this section, we show that the quantities $\ell_2(\mathcal{D})$, $s_\ell(\mathcal{D}, \mathfrak{m})$ and $e(\mathcal{D}, w, \mathfrak{m})$ defined in the previous section for a Heegaard diagram $\mathcal{D}$ equipped with a matching $\mathfrak{m}$ and an exterior point w only depend on their arguments (e.g. on $\mathcal{D}$, for $\ell_2(\mathcal{D})$ …) and not on extra data used to define them like numberings or orientations of the diagram curves. We also give alternative definitions of $\ell_2(\mathcal{D})$ and $s_\ell(\mathcal{D}, \mathfrak{m})$.

2.1 More on $e(\mathcal{D}, w, \mathfrak{m})$

Recall the notation and definitions of Section 1.5 on p. 23 with respect to a fixed matching $\mathfrak{m} = \{m_i; i \in \underline{g}\}$ where $m_i \in \alpha_{\rho^{-1}(i)} \cap \beta_i$, for a permutation ρ of $\underline{g}$.

Lemma 1 – *The number*

$$e(\mathcal{D}, w, \mathfrak{m}) = \sum_{c \in \mathcal{C}} \mathcal{J}_{j(c)i(c)} \sigma(c) d_e(c)$$

depends neither on our specific way of drawing the diagram with our conventions, nor on the orientations of the diagram's curves. It only depends on $\mathcal{D}$, w and $\mathfrak{m}$.

The topological interpretation of $e(\mathcal{D}, w, \mathfrak{m})$ as an Euler class given in Corollary 2 on p. 37 yields a conceptual proof of this lemma. We nevertheless give a purely combinatorial proof below.

We use the Kronecker symbol δ_{cd}, which is 1 if $c = d$ and 0 otherwise. We first prove the following lemma.

Lemma 2 – *A full positive twist of a curve α'_i or a curve α''_i in Figure 3 on p. 23 changes $d_e(c)$ to $d_e(c) + \frac{1}{2}\delta_{i(c)i} - \frac{1}{2}\delta_{\rho(i)j(c)}$.*

Proof. When a crossing is moved counterclockwise along a curve α, (like along α''_i in Figure 15 on p. 50) the degree increases (by 1 for a full loop) when the crossing enters (the disk bounded by) α in Figure 3 on p. 23 and decreases when the crossing goes out. Furthermore the positive crossings enter α'_i and the negative ones enter α''_i. Then letting all the crossings make a full positive loop around α''_i (resp. around α'_i) changes $d_e(\beta_s)$ to $d_e(\beta_s) - \langle \alpha_i, \beta_s \rangle$ (resp. to $d_e(\beta_s) + \langle \alpha_i, \beta_s \rangle$). Now, for a full positive loop around α''_i, $d_e(|m_{j(c)}, c|_\beta)$ is changed to

$$d_e(|m_{j(c)}, c|_\beta) - \langle \alpha_i,]m_{j(c)}, c[_\beta \rangle - \delta_{i(c)i}\delta_{(-1)\sigma(c)}\sigma(c) - \delta_{\rho(i)j(c)}\delta_{1\sigma(m_{j(c)})}\sigma(m_{j(c)}).$$

Indeed, right before c, $\beta_{j(c)}$ hits α''_i if and only if $\sigma(c) = -1$ and $i(c) = i$. Similarly, after $m_{j(c)}$, $\beta_{j(c)}$ exits α''_i if and only if $\sigma(m_{j(c)}) = 1$ and $\rho^{-1}(j(c)) = i$. This expression can be rewritten as

$$d_e(|m_{j(c)}, c|_\beta) - \langle \alpha_i, |m_{j(c)}, c|_\beta \rangle + \frac{1}{2}\delta_{i(c)i} - \frac{1}{2}\delta_{\rho(i)j(c)}.$$

Similarly, for a full positive loop around α'_i, $d_e(|m_{j(c)}, c|_\beta)$ is changed to

$$d_e(|m_{j(c)}, c|_\beta) + \langle \alpha_i, |m_{j(c)}, c|_\beta \rangle + \frac{1}{2}\delta_{i(c)i} - \frac{1}{2}\delta_{\rho(i)j(c)}.$$

Now, since

$$\sum_{(r,s) \in \underline{g}^2} \mathcal{J}_{sr} \langle \alpha_r, |m_{j(c)}, c|_\beta \rangle \langle \alpha_i, \beta_s \rangle = \langle \alpha_i, |m_{j(c)}, c|_\beta \rangle,$$

$d_e(c)$ is changed to $d_e(c) + \frac{1}{2}\delta_{i(c)i} - \frac{1}{2}\delta_{\rho(i)j(c)}$ in both cases. □

Proof (of Lemma 1 on the previous page). Note that $e(\mathcal{D},w,\mathfrak{m})$ does not depend on the numberings of the diagram curves. We prove that $e(\mathcal{D},w,\mathfrak{m})$ does not depend on our specific way of drawing the diagram with our conventions when the orientations of the diagram curves are fixed. When the curves α'_i and α''_i move in the plane without being twisted, the $d_e(c)$ stay in $\frac{1}{2}\mathbb{Z}$ and are therefore invariant. Therefore it suffices to prove that $e(\mathcal{D},w,\mathfrak{m})$ is invariant under a full twist of a curve α'_i or a curve α''_i. Since

$$\sum_{c\in\mathcal{C}}\mathcal{J}_{j(c)i(c)}\sigma(c)(\delta_{i(c)i}-\delta_{\rho(i)j(c)})=\sum_{c\in\alpha_i}\mathcal{J}_{j(c)i}\sigma(c)-\sum_{c\in\beta_{\rho(i)}}\mathcal{J}_{\rho(i)i(c)}\sigma(c)=1-1=0,$$

$e(\mathcal{D},w,\mathfrak{m})$ does not vary under these moves, thanks to Lemma 2 on the previous page. It is not hard to prove that $e(\mathcal{D},w,\mathfrak{m})$ does not depend on the orientations of the curves β. Changing the orientation of a curve α permutes α'_i and α''_i and does not modify $e(\mathcal{D},w,\mathfrak{m})$ either so that the lemma is proved. □

We will see that $e(\mathcal{D},w,\mathfrak{m})$ is also unchanged when the roles of the curves α and the curves β are permuted, in Corollary 3 on p. 37.

2.2 More on $s_\ell(\mathcal{D},\mathfrak{m})$

Fix a point a_i inside each disk $D(\alpha_i)$ and a point b_j inside each disk $D(\beta_j)$. Then join a_i to each crossing c of α_i by a segment $[a_i,c]_{D(\alpha_i)}$ oriented from a_i to c in $D(\alpha_i)$, so that these segments only meet at a_i for different c. Similarly define segments $[c,b_{j(c)}]_{D(\beta_{j(c)})}$ from c to $b_{j(c)}$ in $D(\beta_{j(c)})$. Then for each c, define the *flow line* $\gamma(c)=[a_{i(c)},c]_{D(\alpha_{i(c)})}\cup[c,b_{j(c)}]_{D(\beta_{j(c)})}$. When $\gamma(c)$ is smooth, $\gamma(c)$ is a flow line closure of a Morse function f_M giving birth to $\mathcal{D}$ discussed in the introduction.

For each point a_i in the disk $D(\alpha_i)$ as in Section 1.1 on p. 18, choose a point a_i^+ and a point a_i^- close to a_i outside $D(\alpha_i)$ so that a_i^+ is on the positive side of $D(\alpha_i)$ (the side of the positive normal) and a_i^- is on the negative side of $D(\alpha_i)$. Similarly fix points b_j^+ and b_j^- close to the b_j and outside the $D(\beta_j)$.

Then for a crossing $c\in\alpha_{i(c)}\cap\beta_{j(c)}$, $\gamma(c)_{\|}$ will denote the following chain. Consider a small meridian curve $m(c)$ of $\gamma(c)$ on $\partial H_{\mathcal{A}}$, it intersects $\beta_{j(c)}$ at two points: $c^+_{\mathcal{A}}$ on the positive side of $D(\alpha_{i(c)})$ and $c^-_{\mathcal{A}}$ on the negative side of $D(\alpha_{i(c)})$. The meridian $m(c)$ also intersects $\alpha_{i(c)}$ at $c^+_{\mathcal{B}}$ on the positive side of $D(\beta_{j(c)})$ and $c^-_{\mathcal{B}}$ on the negative side of $D(\beta_{j(c)})$. Let $[c^+_{\mathcal{A}},c^+_{\mathcal{B}}]$, $[c^+_{\mathcal{A}},c^-_{\mathcal{B}}]$, $[c^-_{\mathcal{A}},c^+_{\mathcal{B}}]$ and $[c^-_{\mathcal{A}},c^-_{\mathcal{B}}]$ denote the four quarters of $m(c)$ with the natural ends and orientations associated with the notation, as in Figure 5 on the next page.

Let $\gamma^+_{\mathcal{A}}(c)$ (resp. $\gamma^-_{\mathcal{A}}(c)$) be an arc parallel to $[a_{i(c)},c]_{D(\alpha_{i(c)})}$ from $a^+_{i(c)}$ to $c^+_{\mathcal{A}}$ (resp. from $a^-_{i(c)}$ to $c^-_{\mathcal{A}}$) that does not meet $D(\alpha_{i(c)})$. Let $\gamma^+_{\mathcal{B}}(c)$ (resp. $\gamma^-_{\mathcal{B}}(c)$) be an arc parallel to $[c,b_{j(c)}]_{D(\beta_{j(c)})}$ from $c^+_{\mathcal{B}}$ to $b^+_{j(c)}$ (resp. from $c^-_{\mathcal{B}}$ to $b^-_{j(c)}$) that does not meet $D(\beta_{j(c)})$.

$$\gamma(c)_{\|}=\frac{1}{2}(\gamma^+_{\mathcal{A}}(c)+\gamma^-_{\mathcal{A}}(c))+\frac{1}{4}([c^+_{\mathcal{A}},c^+_{\mathcal{B}}]+[c^+_{\mathcal{A}},c^-_{\mathcal{B}}]+[c^-_{\mathcal{A}},c^+_{\mathcal{B}}]+[c^-_{\mathcal{A}},c^-_{\mathcal{B}}])+\frac{1}{2}(\gamma^+_{\mathcal{B}}(c)+\gamma^-_{\mathcal{B}}(c)).$$

Figure 5 – $m(c)$, $c^+_{\mathcal{A}}$, $c^-_{\mathcal{A}}$, $c^+_{\mathcal{B}}$ and $c^-_{\mathcal{B}}$

Set $a_{i\|} = \frac{1}{2}(a_i^+ + a_i^-)$ and $b_{j\|} = \frac{1}{2}(b_j^+ + b_j^-)$. Then $\partial\gamma(c)_\| = b_{j(c)\|} - a_{i(c)\|}$.

Recall our matching $\mathfrak{m} = \{m_i; i \in \underline{g}\}$ where $m_i \in \alpha_{\rho^{-1}(i)} \cap \beta_i$, for a permutation ρ of $\underline{g}$, so that $\gamma_i = \gamma(m_i)$ goes from $a_{\rho^{-1}(i)}$ to b_i.

Set

$$L(\mathcal{D},\mathfrak{m}) = \sum_{i=1}^{g} \gamma_i - \sum_{c\in\mathcal{C}} \mathcal{J}_{j(c)i(c)}\sigma(c)\gamma(c).$$

Note that $L(\mathcal{D},\mathfrak{m})$ is a cycle since

$$\partial L(\mathcal{D},\mathfrak{m}) = \sum_{i=1}^{g}(b_i - a_i) - \sum_{(i,j)\in\underline{g}^2} \mathcal{J}_{ji}\langle\alpha_i,\beta_j\rangle_{\partial H_A}(b_j - a_i) = 0.$$

Set $L(\mathcal{D},\mathfrak{m})_\| = \sum_{i=1}^{g}\gamma_i - \sum_{c\in\mathcal{C}}\mathcal{J}_{j(c)i(c)}\sigma(c)\gamma(c)_\|$.

In this subsection, we prove the following proposition.

Proposition 1 – *For any Heegaard diagram $\mathcal{D}$ equipped with a matching $\mathfrak{m}$,*

$$s_\ell(\mathcal{D},\mathfrak{m}) = lk(L(\mathcal{D},\mathfrak{m}), L(\mathcal{D},\mathfrak{m})_\|).$$

This proposition has the following easy corollary.

Corollary 1 – *The real number $s_\ell(\mathcal{D},\mathfrak{m})$ is an invariant of the Heegaard diagram $\mathcal{D}$ equipped with $\mathfrak{m}$, which does not depend on the orientations and numberings of the curves α_i and β_j, and which does not change when the roles of the α-curves or the β-curves are permuted.*

We first prove the following lemma, which will be useful later, too.

Lemma 3 – *For any curve α_i (resp. β_j), choose a basepoint $p(\alpha_i)$ (resp. $p(\beta_j)$). These choices being made, for any crossing c of $\mathcal{C}$, define the triangle $T_\beta(c)$ in the disk $D(\beta_{j(c)})$ such that*

$$\partial T_\beta(c) = [p(\beta(c)),c]_\beta + (\gamma(c)\cap H_{\mathcal{B}}) - (\gamma(p(\beta(c)))\cap H_{\mathcal{B}}).$$

Similarly, define the triangle $T_\alpha(c)$ in the disk $D(\alpha_{i(c)})$ such that

$$\partial T_\alpha(c) = -[p(\alpha(c)), c]_\alpha + (\gamma(c) \cap H_{\mathcal{A}}) - (\gamma(p(\alpha(c))) \cap H_{\mathcal{A}}).$$

Let $K = \sum_{c \in \mathcal{C}} k_c \gamma(c)$ be a cycle of M.
Let $\Sigma_T(K) = \sum_{c \in \mathcal{C}} k_c (T_\alpha(c) + T_\beta(c))$ and

$$\Sigma_D(K) = \sum_{(i,j,c) \in \underline{g}^2 \times \mathcal{C}} \mathcal{J}_{ji} k_c \Big(\langle |p(\alpha(c)), c|_\alpha, \beta_j \rangle D(\alpha_i) - \langle \alpha_i, |p(\beta(c)), c|_\beta \rangle D(\beta_j) \Big).$$

There exists a 2-chain $\Sigma_\Sigma(K)$ in $\partial H_{\mathcal{A}}$ whose boundary $\partial \Sigma_\Sigma(K)$ is

$$\sum_{(i,j,c) \in \underline{g}^2 \times \mathcal{C}} \mathcal{J}_{ji} k_c \Big(\langle \alpha_i, |p(\beta(c)), c|_\beta \rangle \beta_j - \langle |p(\alpha(c)), c|_\alpha, \beta_j \rangle \alpha_i \Big) + \sum_{c \in \mathcal{C}} k_c ([p(\alpha(c)), c]_\alpha - [p(\beta(c)), c]_\beta)$$

so that the boundary of

$$\Sigma(K) = \Sigma_\Sigma(K) + \Sigma_D(K) + \Sigma_T(K)$$

is K.

Though it is not visible from the notation, the surfaces depend on the basepoints.

Proof (of Lemma 3 on the previous page).

$$\partial \Sigma_T(K) - K = \sum_{c \in \mathcal{C}} k_c ([p(\beta(c)), c]_\beta - [p(\alpha(c)), c]_\alpha)$$

is a cycle. Any 1-cycle σ of $\partial H_{\mathcal{A}}$ is homologous to $\sum_{(i,j) \in \underline{g}^2} \mathcal{J}_{ji} (\langle \sigma, \beta_j \rangle \alpha_i + \langle \alpha_i, \sigma \rangle \beta_j)$. Therefore by pushing $(\partial \Sigma_T(K) - K)$ in the directions of the positive and negative normals to the α and the β in $\partial H_{\mathcal{A}}$, and by averaging, we see that $(K - \partial \Sigma_T(K))$ is homologous in $\partial H_{\mathcal{A}}$ to

$$\sum_{(i,j,c) \in \underline{g}^2 \times \mathcal{C}} \mathcal{J}_{ji} k_c \Big(\langle |p(\alpha(c)), c|_\alpha, \beta_j \rangle \alpha_i - \langle \alpha_i, |p(\beta(c)), c|_\beta \rangle \beta_j \Big),$$

which bounds

$$\Sigma_D(K) = \sum_{(i,j,c) \in \underline{g}^2 \times \mathcal{C}} \mathcal{J}_{ji} k_c \Big(\langle |p(\alpha(c)), c|_\alpha, \beta_j \rangle D(\alpha_i) - \langle \alpha_i, |p(\beta(c)), c|_\beta \rangle D(\beta_j) \Big).$$

□

Proposition 2 – *For any curve α_i (resp. β_j), choose a basepoint $p(\alpha_i)$ (resp. $p(\beta_j)$). These choices being fixed, set*

$$\tilde{\ell}(c,d) = \langle |p(\alpha(c)),c|_\alpha, |p(\beta(d)),d|_\beta \rangle - \sum_{(i,j)\in \underline{g}^2} \mathcal{J}_{ji} \langle |p(\alpha(c)),c|_\alpha, \beta_j\rangle \langle \alpha_i, |p(\beta(d)),d|_\beta\rangle.$$

Let $K = \sum_{c\in\mathcal{C}} k_c\gamma(c)$ and $L = \sum_{d\in\mathcal{C}} g_d\gamma(d)$ be two 1-cycles of M. Then

$$lk(K,L_{\|}) = lk(L,K_{\|}) = \sum_{(c,d)\in\mathcal{C}^2} k_c g_d \tilde{\ell}(c,d).$$

Proof. The first equality comes from the symmetry of the linking number and from the observation that $lk(K,L_{\|}) = lk(K_{\|},L)$. Compute $lk(K,L_{\|})$ as the intersection of $L_{\|}$ with the surface bounded by K provided by Lemma 3 on p. 27. Thus $lk(K,L_{\|}) = \langle \Sigma_\Sigma(K), L_{\|}\rangle$. Now, since $L = \sum_{d\in\mathcal{C}} g_d\gamma(d)$ is a cycle,

$$L = \sum_{d\in\mathcal{C}} g_d(\gamma(d) - \gamma(p(\beta(d))))$$

and it suffices to prove the result when $L = \gamma(d) - \gamma(p(\beta(d)))$. For any path $[x,y]$ from a point x to a point y in $\partial H_{\mathcal{A}}$, when x and y are outside $\partial\Sigma_\Sigma(K)$,

$$\langle x-y, \Sigma_\Sigma(K)\rangle_\Sigma = \langle [x,y], \partial\Sigma_\Sigma(K)\rangle_{\partial H_{\mathcal{A}}}.$$

Thus by averaging,

$$\langle \gamma(d)_{\|} - \gamma(p(\beta(d)))_{\|}, \Sigma_\Sigma(K)\rangle = \langle \partial\Sigma_\Sigma(K), |p(\beta(d)),d|_\beta\rangle_{\partial H_{\mathcal{A}}}.$$

This is

$$\begin{aligned}&\sum_{c\in\mathcal{C}} k_c \langle |p(\alpha(c)),c|_\alpha, |p(\beta(d)),d|_\beta\rangle_{\partial H_{\mathcal{A}}} \\ &\qquad - \sum_{(i,j,c)\in\underline{g}^2\times\mathcal{C}} k_c\mathcal{J}_{ji}\left(\langle |p(\alpha(c)),c|_\alpha, \beta_j\rangle\langle\alpha_i, |p(\beta(d)),d|_\beta\rangle_{\partial H_{\mathcal{A}}}\right) \\ &= \sum_{c\in\mathcal{C}} k_c\left(\tilde{\ell}(c,d) - \tilde{\ell}(c,p(\beta(d)))\right).\end{aligned}$$

□

Proof (of Proposition 1 on p. 27). Apply Proposition 2 with the basepoints of $\mathfrak{m}$ so that $\tilde{\ell} = \tilde{\ell}_{\mathfrak{m}}$. □

2.3 More on $\ell_2(\mathcal{D})$

Proposition 3 – *For any curve* α_i *(resp.* β_j*), choose a basepoint* $p(\alpha_i)$ *(resp.* $p(\beta_j)$*). These choices being made, for two crossings* c *and* d *of* $\mathcal{C}$*, set*

$$\ell(c,d) = \langle [p(\alpha(c)),c|_\alpha, [p(\beta(d)),d|_\beta\rangle - \sum_{(i,j)\in \underline{g}^2} \mathcal{J}_{ji}\langle [p(\alpha(c)),c|_\alpha, \beta_j\rangle\langle \alpha_i, [p(\beta(d)),d|_\beta\rangle$$

and $\tilde{\ell}(c,d) = \langle |p(\alpha(c)),c|_\alpha, |p(\beta(d)),d|_\beta\rangle - \sum_{(i,j)\in \underline{g}^2} \mathcal{J}_{ji}\langle |p(\alpha(c)),c|_\alpha, \beta_j\rangle\langle \alpha_i, |p(\beta(d)),d|_\beta\rangle$*. Then, for any 2-cycle* $G = \sum_{(c,d)\in\mathcal{C}^2} g_{cd}(\gamma(c)\times\gamma(d)_{||})$ *of* M^2*,*

$$\ell^{(2)}(G) = \sum_{(c,d)\in\mathcal{C}^2} g_{cd}\ell(c,d) = \sum_{(c,d)\in\mathcal{C}^2} g_{cd}\tilde{\ell}(c,d).$$

Furthermore, $\ell^{(2)}(G)$ *is independent of the choices of the basepoints* $p(\alpha_i)$ *or* $p(\beta_j)$*, and of the numberings and orientations of the curves* α_i *or* β_j*.*

Proof. Let ℓ' be defined as ℓ except that the basepoint $p_i = p(\alpha_i)$ of α_i is changed to a basepoint q_i. When $c \in \alpha_i \setminus [p_i, q_i[_\alpha$,

$$\ell'(c,d)-\ell(c,d) = -\langle [p_i,q_i[_\alpha, [p(\beta(d)),d|_\beta\rangle + \sum_{(r,j)\in\underline{g}^2} \mathcal{J}_{jr}\langle [p_i,q_i[_\alpha, \beta_j\rangle\langle \alpha_r, [p(\beta(d)),d|_\beta\rangle \quad (1)$$

When $c \in [p_i,q_i[_\alpha$, $[q_i,c|_\alpha \setminus [p_i,c|_\alpha = [q_i,p_i[_\alpha = \alpha_i \setminus [p_i,q_i[_\alpha$. Since

$$\langle \alpha_i, [p(\beta(d)),d|_\beta\rangle = \sum_{(r,j)\in\underline{g}^2} \mathcal{J}_{jr}\langle \alpha_i,\beta_j\rangle\langle\alpha_r, [p(\beta(d)),d|_\beta\rangle,$$

$\ell'(c,d)-\ell(c,d)$ is given by formula (1), which does not depend on $c\in\alpha_i$ in this case either. Then

$$\sum_{(c,d)\in\mathcal{C}^2} g_{cd}(\ell'(c,d)-\ell(c,d)) = \sum_{(c,d)\in\alpha_i\times\mathcal{C}} g_{cd}(\ell'(c,d)-\ell(c,d)).$$

For any $d\in\mathcal{C}$, since

$$\partial(\gamma(c)\times\gamma(d)_{||}) = (b_{j(c)} - a_{i(c)})\times\gamma(d)_{||} - \gamma(c)\times(b_{j(d)||} - a_{i(d)||}),$$

$\sum_{c\in\alpha_i} g_{cd} = 0$. Since the right-hand side of formula (1) does not depend on $c\in\alpha_i$, this shows that $\sum_{(c,d)\in\mathcal{C}^2} g_{cd}\ell(c,d)$ does not depend on the basepoint choice on α_i. Similarly, it does not depend on the choices of the basepoints on the β_j.

Similarly, $\sum_{(c,d)\in\mathcal{C}^2} g_{cd}\ell(c,d) = \sum_{(c,d)\in\mathcal{C}^2} g_{cd}\tilde{\ell}(c,d)$.

Using $\tilde{\ell}$, changing the orientation of α_i changes $|p(\alpha(c)),c|_\alpha$ to $-\alpha_i + |p(\alpha(c)),c|_\alpha$ for $c\in\alpha_i$, and does not change $\tilde{\ell}(c,d)$. □

Remark 1 – Let $[S]$ be the homology class of $\{x\}\times\partial B_x$ in $\check{M}^2\setminus$ diagonal, where B_x is a ball of $\check{M}$ and x is a point inside B_x. Then $H_2(\check{M}^2\setminus \text{diagonal};\mathbb{Q})=\mathbb{Q}[S]$, and it is proved in Lescop (2015a, Proposition 3.4) that the class of a 2-cycle

$$G=\sum_{(c,d)\in\mathcal{C}^2} g_{cd}(\gamma(c)\times\gamma(d)_{\|})$$

in $H_2(\check{M}^2\setminus \text{diagonal};\mathbb{Q})$ is $\ell^{(2)}(G)[S]$. Furthermore, for two disjoint 1-cycles K and L of $\check{M}$, the class of $K\times L$ in $H_2(\check{M}^2\setminus \text{diagonal};\mathbb{Q})$ is $lk(K,L)[S]$ so that Proposition 2 on p. 29 provides an alternative proof of Lescop (2015a, Proposition 3.4) when G is the product of two 1-cycles. This is the needed case to produce combinatorial expressions of linking numbers involved in the variations of p_1, which we are going to study later.

Proposition 4 – *Set*

$$G(\mathcal{D})=\sum_{(c,d)\in\mathcal{C}^2}\mathcal{J}_{j(c)i(d)}\mathcal{J}_{j(d)i(c)}\sigma(c)\sigma(d)(\gamma(c)\times\gamma(d)_{\|})-\sum_{c\in\mathcal{C}}\mathcal{J}_{j(c)i(c)}\sigma(c)(\gamma(c)\times\gamma(c)_{\|}).$$

Then $G(\mathcal{D})$ is a 2-cycle of M^2. Let $\ell_2(\mathcal{D})=\ell^{(2)}(G(\mathcal{D}))$. Then $\ell_2(\mathcal{D})$ is an invariant of the Heegaard diagram, which does not depend on the orientations and numberings of the curves α_i and β_j. It does not change when the roles of the α-curves or the β-curves are permuted either.

Proof. It is easy to prove that $G(\mathcal{D})$ is a 2-cycle[7].

Permuting the roles of the α_i and the β_j reverses the orientation of $\partial H_{\mathcal{A}}$ and changes $\mathcal{J}$ to the transposed matrix. It does not change $\ell_2(\mathcal{D})$ because of the symmetry in the definition of $\ell^{(2)}$. □

3 The ∞-combings $X(w,\mathfrak{m})$ and their p_1

3.1 On the ∞-combing $X(w,\mathfrak{m})$

In order to finish our description of $X(w,\mathfrak{m})$ started in the introduction, we need to describe the vector field that replaces the gradient field X_{f_M} in regular neighborhoods $N(\gamma_i=\gamma(m_i))$ of the flow lines γ_i associated with a matching $\mathfrak{m}$ of $\mathcal{D}$. Up to renumbering and reorienting the β_j, assume that $m_i\in\alpha_i\cap\beta_i$ to simplify notation.

Choose a natural trivialization (X_1,X_2,X_3) of $T\check{M}$ on a regular neighborhood $N(\gamma_i)$ of γ_i, such that:

[7] See Lescop, 2015a, "A formula for the Θ-invariant from Heegaard diagrams", proof of Proposition 3.2.

- γ_i is directed by X_1,
- the other flow lines never have X_1 as an oriented tangent vector,
- (X_1, X_2) is tangent to the ascending manifold $\mathcal{A}_i$ of a_i (except on the parts of $\mathcal{A}_i$ near b_i that come from other crossings of $\alpha_i \cap \beta_i$), and (X_1, X_3) is tangent to the descending manifold $\mathcal{B}_i$ of b_i (except on the parts of $\mathcal{B}_i$ near a_i that come from other crossings of $\alpha_i \cap \beta_i$).

This parallelization identifies the unit tangent bundle $UN(\gamma_i)$ of $N(\gamma_i)$ with $S^2 \times N(\gamma_i)$.

There is a homotopy $h\colon [0,1] \times (N(\gamma_i) \setminus \gamma_i) \to S^2$, such that

- $h(0,.)$ is the unit vector with the same direction as the gradient vector of the underlying Morse function f_M,
- $h(1,.)$ is the constant map to $(-X_1)$ and
- $h(t,y)$ goes from $h(0,y)$ to $(-X_1)$ along the shortest geodesic arc $[h(0,y), -X_1]$ of S^2 from $h(0,y)$ to $(-X_1)$.

Let 2η be the distance between γ_i and $\partial N(\gamma_i)$ and let $X(y) = h(\max(0, 1-d(y,\gamma_i)/\eta), y)$ on $N(\gamma_i) \setminus \gamma_i$, and $X = -X_1$ along γ_i.

Note that X is tangent to $\mathcal{A}_i$ on $N(\gamma_i)$ (except on the parts of $\mathcal{A}_i$ near b_i that come from other crossings of $\alpha_i \cap \beta_i$), and that X is tangent to $\mathcal{B}_i$ on $N(\gamma_i)$ (except on the parts of $\mathcal{B}_i$ near a_i that come from other crossings of $\alpha_i \cap \beta_i$). More generally, project the normal bundle of γ_i to $\mathbb{R}^2$ in the X_1-direction by sending γ_i to 0, $\mathcal{A}_i$ to an axis $\mathcal{L}_i(A)$ and $\mathcal{B}_i$ to an axis $\mathcal{L}_i(B)$. Then the projection of X goes towards 0 along $\mathcal{L}_i(B)$ and starts from 0 along $\mathcal{L}_i(A)$, it has the direction of $s_a(y)$ at a point y of $\mathbb{R}^2$ near 0, where s_a is the planar reflexion that fixes $\mathcal{L}_i(A)$ and reverses $\mathcal{L}_i(B)$. See Figure 6.

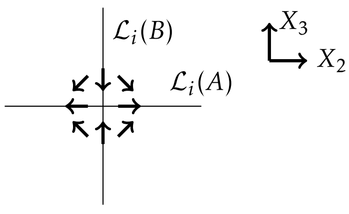

Figure 6 – Projection of X

Then $X(y)$ is on the half great circle that contains $s_a(y)$, X_1 and $(-X_1)$. In Figure 7 on the next page, γ_i is a vertical segment, all the other flow lines corresponding to crossings involving α_i go upward from a_i, and X is simply the upward vertical field in a neighborhood of $\gamma_i \cup D(\alpha_i)$.

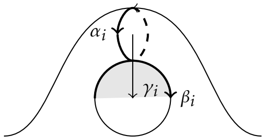

Figure 7 – γ_i

3.2 On $p_1(X(w,\mathfrak{m}))$

A *combing* of a rational homology 3-sphere M is a homotopy class of sections of the unit tangent bundle UM of M. Recall that ∞-combings are defined in the introduction. Invariants p_1 of ∞-combings and combings of rational homology 3-spheres M are invariants valued in $\mathbb{Q}$, which have been introduced and studied in Lescop (2015b) as extensions of a relative first Pontrjagin class from parallelizations to combings.

For a combing that extends to a parallelization τ, the map p_1 coincides with the Hirzebruch defect (or Pontrjagin number) of the parallelization τ, studied in Hirzebruch (1973), Kirby and Melvin (1999), Lescop (2004a), and Lescop (2013). For a parallelization $\tau\colon M\times\mathbb{R}^3\to TM$ of a 3-manifold M that bounds a connected oriented 4-dimensional manifold W with signature 0, $p_1(\tau)$ is defined as the evaluation at the fundamental class of $[W,\partial W]$ of the relative first Pontrjagin class of TW equipped with the trivialization of $TW_{|\partial W}$ that is the stabilization by the "outward normal exterior first" of τ. For ∞-combings that extend to parallelizations standard near ∞, p_1 is defined similarly by replacing W by a connected oriented signature 0 cobordism W_c with corners between $B(1)$ and the rational homology ball B_M. A neighborhood of the boundary

$$\partial W_c = -B(1) \bigcup_{\partial B(1)\sim 0\times B(1)} (-[0,1]\times\partial B(1)) \bigcup_{\partial B_M\sim 1\times\partial B(1)} B_M,$$

of such a cobordism is naturally identified with an open subspace of one of the products $[0,1[\times B(1)$ or $]0,1]\times B_M$ near ∂W_c, so that the standard parallelization of $\mathbb{R}^3$ and τ induce a trivialization of $TW_{c|\partial W_c}$ by stabilizing by the "tangent vector to $[0,1]$ first". For more details, see Lescop (2004a, Section 1.5).

Recall that any smooth compact oriented 3-manifold M can be equipped with a parallelization τ. When such a parallelization τ of M is given, for $\tilde{M}=M$ or $\check{M}$, two sections X and Y of $U\tilde{M}$ induce a map $(X,Y)\colon\tilde{M}\to S^2\times S^2$. Such sections are said to be *transverse* if the graphs of the induced maps (X,Y) and $(X,-Y)$ are transverse to $\tilde{M}\times\mathrm{diag}(S^2\times S^2)$ in $\tilde{M}\times S^2\times S^2$, where $-Y$ denotes the section opposite to Y. This is generic and independent of τ. For two transverse sections X and Y, let $L_{X=Y}$ be the preimage of the diagonal of S^2 under the map (X,Y). Thus $L_{X=Y}$ is an oriented link, which is cooriented by the fiber of the normal bundle to the diagonal of $(S^2)^2$. In Lescop (2015b, Theorem 1.2), we proved that our extensions p_1 satisfy the following property, which finishes defining them, unambiguously.

Theorem 1 – *When X and Y are two transverse representative sections of ∞-combings (resp. combings) of a rational homology 3-sphere M,*

$$p_1(Y) - p_1(X) = 4lk(L_{X=Y}, L_{X=-Y}).$$

In Lescop (2015b, Section 4.3), we also proved that p_1 coincides with the invariant θ_G defined by Gompf in Gompf (1998, Section 4), for combings of rational homology 3-spheres.

The following properties of p_1 are easy to deduce from its definition.

Proposition 5 – *The map p_1 has the following properties.*

- *A constant nonzero section N of $T\mathbb{R}^3$ represents an ∞-combing $[N]$ of S^3 such that $p_1([N]) = p_1(N) = 0$.*
- *Let M be a rational homology 3-sphere equipped with a representative section X of an ∞-combing (resp. of a combing). Let M' be a rational homology 3-sphere equipped with a representative section X' of an ∞-combing. Assume that X' coincides with a constant section N of $B(1)$ on $\partial B_{M'}$ and that there is a standard ball $B(1)$ embedded in $\check{M}$ where X coincides with N. Replacing this embedded ball $(B(1), N)$ by $(B_{M'}, X')$ gives rise to a representative section of an ∞-combing (resp. of a combing) X'' of the obtained manifold such that $p_1(X'') = p_1(X) + p_1(X')$.*
- *Changing the orientation of M changes $p_1(X)$ to $-p_1(X)$.*

Let ξ be an oriented plane bundle over a compact oriented surface S and let σ be a nowhere vanishing section of ξ on ∂S. The *relative Euler number* $e(\xi, S, \sigma)$ of σ is the algebraic intersection of an extension of σ to S with the zero section of ξ. When S is connected, it is the obstruction to extending σ as a nowhere vanishing section of ξ. The following proposition is a direct corollary of consequences of Theorem 1 derived in Lescop (2015b).

Proposition 6 – *Let $\mathfrak{m}$ and $\mathfrak{m}'$ be two matchings of $\mathcal{D}$. Let $L(\mathfrak{m}', \mathfrak{m}) = L(\mathcal{D}, \mathfrak{m}') - L(\mathcal{D}, \mathfrak{m})$, and let $\Sigma(L(\mathfrak{m}', \mathfrak{m}))$ be a compact oriented surface bounded by $L(\mathfrak{m}', \mathfrak{m})$ in $M \setminus (S^3 \setminus B(1))$. Consider the four following fields Y^{++}, Y^{+-}, $(Y^{-+} = -Y^{+-})$ and $(Y^{--} = -Y^{++})$ in a neighborhood of the $\gamma(c)$. Y^{++} and Y^{+-} are positive normals for $\mathcal{A}_i$ (which is oriented like $D(\alpha_i)$) on $\mathcal{A}_i \cap f_M^{-1}(]-\infty, 3])$, and Y^{++} and Y^{-+} are positive normals for $\mathcal{B}_j$ on $\mathcal{B}_j \cap f_M^{-1}([3, +\infty[)$. These four fields are orthogonal to $X(w, \mathfrak{m})$ over $L(\mathfrak{m}', \mathfrak{m})$ and they define parallels $L(\mathfrak{m}', \mathfrak{m})_{\| Y^{\varepsilon,\eta}}$ of $L(\mathfrak{m}', \mathfrak{m})$ obtained by pushing in the $Y^{\varepsilon,\eta}$-direction. Then*

$$p_1(X(w, \mathfrak{m}')) - p_1(X(w, \mathfrak{m})) = -\sum_{(\varepsilon,\eta) \in \{+,-\}^2} lk(L(\mathfrak{m}', \mathfrak{m}), L(\mathfrak{m}', \mathfrak{m})_{\| Y^{\varepsilon,\eta}}) + E(w, \mathfrak{m}', \mathfrak{m})$$

where

$$E(w,\mathfrak{m}',\mathfrak{m}) = -\sum_{(\varepsilon,\eta)\in\{+,-\}^2} e(X(w,\mathfrak{m})^{\perp},\Sigma(L(\mathfrak{m}',\mathfrak{m})),Y^{\varepsilon,\eta})$$

Proof. Set $L = L(\mathfrak{m}',\mathfrak{m})$. Construct a cable L_2 of L locally obtained by pushing one copy of L in each direction normal to the $\mathcal{B}_j$, except near the a_i where L_2 sits in $\mathcal{A}_i$. Define the field Z over L_2 such that, at a point k of L_2, Z has the direction of the vector from the closest point to k on L towards k. Thus $X(w,\mathfrak{m}') = D(X(w,\mathfrak{m}),L,L_2,Z,-1)$ with the notation of Proposition 4.21 in Lescop (2015b).

Then $((L_{\|Y^{+,+}},L_{\|Y^{-,-}}),(Y^{+,+},Y^{-,-}))$ is obtained from (L_2,Z) by some half-twists and

$$((L_{\|Y^{+,-}},L_{\|Y^{-,+}}),(Y^{+,-},Y^{-,+}))$$

is obtained from (L_2,Z) by the opposite half-twists. Then according to Proposition 4.21 in Lescop (2015b), with the notation of Lescop (2015b, Definition 4.16),

$$\begin{aligned} p_1(X(w,\mathfrak{m}')) = &\frac{1}{2}p_1(D(X(w,\mathfrak{m}),L,L_{\|Y^{+,+}},Y^{+,+},-1)) \\ &+\frac{1}{2}p_1(D(X(w,\mathfrak{m}),L,L_{\|Y^{+,-}},Y^{+,-},-1)). \end{aligned}$$

Thus $p_1(X(w,\mathfrak{m}')) = \frac{1}{4}\sum_{(\varepsilon,\eta)\in\{+,-\}^2} p_1(D(X(w,\mathfrak{m}),L,L_{\|Y^{\varepsilon,\eta}},Y^{\varepsilon,\eta},-1))$ and, according to Lescop (2015b, Proposition 4.18 and Lemma 4.14),

$$\begin{aligned} p_1(X(w,\mathfrak{m}')) - p_1(X(w,\mathfrak{m})) = &-\sum_{(\varepsilon,\eta)\in\{+,-\}^2} lk(L(\mathfrak{m}',\mathfrak{m}),L(\mathfrak{m}',\mathfrak{m})_{\|Y^{\varepsilon,\eta}}) \\ &-\sum_{(\varepsilon,\eta)\in\{+,-\}^2} e(X(w,\mathfrak{m})^{\perp},\Sigma(L(\mathfrak{m}',\mathfrak{m})),Y^{\varepsilon,\eta}). \end{aligned}$$ □

Combinatorial expressions for $\sum_{(\varepsilon,\eta)\in\{+,-\}^2} lk(L(\mathfrak{m}',\mathfrak{m}),L(\mathfrak{m}',\mathfrak{m})_{\|Y^{\varepsilon,\eta}})$ may be deduced from Propositions 2 and 3 on p. 29 and on p. 30. A combinatorial expression for $E(w,\mathfrak{m}',\mathfrak{m})$ will be given in Proposition 7 on p. 37.

The following theorem will be proved in Section 4.5 on p. 43.

Theorem 2 – *Let $L(w',w)$ be the union of the closures of the flow line through w' and the reversed flow line through w.*

$$p_1(X(w',\mathfrak{m})) - p_1(X(w,\mathfrak{m})) = 8lk(L(\mathcal{D},\mathfrak{m}),L(w',w)).$$

Proposition 10 on p. 47 together with the definition of $L(\mathcal{D},\mathfrak{m})$ before Proposition 1 on p. 27 will provide a combinatorial expression for $lk(L(\mathcal{D},\mathfrak{m}),L(w',w))$. Proposition 9 and Corollary 4 on p. 41 and on p. 48 provide other ones.

4 On the variations of $p_1(X(w,\mathfrak{m}))$

4.1 More on the variation of p_1 when $\mathfrak{m}$ changes

Lemma 4 – *Let $K=\sum_{c\in\mathcal{C}}k_c\gamma(c)$ be a cycle of M, and let $\Sigma(K)$ be a surface bounded by K in $\check{M}$. For $(\varepsilon,\eta)\in\{+,-\}^2$, let $Y^{\varepsilon,\eta}$ be the field defined in Proposition 6 on p. 34 along the $\gamma(c)$. Then*

$$\sum_{(\varepsilon,\eta)\in\{+,-\}^2} e(X(w,\mathfrak{m})^\perp,\Sigma(K),Y^{\varepsilon,\eta}) = -4\sum_{c\in\mathcal{C}}k_c d_e(c)$$

where d_e is defined before Lemma 1 on p. 25 with respect to our initial data, which involve $(w,\mathfrak{m})$.

Proof. Set $X(\mathfrak{m})=X(w,\mathfrak{m})$. Since M is a rational homology 3-sphere, the Euler number $e(X(\mathfrak{m})^\perp,\Sigma(K),Y^{\varepsilon,\eta})$ does not depend on the surface $\Sigma(K)$. Choose the surface constructed in Lemma 3 on p. 27 with the points of $\mathfrak{m}$ as basepoints. After removing the neighborhood $N(\gamma(w))$ of the flow line through w, $f_M^{-1}(]-\infty,0])$ behaves as a product by the rectangle $R_{\mathcal{D}}$ of Figure 3 on p. 23 and has the product parallelization induced by the vertical vector field and the parallelization of $R_{\mathcal{D}}$. This parallelization extends to the one-handles of H_A as the standard parallelization of $\mathbb{R}^3$ in Figure 7 on p. 33 so that it naturally extends to $f_M^{-1}(]-\infty,3])$, it furthermore extends to the neighborhood of the favourite flow lines in Figure 7 on p. 33. The first vector of this parallelization is $X(\mathfrak{m})$ and its second vector is everywhere orthogonal to $D(\alpha_i)$. It can be chosen to be $Y^{\varepsilon,\eta}$. In a symmetric way, $X(\mathfrak{m})^\perp$ has a unit section that coincides with the second vector of the above parallelization on the neighborhoods of the favourite flow lines in Figure 7 on p. 33 and that is orthogonal to $D(\beta_i)$ on $f_M^{-1}([4,\infty[)\setminus N(\gamma(w))$. Thus $e(X(\mathfrak{m})^\perp,\Sigma(K),Y^{\varepsilon,\eta})$ reads

$$\begin{aligned}&\sum_{c\in\mathcal{C}}k_c e(X(\mathfrak{m})^\perp,|m_{j(c)},c|_\beta\times[3,4],Y^{\varepsilon,\eta})\\ &\quad-\sum_{(i,j,c)\in\underline{g}^2\times\mathcal{C}}\mathcal{J}_{ji}k_c\langle\alpha_i,|m_{j(c)},c|_\beta\rangle e(X(\mathfrak{m})^\perp,\beta_j\times[3,4],Y^{\varepsilon,\eta})\end{aligned}$$

where

$$\begin{aligned}d_e(|m_{j(c)},c|_\beta)&=-\frac{1}{4}\sum_{(\varepsilon,\eta)\in\{+,-\}^2}e(X(\mathfrak{m})^\perp,|m_{j(c)},c|_\beta\times[3,4],\tilde{Y}^{\varepsilon,\eta})\\ d_e(\beta_s)&=-\frac{1}{4}\sum_{(\varepsilon,\eta)\in\{+,-\}^2}e(X(\mathfrak{m})^\perp,\beta_s\times[3,4],\tilde{Y}^{\varepsilon,\eta})\end{aligned}$$

with respect to our partial extensions $\tilde{Y}^{\varepsilon,\eta}$ of $Y^{\varepsilon,\eta}$, as in Lescop (2015a, Lemma 7.5). □

We get the following proposition as a direct corollary of Lemma 4 on the preceding page :

Proposition 7 – *Under the hypotheses of Proposition 6 on p. 34, if* $\mathfrak{m}' = \{d_j\}_{j\in\underline{g}}$, *then*

$$E(w,\mathfrak{m}',\mathfrak{m}) = 4\sum_{j=1}^{g} d_e(d_j)$$

where d_e *is defined with respect to our initial data, which involve* $(w,\mathfrak{m})$.

Note that Lemma 2 on p. 25 independently implies that $\sum_{j=1}^{g} d_e(d_j)$ only depends on $(w,\mathfrak{m},\mathfrak{m}')$.

Lemma 4 on the preceding page also yields the following second corollary, which is Lescop (2015a, Proposition 7.2), which in turn yields Corollary 3.

Corollary 2 – *Let* $\Sigma(L(\mathcal{D},\mathfrak{m}))$ *be a surface bounded by* $L(\mathcal{D},\mathfrak{m})$ *in* $\check{M}$.

$$e(\mathcal{D},w,\mathfrak{m}) = \frac{1}{4}\sum_{(\varepsilon,\eta)\in\{+,-\}^2} e(X(w,\mathfrak{m})^{\perp},\Sigma(L(\mathcal{D},\mathfrak{m})),Y^{\varepsilon,\eta})$$

Corollary 3 – $e(\mathcal{D},w,\mathfrak{m})$ *is unchanged when the roles of the curves* α *and the curves* β *are permuted.*

Proof. Permuting the roles of the curves α and the curves β reverses the orientation of $L(\mathcal{D},\mathfrak{m})$ and changes $X(w,\mathfrak{m})$ to its opposite while the set $\{Y^{\varepsilon,\eta}\}_{(\varepsilon,\eta)\in\{+,-\}^2}$ is preserved. □

4.2 Associating a closed combing to a combing

The Heegaard surface $f_M^{-1}(0)$ of our Morse function f_M is obtained by gluing the complement D_R of a rectangle in a sphere S^2 to the boundary of the rectangle $R_{\mathcal{D}}$ of Figure 3 on p. 23. Let $D_R\times[-2,7]$ denote the intersection of $f_M^{-1}([-2,7])$ with the flow lines through D_R so that f_M is the projection to $[-2,7]$ on $D_R\times[-2,7]$ and the flow lines read $\{x\}\times[-2,7]$ there. Similarly, our Morse function f_M reads as the projection on the interval on

$$f_M^{-1}([-2,0]\cup[6,8]) = \left(S^2\times[-2,0]\right)\cup\left(S^2\times[6,8]\right)$$

while $f_M^{-1}([-3,-2])$ and $f_M^{-1}([7,9])$ are balls centered at a minimum and a maximum mapped to -3 and 9, respectively.

The combing $X(w,\mathfrak{m})$ of Section 3.1 on p. 31 of B_M can be extended as a closed combing $X(M,w,\mathfrak{m})$, which is obtained from the tangent X_ϕ to the flow lines outside B_M by reversing it along the line $\overline{\{w\}\times]-3,9[}$ as follows:

Let us first describe $X(M,w,\mathfrak{m})$ on $D_R \times [-2,8]$. Let D be a small disk of D_R centered at w. Reverse the flow on $\{w\} \times [-2,8]$ so that it coincides with the tangent X_ϕ to the flow outside $D \times [-2,8]$, and so that on a ray of $D \times \{t\}$ directed by a vector Z from the center, it describes the half great circle $[-X_\phi, X_\phi]_Z$ from $(-X_\phi)$ to X_ϕ through Z, if $t \in [-2,7]$. Then on $S^2 \times \{-2\}$, X is naturally homotopic to the restriction to the boundary of a constant field of B^3. See Figure 8 for a vertical section of the ball centered at the minimum where the constant vector field points downward. We extend it as such.

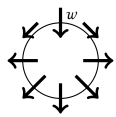

Figure 8 – The vector field near a minimum in a planar section of $f_M^{-1}([-3,-2])$

Now, on $S^2 \times \{7\}$, X looks like in Figure 9. It would naturally be homotopic to the restriction to the boundary of a constant field of B^3 if the half great circle $[-X_\phi, X_\phi]_Z$ from $(-X_\phi)$ to X_ϕ through Z went through $(-Z)$. Let $\rho_{X_\phi,\theta}$ denote the rotation with axis X_ϕ and with angle θ. For $t \in [7,8]$, on a ray of $D \times \{t\}$ directed by a vector Z from the center, let X describe the half great circle $[-X_\phi, X_\phi]_{\rho_{X_\phi,(t-7)\pi}(Z)}$ from $(-X_\phi)$ to X_ϕ through $\rho_{X_\phi,(t-7)\pi}(Z)$. Now, we extend X as the constant field of B^3, which we see near the maximum, to obtain the closed combing $X(M,w,\mathfrak{m})$.

On $S^2 \times \{7\}$ On $S^2 \times \{8\}$

Figure 9 – The vector field near a maximum

Lemma 5 – $(p_1(X(M,w,\mathfrak{m})) - p_1(X(w,\mathfrak{m})))$ *is a constant independent of* M, w *and* $\mathfrak{m}$.

Proof. This follows from the second item in Proposition 5 on p. 34 since the combing in the outside ball is unambiguously defined. $\square$

4.3 An abstract expression for the variation of p_1 when w varies

This section is devoted to the proof of the following proposition, which describes the variation of the Pontrjagin class $p_1(X(w,\mathfrak{m}))$ when w varies.

Proposition 8 – *Let w and w' be two exterior points of $\mathcal{D}$. Let $[w,w']_\alpha$ be a path on $\partial H_{\mathcal{A}}$ from w to w' disjoint from the α_i and let $[w',w]_\beta$ be a path on $\partial H_{\mathcal{A}}$ from w' to w disjoint from the β_j. Set $[w,w']_\beta = -[w',w]_\beta$. Assume that the tangent vectors of $[w,w']_\alpha$ and $[w,w']_\beta$ at w and w' coincide. Let*

$$L([w,w']_\alpha,[w',w]_\beta) = ([w,w']_\alpha \times \{2\}) \cup (\{w'\} \times [2,4]) \cup \Big([w',w]_\beta \times \{4\}\Big) \cup (\{w\} \times [4,2]).$$

Let $\varepsilon = \pm 1$. Let Y be a vector field defined on $L([w,w']_\alpha,[w',w]_\beta)$ that is tangent to the Morse levels $\partial H_{\mathcal{A}} \times \{t\}$ and that is an ε-normal (positive if $\varepsilon = 1$ and negative otherwise) to $[w,w']_\alpha \times \{2\}$ and a $(-\varepsilon)$ normal to $[w',w]_\beta \times \{4\}$. Let $L([w,w']_\alpha,[w',w]_\beta)_{\parallel Y}$ be the induced parallel of $L([w,w']_\alpha,[w',w]_\beta)$. Let Σ be a surface bounded by $L([w,w']_\alpha,[w',w]_\beta)$. Then

$$\begin{aligned} &p_1(X(w',\mathfrak{m}))) - p_1(X(w,\mathfrak{m}))) \\ &\qquad = 4e(X(w,\mathfrak{m})^\perp,\Sigma,Y) - 4lk(L([w,w']_\alpha,[w',w]_\beta), L([w,w']_\alpha,[w',w]_\beta)_{\parallel Y}). \end{aligned}$$

Proof. First note that $X(M,w,\mathfrak{m})$ directs $\{w'\} \times [2,4]$ and $\{w\} \times [4,2]$ so that the right-hand side of the equality above is independent of the field Y that satisfies the conditions of the statement. Let $L(w',w)$ be the knot of M that is the union of the closures of $\{w'\} \times]-3,9[$ and $\{w\} \times (-]-3,9[)$. Let $\tilde{X}(M,w',\mathfrak{m})$ be obtained from $X(M,w,\mathfrak{m})$ by reversing $X(M,w,\mathfrak{m})$ along $L(w',w)$, where $X(M,w,\mathfrak{m})$ is tangent to $L(w',w)$. In this situation, there is a standard way of reversing (namely the one that was used along $\{w\} \times [-2,7]$ in Section 4.2 on p. 37) by choosing a framing that determines both the parallel and the orthogonal field.

Proposition 8 is the direct consequence of Lemma 5 on the preceding page and of the following three lemmas.

Lemma 6 – *There exists a constant C_0 independent of $(M,w,w',\mathfrak{m})$ such that*

$$\begin{aligned} p_1(\tilde{X}(M,w',\mathfrak{m})) - p_1(X(M,w,\mathfrak{m})) &= 4e(X(w,\mathfrak{m})^\perp,\Sigma,Y) + 4C_0 \\ &\quad - 4lk(L([w,w']_\alpha,[w',w]_\beta), L([w,w']_\alpha,[w',w]_\beta)_{\parallel Y}). \end{aligned}$$

Lemma 7 – *There exists a constant C_1 independent of $(M,w,w',\mathfrak{m})$ such that*

$$p_1(\tilde{X}(M,w',\mathfrak{m})) - p_1(X(M,w',\mathfrak{m})) = 4C_1.$$

Lemma 8 – *The constants C_0 and C_1 coincide.*

Proof (of Lemma 6). Let $T([w,w']_\alpha)$ be the (closure of the) past of $[w,w']_\alpha \times \{2\}$ under the flow. This is a triangle and we can assume that it is smoothly embedded (near the minimum). Similarly, let $T([w',w]_\beta)$ be the future of $[w',w]_\beta \times \{4\}$ under the flow, assume without loss that it intersects $S^2 \times \{7\}$ as a half-great circle, so that it intersects $f_M^{-1}([7,9])$ as a hemidisk denoted by $T_7([w',w]_\beta)$. Orient $T([w,w']_\alpha)$ and $T([w',w]_\beta)$ so that

$$\partial(\Sigma + T([w,w']_\alpha) + T([w',w]_\beta)) = L(w',w).$$

Then Y extends to $T([w,w']_\alpha)$ as the (ε)-normal on $T([w,w']_\alpha)$, which is in $X(M,w,\mathfrak{m})^\perp$. Similarly, Y extends to $T([w',w]_\beta)$ as the (ε)-normal on $T([w',w]_\beta)$, it is a unit vector field, which is in $X(M,w,\mathfrak{m})^\perp$ outside the interior of $T_7([w',w]_\beta)$. Use Y to frame $L(w',w)$. Then, according to Lescop (2015b, Proposition 4.18 and Lemma 4.14) where $\eta = 1$,

$$\begin{aligned} & p_1(\tilde{X}(M,w',\mathfrak{m})) - p_1(X(M,w,\mathfrak{m})) \\ & \quad = 4e(X(M,w,\mathfrak{m})^\perp, \Sigma + T_7([w',w]_\beta), Y) - 4lk(L(w',w), L(w',w)_{\|Y}) \end{aligned}$$

where

$$e(X(M,w,\mathfrak{m})^\perp, T_7([w',w]_\beta, Y) = C_0$$

for a constant C_0 independent of $(M,w,w',\mathfrak{m})$, and

$$lk(L(w',w), L(w',w)_{\|Y}) = lk(L([w,w']_\alpha, [w',w]_\beta), L([w,w']_\alpha, [w',w]_\beta)_{\|Y}). \qquad \square$$

Proof (of Lemma 7 on the previous page). Recall that D is a small disk of $\partial H_{\mathcal{A}}$ centered at w. The vector fields $\tilde{X}(M,w',\mathfrak{m})$ and $X(M,w',\mathfrak{m})$ coincide outside $f_M^{-1}([-3,-2] \cup [7,9]) \cup D \times [-2,7]$. This is a ball where the definition of these fields is unambiguous and independent of $(M,w,w',\mathfrak{m})$. □

Proof (of Lemma 8 on the previous page). According to the previous lemmas, for any $(M,w,w',\mathfrak{m})$,

$$\begin{aligned} & p_1(X(M,w',\mathfrak{m})) - p_1(X(M,w,\mathfrak{m})) \\ & \qquad = -4lk(L([w,w']_\alpha, [w',w]_\beta), L([w,w']_\alpha, [w',w]_\beta)_{\|Y}) \\ & \qquad\quad + 4e(X(w,\mathfrak{m})^\perp, \Sigma, Y) + 4(C_0 - C_1). \end{aligned}$$

When M is S^3 equipped with a Morse function with 2 extrema and no other critical points, and when w and w' are two points of S^2 related by a geodesic arc $[w,w']_\alpha = -[w',w]_\beta$, it is easy to see that the first two terms of the right-hand side add up to zero, so that $(C_0 - C_1) = 0$. □

4.4 A combinatorial formula for the variation of p_1 when w varies

Now, we give an explicit formula for the right-hand side of Proposition 8 on the previous page.

Proposition 9 – *Assume that w is on the upper side of the rectangle $R_{\mathcal{D}}$ of Figure 3 on p. 23. Assume that $[w,w']_\alpha$ and $[w,w']_\beta = -[w',w]_\beta$ point downward near w and w' and that $[w,w']_\beta$ is on the same side of $[w,w']_\alpha$ near w and w' as in Figure 10. Let $d_e^{(w)}([w,w']_\alpha)$ be the degree of the tangent map to $[w,w']_\alpha$ on the rectangle $R_{\mathcal{D}}$ of Figure 3 on p. 23. Let $d_e^{(w)}([w,w']_\beta)$ be the degree of the tangent map to $[w,w']_\beta$ on $R_{\mathcal{D}}$, where $[w,w']_\beta$ intersects the α'_j and the α''_j on their vertical portions opposite to the crossings of $\mathfrak{m}$, with horizontal tangencies. Then*

$$p_1(X(w',\mathfrak{m})) - p_1(X(w,\mathfrak{m})) = p'_1(\mathfrak{m};w,w')$$

where

$$\begin{aligned} p'_1(\mathfrak{m};w,w') = 4d_e^{(w)}([w,w']_\alpha) - 4d_e^{(w)}([w,w']_\beta) \\ + 4\sum_{(i,j)\in\underline{g}^2} \mathcal{J}_{ji}\langle\alpha_i,[w,w']_\beta\rangle d_e^{(w)}(\beta_j) \\ - 4\langle]w,w'[_\alpha,]w,w'[_\beta\rangle \\ + 4\sum_{(i,j)\in\underline{g}^2} \mathcal{J}_{ji}\langle\alpha_i,[w,w']_\beta\rangle\langle[w,w']_\alpha,\beta_j\rangle. \end{aligned}$$

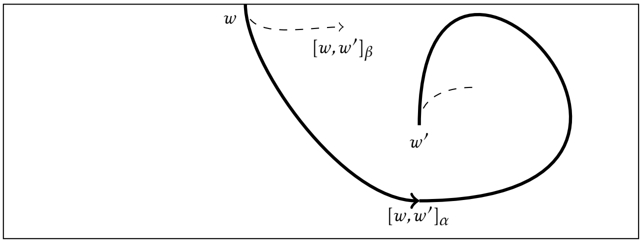

Figure 10 – $[w,w']_\alpha$ and $[w,w']_\beta$

Proof. Define the field Y of Proposition 8 on p. 39 along $\{w'\}\times[2,4]$ and $\{w\}\times[4,2]$, as the field pointing to the right in Figure 10, which is preserved by the flow along $\{w'\}\times[2,4]$ and $\{w\}\times[4,2]$, so that it is always normal to $[w,w']_\alpha\times[2,4]$ or $[w,w']_\beta\times[2,4]$ along $\{w'\}\times[2,4]$ and $\{w\}\times[4,2]$. Let $L = L([w,w']_\alpha,[w',w]_\beta)$ and let $L_\parallel = L_{\parallel Y}$. The proposition follows by applying Proposition 8 on p. 39, with the computations of Lemmas 9 and 11 on the current page and on the next page (replacing $[w,w']_\beta = -[w',w]_\beta$). □

Lemma 9 – *We have*

$$lk\left(L,L_\parallel\right) = -\langle]w,w'[_\alpha,]w',w[_\beta\rangle + \sum_{(i,j)\in\underline{g}^2} \mathcal{J}_{ji}\langle\alpha_i,[w',w]_\beta\rangle\langle[w,w']_\alpha,\beta_j\rangle.$$

In order to prove Lemma 9 on the previous page, we will use the following lemma.

Lemma 10 – *There is a surface* $\Sigma([w,w']_\alpha,[w',w]_\beta)$ *in* $\partial H_{\mathcal{A}} \setminus \mathring{D}_R$ *such that*

$$\partial\Sigma([w,w']_\alpha,[w',w]_\beta) = [w,w']_\alpha - \sum_{(i,j)\in\underline{g}^2} \mathcal{J}_{ji}\langle [w,w']_\alpha, \beta_j\rangle \alpha_i + [w',w]_\beta - \sum_{(i,j)\in\underline{g}^2} \mathcal{J}_{ji}\langle \alpha_i, [w',w]_\beta\rangle \beta_j.$$

Let w'_E *be a point very close to* w' *on its right-hand side. Then*

$$\langle \Sigma([w,w']_\alpha,[w',w]_\beta), w'_E\rangle_{\partial H_{\mathcal{A}}} = -\langle]w,w'[_\alpha,]w',w[_\beta\rangle + \sum_{(i,j)\in\underline{g}^2} \mathcal{J}_{ji}\langle \alpha_i, [w',w]_\beta\rangle\langle [w,w']_\alpha, \beta_j\rangle.$$

Proof. Since the prescribed boundary $\partial\Sigma([w,w']_\alpha,[w',w]_\beta)$ is a cycle that does not intersect the α_i and the β_j, algebraically, the surface $\Sigma([w,w']_\alpha,[w',w]_\beta)$ exists. Let w_E be a point very close to w on its right-hand side. Along a path $]w_E,w'_E[_\alpha$ parallel to $]w,w'[_\alpha$, the intersection of a point with $\Sigma([w,w']_\alpha,[w',w]_\beta)$ starts with the value 0 and varies when the path meets $\partial\Sigma([w,w']_\alpha,[w',w]_\beta)$ so that

$$\begin{aligned}\langle \Sigma([w,w']_\alpha,[w',w]_\beta), w'_E\rangle_{\partial H_{\mathcal{A}}} &= -\langle]w_E,w'_E[_\alpha, \partial\Sigma([w,w']_\alpha,[w',w]_\beta)\rangle \\ &= -\langle]w_E,w'_E[_\alpha,]w',w[_\beta\rangle \\ &\quad + \sum_{(i,j)\in\underline{g}^2} \mathcal{J}_{ji}\langle \alpha_i, [w',w]_\beta\rangle\langle]w_E,w'_E[_\alpha, \beta_j\rangle. \qquad \square\end{aligned}$$

Proof (of Lemma 9 on the previous page). L bounds

$$\Sigma_0 = \Sigma([w,w']_\alpha,[w',w]_\beta) + ([w,w']_\alpha \times [2,3]) - ([w',w]_\beta \times [3,4]) + \sum_{(i,j)\in\underline{g}^2} \mathcal{J}_{ji}\langle [w,w']_\alpha, \beta_j\rangle D(\alpha_i) + \sum_{(i,j)\in\underline{g}^2} \mathcal{J}_{ji}\langle \alpha_i, [w',w]_\beta\rangle D(\beta_j)$$

The link $L_{\|Y} = L([w,w']_\alpha,[w',w]_\beta)_{\|Y}$ does not meet the $D(\alpha_i)$ and the $D(\beta_j)$. Therefore its intersection with Σ_0 is the intersection of w'_E with $\Sigma([w,w']_\alpha,[w',w]_\beta)$ so that Lemma 10 yields the conclusion. □

Lemma 11 – *Let* Σ_1 *be a surface bounded by* L *in* M*. Then*

$$e(X(w,\mathfrak{m})^\perp,\Sigma_1,Y) = d_e^{(w)}([w,w']_\alpha) - d_e^{(w)}([w,w']_\beta) - \sum_{(i,j)\in\underline{g}^2} \mathcal{J}_{ji}\langle \alpha_i, [w',w]_\beta\rangle d_e^{(w)}(\beta_j).$$

Proof. Let $\Sigma_2 = \Sigma([w,w']_\alpha, [w',w]_\beta) \times \{2\}$ with the surface $\Sigma([w,w']_\alpha, [w',w]_\beta) \subset \partial H_{\mathcal{A}}$ of Lemma 10 on the preceding page. The link L bounds

$$\Sigma_1 = \Sigma_2 - [w',w]_\beta \times [2,4] + \sum_{(i,j)\in \underline{g}^2} \mathcal{J}_{ji}\langle [w,w']_\alpha, \beta_j\rangle D_{\leq 2}(\alpha_i)$$

$$+ \sum_{(i,j)\in \underline{g}^2} \mathcal{J}_{ji}\langle \alpha_i, [w',w]_\beta\rangle D_{\geq 2}(\beta_j)$$

where $D_{\leq 2}(\alpha_i) = D(\alpha_i) \cap f_M^{-1}([-3,2])$ and $D_{\geq 2}(\beta_j) = D(\beta_j) \cup \beta_j \times [2,3]$.

Define a field Y_E on L such that Y_E and Y coincide on $L \setminus ([w,w']_\alpha \times \{2\})$ and Y_E points East or to the right in Figure 3 on p. 23 along $[w,w']_\alpha \times \{2\}$ so that

$$e(X(w,\mathfrak{m})^\perp, \Sigma_1, Y) = e(X(w,\mathfrak{m})^\perp, \Sigma_1, Y_E) + d_e^{(w)}([w,w']_\alpha).$$

Extend Y_E to the product by $[2,4]$ of a short vertical segment $[w, w^{(S)}]$ from w to some point $w^{(S)}$ below w, such that $X(w,\mathfrak{m})$ directs $w \times [4,2]$ and $w^{(S)} \times [2,4]$, and $X(w,\mathfrak{m})$ is tangent to $[w,w^{(S)}] \times [2,4]$. Truncate the rectangle of Figure 10 on p. 41 so that $w^{(S)}$ is on its boundary and $w^{(S)}$ replaces w in the right-hand side of the equality of the statement without change. Now, $X(w,\mathfrak{m})$ is orthogonal to this rectangle.

In order to compute $e(X(w,\mathfrak{m})^\perp, \Sigma_1, Y_E)$, we will first define extensions of Y_E on the pieces of Σ_1, independently, and we will next compare our extensions on the boundary's pieces where they do not match.

On one hand, extend Y_E to $f_M^{-1}([0,2]) \setminus (D_R \times [0,2])$ as the field Y_E that points East or to the right in Figures 3, 7 and 10 on p. 23, on p. 33 and on p. 41 so that it is normal to the $D_{\leq 2}(\alpha_i)$. Use this extension on the $D_{\leq 2}(\alpha_i)$ and on Σ_2. On the other hand, extend Y_E to $D_{\geq 2}(\beta_j)$ and to $[w',w]_\beta \times [2,4]$ as a field normal to these surfaces.

Now, compute the Euler class of Y_E with respect to Σ_1, by comparing these two extensions to the standard one above, on $[w,w']_\beta \times \{2\} + \sum_{(i,j)\in \underline{g}^2} \mathcal{J}_{ji}\langle \alpha_i, [w',w]_\beta\rangle (\beta_j \times \{2\})$.

$$e(X(w,\mathfrak{m})^\perp, \Sigma_1, Y_E) = -d_e^{(w)}([w,w']_\beta) - \sum_{(i,j)\in \underline{g}^2} \mathcal{J}_{ji}\langle \alpha_i, [w',w]_\beta\rangle d_e^{(w)}(\beta_j).$$

□

4.5 Proof of Theorem 2 on p. 35

Thanks to Proposition 9 on p. 41, in order to prove Theorem 2 on p. 35, we are left with the proof that

$$p_1'(\mathfrak{m}; w, w') = 8lk(L(\mathcal{D},\mathfrak{m}), L(w',w))$$

where $p_1'(\mathfrak{m}; w, w')$ is defined in the statement of Proposition 9 on p. 41 and $L(w',w)$ is the union of the closures of the flow line through w' and the reversed flow line through w. In order to prove this, fix an exterior point w_0 of $\mathcal{D}$, and define $p_1''(w)$, for any exterior point w of $\mathcal{D}$, as

$$p_1''(w) = p_1'(\mathfrak{m}; w_0, w) - 8lk(L(\mathcal{D}, \mathfrak{m}), L(w, w_0)).$$

Lemma 12 – *p_1'' satisfies the following properties:*

- *$p_1''(w)$ only depends on the connected component of w in the complement of the α_i and the β_j in the closed surface $\partial H_{\mathcal{A}}$,*
- *$p_1''(w_0) = 0$,*
- *For any 4 points w, S, E, N located around a crossing $d \notin \mathfrak{m}$, as in Figure 11*

$$p_1''(N) + p_1''(S) = p_1''(w) + p_1''(E).$$

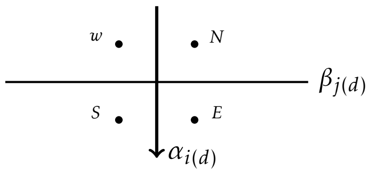

Figure 11 – Near d

Proof. The first two properties come from the definition. Let us prove the third one.
Set

$$D = (p_1''(N) + p_1''(S) - (p_1''(w) + p_1''(E)))$$

Note that D is independent of w_0, thanks to Proposition 9 on p. 41, and that it reads $D = D_1 - 8D_2$ with

$$D_1 = p_1'(\mathfrak{m}; w, N) + p_1'(\mathfrak{m}; w, S) - p_1'(\mathfrak{m}; w, E) \text{ and } D_2 = lk(L(N, w) + L(S, E), L(\mathcal{D}, \mathfrak{m}))$$

and

$$\begin{aligned} p_1'(\mathfrak{m}; w, w') = {} & 4d_e^{(w)}([w, w']_\alpha) - 4d_e^{(w)}([w, w']_\beta) \\ & + 4 \sum_{(i,j) \in \underline{g}^2} \mathcal{J}_{ji} \langle \alpha_i, [w, w']_\beta \rangle d_e^{(w)}(\beta_j) \\ & - 4\langle \Sigma([w, w']_\alpha, [w', w]_\beta), w'_E \rangle_{\partial H_A} \end{aligned}$$

according to Proposition 9 on p. 41 and Lemma 10 on p. 42.
We are going to prove that

$$D_1 = 8D_2 = -8\sigma(d)\mathcal{J}_{j(d)i(d)}.$$

Let us first compute D_1. Its computation involves paths $[w,w']_\alpha$ and $[w,w']_\beta$ starting from w on the upper side of the rectangle $R_{\mathcal{D}}$ of Figure 3 on p. 23, before reaching a point $w' = N$, S or E. We assume that all these paths begin by following a first path $[w,\tilde{w}]$ that connects w to a point $\tilde{w}$ near d in the complement of the curves α_i and β_j in $R_{\mathcal{D}}$ and that this path $[w,\tilde{w}]$ has tangent vectors pointing downward at its ends. The degree of the path $[w,\tilde{w}]$ does not matter since it is counted twice with opposite sign in $(d_e^{(w)}([w,w']_\alpha) - d_e^{(w)}([w,w']_\beta))$. Thus we may change w to $\tilde{w}$ in $p_1'(\mathfrak{m};w,w')$ or equivalently assume that w arises near d as in Figure 11 on the preceding page split along $\alpha_{i(d)}$ and embedded in Figure 3 on p. 23 as soon as we translate our initial conventions for tangencies near the boundaries. Now (keeping the first composition by $[w,\tilde{w}]$ in mind) we can draw our paths $[\tilde{w},S]_\alpha$ and $[\tilde{w},N]_\beta$ in Figure 12 where $\tilde{w}$ is denoted by w. These paths together with the other drawn paths $[N,E]_\alpha$ and $[S,E]_\beta$ bound a "square" C around d. In Figure 12, there are also dashed paths $[w,N]_\alpha$ and $[w,S]_\beta$, which may be complicated outside the pictured neighborhood of our square, but which meet this neighborhood as in the figure. We choose $[w,E]_\alpha$ (resp. $[w,E]_\beta$) to be the path composition of $[w,N]_\alpha$ and $[N,E]_\alpha$ (resp. $[w,S]_\beta$ and $[S,E]_\beta$).

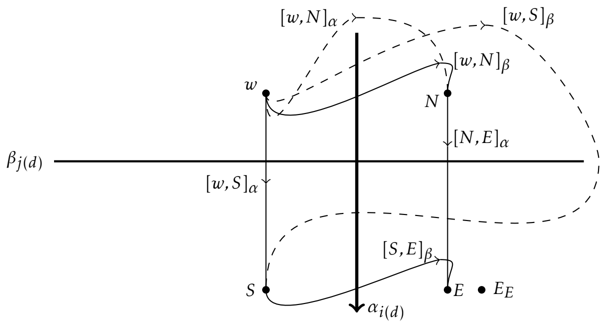

Figure 12 – Near d

With these choices, the contribution to D_1 of the parts

$$d_e^{(w)}([w,w']_\alpha) - d_e^{(w)}([w,w']_\beta) + \sum_{(i,j)\in \underline{g}^2} \mathcal{J}_{ji}\langle \alpha_i, [w,w']_\beta\rangle d_e^{(w)}(\beta_j)$$

cancel. When w' is E, N or S, let $\Sigma(w') = \Sigma([w,w']_\alpha, [w',w]_\beta)$, with the notation of Lemma 10 on p. 42. Then

$$\begin{aligned} D_1 &= 4\langle \Sigma(E), E_E\rangle - 4\langle \Sigma(N), N_E\rangle - 4\langle \Sigma(S), S_E\rangle \\ &= -4\langle \Sigma(N) + \Sigma(S) - \Sigma(E), E_E\rangle - 4\langle [N_E, E_E], \partial\Sigma(N)\rangle - 4\langle [S_E, E_E], \partial\Sigma(S)\rangle \end{aligned}$$

where $\partial(\Sigma(N) + \Sigma(S) - \Sigma(E)) = [w,S]_\alpha - [N,E]_\alpha + [N,w]_\beta - [E,S]_\beta$ so that $(\Sigma(N) +$

$\Sigma(S)-\Sigma(E))$ is our square and $\langle \Sigma(N)+\Sigma(S)-\Sigma(E), E_E\rangle = 0$,

$$\begin{aligned}\langle [N_E,E_E]_\alpha, \partial\Sigma(N)\rangle &= \langle [N_E,E_E]_\alpha, [N,w]_\beta - \sum_{(i,j)\in \underline{g}^2} \mathcal{J}_{ji}\langle \alpha_i, [N,w]_\beta\rangle \beta_j\rangle \\ &= \sigma(d)\mathcal{J}_{j(d)i(d)} \\ \langle [S_E,E_E]_\beta, \partial\Sigma(S)\rangle &= \langle -[w,S]_\alpha + \sum_{(i,j)\in \underline{g}^2} \mathcal{J}_{ji}\langle [w,S]_\alpha, \beta_j\rangle \alpha_i, [S_E,E_E]_\beta\rangle \\ &= \sigma(d)\mathcal{J}_{j(d)i(d)}\end{aligned}$$

Then $D_1 = -8\sigma(d)\mathcal{J}_{j(d)i(d)}$.
In order to compute D_2, construct a Seifert surface for $L(N,w)+L(S,E)$ made of

- two triangles parallel to the $D(\beta)$ with bottom boundaries $[w,N]_\beta$ and $[E,S]_\beta$,
- two triangles parallel to the $D(\alpha)$ with top edges $[S,w]_\alpha$ and $[N,E]_\alpha$,
- our square C bounded by $([N,w]_\beta \cup [w,S]_\alpha \cup [S,E]_\beta \cup [E,N]_\alpha)$, which is a meridian of $\gamma(d)$.

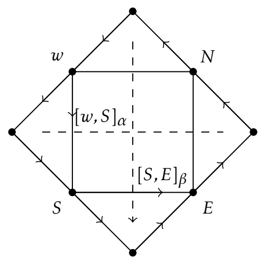

Figure 13 – A Seifert surface of $L(N,w)+L(S,E)$

Therefore $D_2 = -\sigma(d)\mathcal{J}_{j(d)i(d)}$. □

Now, we conclude as follows. According to the above lemma, the variation of p_1'' across a curve α_i or β_j is constant so that the variation of p_1'' along a path γ reads

$$\sum_i v_i\langle\gamma,\alpha_i\rangle + \sum_j w_j\langle\gamma,\beta_j\rangle$$

for some v_i and w_j independent of γ. Since this is zero for any loop γ, the v_i and the w_j vanish, and the function p_1'' is constant. Then it is identically zero and Theorem 2 on p. 35 is proved.

5 Behaviour of $\tilde{\Theta}$ when w and $\mathfrak{m}$ vary

In this section, we compute the variations of $\tilde{\Theta}(w,\mathfrak{m}) = \tilde{\Theta}(\mathcal{D},w,\mathfrak{m})$ when w and $\mathfrak{m}$ change for a fixed $\mathcal{D}$, and we find that these variations coincide with the variations of $\frac{1}{4}p_1(X(w,\mathfrak{m}))$ computed in the previous section. Thus we prove that $\left(\tilde{\Theta}(w,\mathfrak{m}) - \frac{1}{4}p_1(X(w,\mathfrak{m}))\right)$ is independent of $(w,\mathfrak{m})$.

5.1 Changing w

Let us first prove the next proposition, which is similar to Proposition 2 on p. 29.

Proposition 10 – *Let w and w' be two exterior points of $\mathcal{D}$. Let $L(w',w)$ be the union of the closures of the flow line through w' and the reversed flow line through w,*

let $[w,w']_\alpha$ be a path from w to w' outside the α_i. Choose a basepoint $p(\beta_j)$ for any curve β_j. For any 1-cycle $K = \sum_{c\in\mathcal{C}} k_c\gamma(c)$,

$$lk(K,L(w',w)) = \sum_{c\in\mathcal{C}} k_c\langle [w,w']_\alpha, [p(\beta(c)),c|_\beta\rangle - \sum_{(j,i)\in\underline{g}^2}\sum_{c\in\mathcal{C}} k_c\mathcal{J}_{ji}\langle \alpha_i,[p(\beta(c)),c|_\beta\rangle\langle [w,w']_\alpha,\beta_j\rangle$$

where $\beta(c) = \beta_{j(c)}$.

Proof. As in Lemma 3 on p. 27, K bounds a chain

$$\Sigma(K) = \Sigma_\Sigma(K) + \sum_{c\in\mathcal{C}} k_c(T_\beta(c)+T_\alpha(c)) - \sum_{(j,i)\in\underline{g}^2}\sum_{c\in\mathcal{C}} k_c\mathcal{J}_{ji}\Big(\langle\alpha_i,|p(\beta(c)),c|_\beta\rangle D(\beta_j) - \langle |p(\alpha(c)),c|_\alpha,\beta_j\rangle D(\alpha_i)\Big)$$

where $\Sigma_\Sigma(K)$ is a chain of $\partial H_{\mathcal{A}}\setminus\{w\}$ with boundary

$$\partial\Sigma_\Sigma(K) = \sum_{c\in\mathcal{C}} k_c(|p(\alpha(c)),c|_\alpha - |p(\beta(c)),c|_\beta) + \sum_{(j,i)\in\underline{g}^2}\sum_{c\in\mathcal{C}} k_c\mathcal{J}_{ji}\Big(\langle\alpha_i,|p(\beta(c)),c|_\beta\rangle\beta_j - \langle|p(\alpha(c)),c|_\alpha,\beta_j\rangle\alpha_i\Big).$$

Now, $lk(L,L(w',w))$ is the intersection of w' and $\Sigma_\Sigma(K)$, which is $\langle -[w,w']_\alpha, \partial\Sigma_\Sigma(K)\rangle$.□

Lemma 13 – *Let w and w' be two exterior points of $\mathcal{D}$. Let $[w,w']_\alpha$ be a path of $\Sigma\setminus(\cup_{i=1}^g\alpha_i)$ from w to w'. Set*

$$\tilde{\Theta}' = \tilde{\Theta}(w',\mathfrak{m}) - \tilde{\Theta}(w,\mathfrak{m}) = e(\mathcal{D},w,\mathfrak{m}) - e(\mathcal{D},w',\mathfrak{m}).$$

Then

$$\tilde{\Theta}' = 2\sum_{c\in\mathcal{C}} \mathcal{J}_{j(c)i(c)}\sigma(c)\left(\sum_{(r,s)\in\underline{g}^2} \mathcal{J}_{sr}\langle\alpha_r, |m_{j(c)}, c|_\beta\rangle\langle[w,w']_\alpha, \beta_s\rangle - \langle[w,w']_\alpha, |m_{j(c)}, c|_\beta\rangle\right).$$

Proof. Pick a vertical path $[w,w']_\alpha$ from a point w in the boundary of the rectangle of Figure 3 on p. 23 to the point w' that cuts horizontal parts of the β curves. When w is changed to w', the portions or arcs near the intersection points with $[w,w']_\alpha$ are transformed to arcs that turn around the whole picture of Figure 3 on p. 23. This operation adds 2 to the degree of an arc oriented from left to right. See Figure 14.

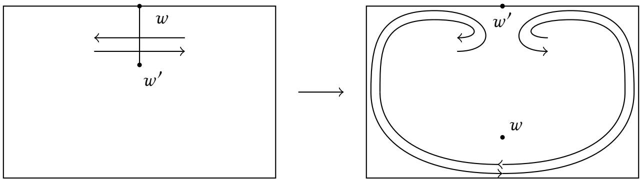

Figure 14 – Changing w to w'

Therefore

$$d_e^{(w')}(\beta_s) - d_e^{(w)}(\beta_s) = 2\langle[w,w']_\alpha, \beta_s\rangle$$

and

$$d_e^{(w')}(|m_{j(c)}, c|_\beta) - d_e^{(w)}(|m_{j(c)}, c|_\beta) = 2\langle[w,w']_\alpha, |m_{j(c)}, c|_\beta\rangle. \qquad \square$$

Corollary 4 – *Let $L(w',w)$ be the union of the closures of the flow line through w' and the reversed flow line through w.*

$$\tilde{\Theta}(w',\mathfrak{m}) - \tilde{\Theta}(w,\mathfrak{m}) = 2lk(L(\mathcal{D},\mathfrak{m}), L(w',w)) = \frac{1}{4}p_1(X(w',\mathfrak{m})) - \frac{1}{4}p_1(X(w,\mathfrak{m})).$$

Proof. This follows from Lemma 13 on the previous page, Proposition 10 on the previous page and Theorem 2 on p. 35. □

5.2 Changing $\mathfrak{m}$

Let $\mathfrak{m}' = \{d_i \in \alpha_i \cap \beta_{\psi^{-1}(i)}\}$ be another matching for a permutation ψ. The matching $\mathfrak{m}'$ replaces our initial matching $\mathfrak{m}$ of positive crossings $m_i \in \alpha_i \cap \beta_i$.

Set $L(\mathfrak{m}) = L(\mathcal{D},\mathfrak{m})$ and $L(\mathfrak{m}') = L(\mathcal{D},\mathfrak{m}')$.

Let $L(\mathfrak{m}',\mathfrak{m}) = L(\mathfrak{m}') - L(\mathfrak{m}) = \sum_{i=1}^{g}(\gamma(d_i) - \gamma_i)$.

This subsection is devoted to the proof the following proposition.

Proposition 11 – *Under the assumptions above,*

$$\tilde{\Theta}(w,\mathfrak{m}') - \tilde{\Theta}(w,\mathfrak{m}) = \frac{1}{4}p_1(X(w,\mathfrak{m}')) - \frac{1}{4}p_1(X(w,\mathfrak{m})).$$

This proposition is a direct corollary of Propositions 6, 7 and 12 on p. 34, on p. 37 and on the current page so that we are left with the proof of Proposition 12.

Proposition 12 – *Under the assumptions above,*

$$\begin{aligned}\tilde{\Theta}(w,\mathfrak{m}') - \tilde{\Theta}(w,\mathfrak{m}) &= lk(L(\mathfrak{m}'),L(\mathfrak{m}')_{\parallel}) - lk(L(\mathfrak{m}),L(\mathfrak{m})_{\parallel}) + e(\mathcal{D},w,\mathfrak{m}) - e(\mathcal{D},w,\mathfrak{m}')\\ &= \sum_{i=1}^{g} d_e(d_i) - lk(L(\mathfrak{m}',\mathfrak{m}),L(\mathfrak{m}',\mathfrak{m})_{\parallel}).\end{aligned}$$

Here, d_e is defined with respect to our initial data, which involve w and $\mathfrak{m}$.

Proposition 12 is a direct consequence of Lemma 14 and Lemma 15, which will be proved at the end of this subsection.

Lemma 14 – *We have*

$$lk(L(\mathfrak{m}'),L(\mathfrak{m}')_{\parallel}) - lk(L(\mathfrak{m}),L(\mathfrak{m})_{\parallel}) = 2lk(L(\mathfrak{m}',\mathfrak{m}),L(\mathfrak{m}')_{\parallel}) - lk(L(\mathfrak{m}',\mathfrak{m}),L(\mathfrak{m}',\mathfrak{m})_{\parallel}).$$

Proof. Use the symmetry of the linking number, and replace $L(\mathfrak{m}) = L(\mathfrak{m}') - L(\mathfrak{m}',\mathfrak{m})$. □

Lemma 15 – *We have*

$$e(\mathcal{D},w,\mathfrak{m}') - e(\mathcal{D},w,\mathfrak{m}) = 2lk(L(\mathfrak{m}',\mathfrak{m}),L(\mathfrak{m}')_{\parallel}) - \sum_{j=1}^{g} d_e(d_j)$$

where $d_e(d_{\psi(j)}) = d_e(|m_j,d_{\psi(j)}|_\beta) - \sum_{s=1}^{g}\sum_{i=1}^{g}\mathcal{J}_{si}\langle\alpha_i,|m_j,d_{\psi(j)}|_\beta\rangle d_e(\beta_s)$.

Lemma 16 – *The number $lk(L(\mathfrak{m}',\mathfrak{m}),L(\mathfrak{m}')_{\parallel})$ is equal to*

$$\begin{aligned}&\sum_{i=1}^{g}\sum_{c\in\mathcal{C}}\sigma(c)\mathcal{J}_{j(c)i(c)}\left(\sum_{(s,r)\in\underline{g}^2}\mathcal{J}_{sr}\langle|m_i,d_i|_\alpha,\beta_s\rangle\langle\alpha_r,|d_{\psi(j(c))},c|_\beta\rangle\right)\\ &\quad-\sum_{i=1}^{g}\sum_{c\in\mathcal{C}}\sigma(c)\mathcal{J}_{j(c)i(c)}\Big(\langle|m_i,d_i|_\alpha,|d_{\psi(j(c))},c|_\beta\rangle\Big).\end{aligned}$$

Proof. Use Proposition 3 on p. 30 with $p(\alpha_i) = m_i$, $p(\beta_j) = d_{\psi(j)}$, and $\tilde{\ell}$. □

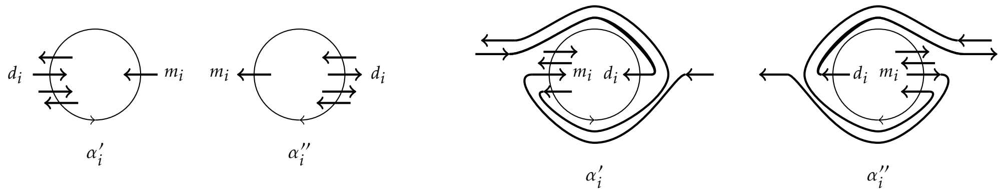

Figure 15 – Making the crossings move around

Proof (of Lemma 15 on the previous page). Move the crossings of $[m_i, d_i]$, counterclockwise along α_i'' and clockwise along α_i' as in Figure 15 so that m_i and d_i make half a loop and the crossings of $]m_i, d_i[$ make a (almost) full loop until they reach the standard position with respect to d_i.

As in the proof of Lemma 1 on p. 25, on both sides of each crossing c of $]m_i, d_i[$ the degree is incremented by $(-\sigma(c))$, and it is incremented by $(-\sigma(m_i)/2)$ on both sides of m_i and by $(-\sigma(d_i)/2)$ on both sides of d_i so that after this modification the degree $d_e'(\beta_j)$ of β_j reads

$$d_e'(\beta_j) = d_e(\beta_j) - 2\sum_{i=1}^{g} \langle |m_i, d_i|_\alpha, \beta_j \rangle.$$

Before this modification, the degree $d_e(|d_{\psi(j(c))}, c|_\beta)$ of the tangent to β_j from $d_{\psi(j(c))}$ to c was

$$\begin{cases} d_e(|d_{\psi(j(c))}, m_{j(c)}|_\beta) + d_e(|m_{j(c)}, c|_\beta) & \text{if } c \in [m_{j(c)}, d_{\psi(j(c))}[_\beta \\ d_e(|d_{\psi(j(c))}, m_{j(c)}|_\beta) + d_e(|m_{j(c)}, c|_\beta) - d_e(\beta_{j(c)}) & \text{if } c \in [d_{\psi(j(c))}, m_{j(c)}[_\beta. \end{cases}$$

After the modification, it reads

$$d_e'(|d_{\psi(j(c))}, c|_\beta) = d_e(|d_{\psi(j(c))}, c|_\beta) - 2\sum_{i=1}^{g} \langle |m_i, d_i|_\alpha, |d_{\psi(j(c))}, c|_\beta \rangle.$$

Now

$$e(\mathcal{D}, w, \mathfrak{m}') = \sum_{c \in \mathcal{C}} \mathcal{J}_{j(c)i(c)} \sigma(c) d_e'(c)$$

where $d_e'(c) = d_e'(|d_{\psi(j(c))}, c|_\beta) - \sum_{(r,s) \in \underline{g}^2} \mathcal{J}_{sr} \langle \alpha_r, |d_{\psi(j(c))}, c|_\beta \rangle d_e'(\beta_s)$. Thus

$$e(\mathcal{D}, w, \mathfrak{m}') - e(\mathcal{D}, w, \mathfrak{m}) = e_1(w, \mathfrak{m}, \mathfrak{m}') - e_2(w, \mathfrak{m}, \mathfrak{m}')$$

where

$$e_1(w, \mathfrak{m}, \mathfrak{m}') = \sum_{c \in \mathcal{C}} \mathcal{J}_{j(c)i(c)} \sigma(c) \left(d_e'(|d_{\psi(j(c))}, c|_\beta) - d_e(|m_{j(c)}, c|_\beta) \right)$$

and $e_2(w,\mathfrak{m},\mathfrak{m}')$ is equal to

$$\sum_{c\in\mathcal{C}} \mathcal{J}_{j(c)i(c)}\sigma(c) \sum_{(r,s)\in\underline{g}^2} \mathcal{J}_{sr}\Big(\langle\alpha_r,|d_{\psi(j(c))},c|_\beta\rangle d'_e(\beta_s)-\langle\alpha_r,|m_{j(c)},c|_\beta\rangle d_e(\beta_s)\Big).$$

We have

$$\begin{aligned}
e_1(w,\mathfrak{m},\mathfrak{m}') &= \sum_{c\in\mathcal{C}} \mathcal{J}_{j(c)i(c)}\sigma(c)d_e(|d_{\psi(j(c))},m_{j(c)}|_\beta) - \sum_{c\in[d_{\psi(j(c))},m_{j(c)}[_\beta} \mathcal{J}_{j(c)i(c)}\sigma(c)d_e(\beta_{j(c)}) \\
&\quad -2\sum_{c\in\mathcal{C}} \mathcal{J}_{j(c)i(c)}\sigma(c)\sum_{i=1}^{g}\langle|m_i,d_i|_\alpha,|d_{\psi(j(c))},c|_\beta\rangle \\
&= \sum_{j=1}^{g} d_e(|d_{\psi(j)},m_j|_\beta) - \sum_{j=1}^{g}\sum_{i=1}^{g}\mathcal{J}_{ji}\langle\alpha_i,[d_{\psi(j)},m_j[_\beta\rangle d_e(\beta_j) \\
&\quad -2\sum_{c\in\mathcal{C}} \mathcal{J}_{j(c)i(c)}\sigma(c)\sum_{i=1}^{g}\langle|m_i,d_i|_\alpha,|d_{\psi(j(c))},c|_\beta\rangle.
\end{aligned}$$

Since $\Big(\langle\alpha_r,|d_{\psi(j(c))},c|_\beta\rangle d'_e(\beta_s)-\langle\alpha_r,|m_{j(c)},c|_\beta\rangle d_e(\beta_s)\Big)$ is equal to

$$\begin{aligned}
-2\langle\alpha_r,|d_{\psi(j(c))},c|_\beta\rangle \sum_{i=1}^{g}\langle|m_i,d_i|_\alpha,\beta_s\rangle + \langle\alpha_r,|d_{\psi(j(c))},m_{j(c)}|_\beta\rangle d_e(\beta_s) \\
-\chi_{[d_{\psi(j(c))},m_{j(c)}[_\beta}(c)\langle\alpha_r,\beta_{j(c)}\rangle d_e(\beta_s),
\end{aligned}$$

where $\chi_{[d_{\psi(j(c))},m_{j(c)}[_\beta}(c) = \begin{cases} 1 & \text{if } c\in[d_{\psi(j(c))},m_{j(c)}[_\beta \\ 0 & \text{otherwise.}\end{cases}$

We have

$$\begin{aligned}
e_2(w,\mathfrak{m},\mathfrak{m}') &= -2\sum_{c\in\mathcal{C}} \mathcal{J}_{j(c)i(c)}\sigma(c) \sum_{(r,s,i)\in\underline{g}^3} \mathcal{J}_{sr}\langle|m_i,d_i|_\alpha,\beta_s\rangle\langle\alpha_r,|d_{\psi(j(c))},c|_\beta\rangle \\
&\quad + \sum_{(r,s,j)\in\underline{g}^3} \mathcal{J}_{sr}\langle\alpha_r,|d_{\psi(j)},m_j|_\beta\rangle d_e(\beta_s) \\
&\quad - \sum_{(r,s,j)\in\underline{g}^3} \mathcal{J}_{sr}\sum_{i=1}^{g}\mathcal{J}_{ji}\langle\alpha_i,[d_{\psi(j)},m_j[_\beta\rangle\langle\alpha_r,\beta_j\rangle d_e(\beta_s) \\
&= -2\sum_{c\in\mathcal{C}} \mathcal{J}_{j(c)i(c)}\sigma(c) \sum_{(r,s,i)\in\underline{g}^3} \mathcal{J}_{sr}\langle|m_i,d_i|_\alpha,\beta_s\rangle\langle\alpha_r,|d_{\psi(j(c))},c|_\beta\rangle \\
&\quad + \sum_{(r,s,j)\in\underline{g}^3} \mathcal{J}_{sr}\langle\alpha_r,|d_{\psi(j)},m_j|_\beta\rangle d_e(\beta_s)
\end{aligned}$$

(Cont. next page)

$$-\sum_{(i,j)\in\underline{g}^2}\mathcal{J}_{ji}\langle\alpha_i,[d_{\psi(j)},m_j[_\beta\rangle d_e(\beta_j).$$

Therefore, according to Lemma 16 on p. 49,

$$e(\mathcal{D},w,\mathfrak{m}')-e(\mathcal{D},w,\mathfrak{m})=2lk(L(\mathfrak{m}',\mathfrak{m}),L(\mathfrak{m}')_{\parallel})+V$$

where

$$\begin{aligned}V&=\sum_{j=1}^{g}d_e(]d_{\psi(j)},m_j[_\beta)-\sum_{(r,s,j)\in\underline{g}^3}\mathcal{J}_{sr}\langle\alpha_r,]d_{\psi(j)},m_j[_\beta\rangle d_e(\beta_s)\\&=-\sum_{j=1}^{g}d_e(]m_j,d_{\psi(j)}[_\beta)+\sum_{(r,s,j)\in\underline{g}^3}\mathcal{J}_{sr}\langle\alpha_r,]m_j,d_{\psi(j)}[_\beta\rangle d_e(\beta_s)\\&\quad+\sum_{j=1}^{g}d_e(\beta_j)-\sum_{(r,s,j)\in\underline{g}^3}\mathcal{J}_{sr}\langle\alpha_r,\beta_j\rangle d_e(\beta_s).\end{aligned}$$

Since the last line vanishes, we get the result. □

Corollary 4 and Proposition 11 on p. 48 and on p. 49 allow us to define the function $\tilde{\lambda}$ of Heegaard diagrams

$$\tilde{\lambda}(\mathcal{D})=\frac{\tilde{\Theta}(\mathcal{D},w,\mathfrak{m})}{6}-\frac{p_1(X(w,\mathfrak{m}))}{24},$$

which does not depend on the orientations and numberings of the curves α_i and β_j, and which is also unchanged by permuting the roles of the α_i and β_j, thanks to Corollary 3 on p. 37.

6 Invariance of $\tilde{\lambda}$

In this section, we are first going to prove that $\tilde{\lambda}$ only depends on the Heegaard decomposition induced by $\mathcal{D}$ of M, and not on the curves α_i and β_j. Then it will be easily observed that $\tilde{\lambda}$ is additive under connected sum of Heegaard decompositions and that $\tilde{\lambda}$ maps the genus one Heegaard decomposition of S^3 to 0. Since according to the so-called Reidemeister-Singer theorem, two Heegaard decompositions of a 3-manifold become diffeomorphic after some connected sums with this Heegaard decomposition of S^3, we will conclude that $\tilde{\lambda}$ is an invariant of rational homology 3-spheres, which is additive under connected sum.

6.1 Systems of meridians of a handlebody

A *handle slide* in a system $\{\alpha_i\}_{i\in\underline{g}}$ of meridians of a curve α_k across a curve α_j, with $j \neq k$, is defined as follows: Choose a path γ in $\partial H_{\mathcal{A}}$ from a point $\gamma(0) \in \alpha_k$ to a point $\gamma(1) \in \alpha_j$ such that $\gamma(]0,1[)$ does not meet $\cup_{i\in\underline{g}}\alpha_i$ and change α_k to the band sum α'_k of α_k and a parallel of α_j on the γ-side as in Figure 16.

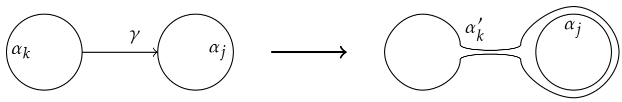

Figure 16 – Handle slide in $\partial H_{\mathcal{A}}$

A *right-handed Dehn twist* about a simple closed curve $K(S^1)$ of a surface F is a homeomorphism of F that fixes the exterior of a collar $K(S^1)\times[-\pi,\pi]$ of K in F pointwise, and that maps $(K(\exp(i\theta)),t)$ to $(K(\exp(i(\theta+t+\pi))),t)$.

In order to prove that $\tilde{\lambda}$ only depends on the Heegaard decompositions and not on the chosen systems $\{\alpha_i\}_{i\in\underline{g}}$ and $\{\beta_j\}_{j\in\underline{g}}$ of meridians of $H_{\mathcal{A}}$ and $H_{\mathcal{B}}$ we will use the following standard theorem.

Theorem 3 – *Up to isotopy, renumbering of meridians, orientation reversals of meridians, two meridian systems of a handlebody are obtained from one another by a finite number of handle slides.*

Proof. Let $\{\alpha_i\}_{i\in\underline{g}}$ and $\{\alpha'_i\}_{i\in\underline{g}}$ be two systems of meridians of $H_{\mathcal{A}}$. There exists an orientation-preserving diffeomorphism of $H_{\mathcal{A}}$ that maps the first system to the second one.

See $H_{\mathcal{A}}$ as the unit ball $B(1)$ of $\mathbb{R}^3$ with embedded handles $D(\alpha_i)\times[0,1]$ attached along $D(\alpha_i)\times\partial[0,1]$, so that there is a rotation ρ of angle $\frac{2\pi}{g}$ of $\mathbb{R}^3$ that maps $H_{\mathcal{A}}$ to itself and that permutes the handles, cyclically. See the meridians disks bounded by the α_i as disks $D(\alpha_i) = D(\alpha_i)\times\{\frac{1}{2}\}$ that cut the handles. Let H_i denote the handle of α_i. In Suzuki (1977, Theorem 4.1), Suzuki proves that the group of isotopy classes of orientation-preserving diffeomorphisms of $H_{\mathcal{A}}$ is generated by 6 generators represented by the following diffeomorphisms

- the rotation ρ above of Suzuki (1977, p. 3.1), which permutes the α_i, cyclically,

and the remaining 5-diffeomorphisms, which fix all the handles H_i, for $i > 2$, pointwise,

- the knob interchange ρ_{12} of Suzuki (1977, p. 3.4), which exchanges H_1 and H_2 and maps α_1 to α_2 and α_2 to α_1,
- the knob twist ω_1 of Suzuki (1977, p. 3.2), which fixes H_2 pointwise, and which maps α_1 to the curve with opposite orientation, (it is the final time

of an ambient isotopy of $\mathbb{R}^3$ that performs a half-twist on a disk of $H_{\mathcal{A}}$ that contains the two feet ($D(\alpha_1)\times\{0\}$ and $D(\alpha_1)\times\{1\}$) of the handle H_1),

- the right-handed Dehn twist τ_1^{-1} of Suzuki (1977, p. 3.3) along a curve parallel to α_1,
- the sliding ξ_{12} of Suzuki (1977, 3.5 and 3.9), which is the final time of an ambient isotopy of $\mathbb{R}^3\times\mathbb{R}$ that fixes the handles H_i, for $i>2$, pointwise, and that lets one foot of H_1 slide along a circle parallel to α_2 once,
- the sliding θ_{12} of Suzuki (1977, 3.5 and 3.8), which is the final time of an ambient isotopy of $\mathbb{R}^3$ that fixes the handles H_i, for $i>2$, pointwise, and that lets one foot of H_1 slide along a circle a_2 that cuts α_2 once and that does not meet the interiors of the H_i, for $i\neq 2$.

All these generators are described more precisely in Suzuki (1977, Section 3). All of them except θ_{12} fix the set of curves α_i seen as unoriented curves, while θ_{12} fixes all the curves α_i, for $i\neq 2$ pointwise. When the foot of H_1 moves along the circle a_2, the curves that cross a_2 move with it, so that the meridian α_2 is changed as in Figure 17, which is a figure of a handle slide of α_2 across α_1. □

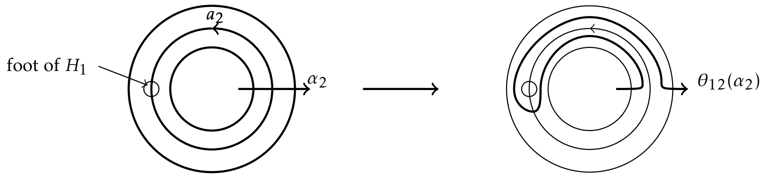

Figure 17 – Action of θ_{12} on α_2

6.2 Isotopies of systems of meridians

When the α_i are fixed on $\partial H_{\mathcal{A}}$, and when the β_j vary by isotopy, the only generic encountered accidents are the births or deaths of bigons, which modify the Heegaard diagram as in Figure 18, which represents the birth of a bigon between an arc of α_i and an arc of β_j.

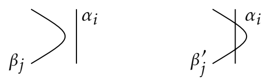

Figure 18 – Birth of a bigon

Therefore, in order to prove that $\tilde{\lambda}$ is invariant when the β_j (or the α_i) are moved by an isotopy, it is enough to prove the following proposition:

Proposition 13 – *For any Heegaard diagram $\mathcal{D}$ and $\mathcal{D}'$ such that $\mathcal{D}'$ is obtained from $\mathcal{D}$ by a birth of a bigon as above.*

$$\tilde{\lambda}(\mathcal{D}') = \tilde{\lambda}(\mathcal{D}).$$

Since we know that changing the orientation of α_i does not modify $\tilde{\lambda}$, we assume that our born bigon is one of the two bigons shown in Figure 19, with two arcs going from a crossing e to a crossing f, without loss.

Figure 19 – The considered two bigons

We fix a matching $\mathfrak{m}$ for $\mathcal{D} = ((\alpha_i), (\beta_j))$ and the same one for $\mathcal{D}'$, and an exterior point w of $\mathcal{D}'$ outside the bigon so that w is also an exterior point of $\mathcal{D}$.

Lemma 17 – *We have*

$$p_1(X(\mathcal{D}, w, \mathfrak{m})) = p_1(X(\mathcal{D}', w, \mathfrak{m}))$$

Proof. The two fields $X(\mathcal{D}, w, \mathfrak{m})$ and $X(\mathcal{D}', w, \mathfrak{m})$ may be assumed to coincide outside a ball that contains the past and the future in $f_M^{-1}([-2,7])$ of a disk of $H_{\mathcal{A}}$ around the bigon, with respect to a flow associated with $\mathcal{D}'$. Since both fields are positive normals to the level surfaces of f_M on this ball they are homotopic. □

Now, Proposition 13 is a direct consequence of Lemmas 18 and 19 on the current page and on the next page.

Lemma 18 – *We have*

$$\ell_2(\mathcal{D}') = \ell_2(\mathcal{D}) + \mathcal{J}_{ji}/2$$
$$s_\ell(\mathcal{D}', \mathfrak{m}) = s_\ell(\mathcal{D}, \mathfrak{m})$$

Proof. Let $\mathcal{C}$ be the set of crossings of $\mathcal{D}$. Note that $\sigma(f) = -\sigma(e)$. With the notations of Proposition 4 on p. 31,

$$\begin{aligned}G(\mathcal{D}') - G(\mathcal{D}) = &\sum_{c \in \mathcal{C}} \mathcal{J}_{j(c)i}\mathcal{J}_{ji(c)}\sigma(c)\sigma(f)\gamma(c) \times (\gamma(f) - \gamma(e))_{\|} \\ &+ \sum_{d \in \mathcal{C}} \mathcal{J}_{ji(d)}\mathcal{J}_{j(d)i}\sigma(d)\sigma(f)(\gamma(f) - \gamma(e)) \times \gamma(d)_{\|} \\ &+ \mathcal{J}_{ji}^2(\gamma(f) - \gamma(e)) \times (\gamma(f) - \gamma(e))_{\|} \\ &- \mathcal{J}_{ji}\sigma(f)(\gamma(f) \times \gamma(f)_{\|} - \gamma(e) \times \gamma(e)_{\|}).\end{aligned}$$

Use Proposition 3 on p. 30 to compute $\ell^{(2)}(G(\mathcal{D}')-G(\mathcal{D}))$ with the basepoints of $\mathfrak{m}$, so that for any $c \in \mathcal{C}$,

$$\ell(c,f)-\ell(c,e)=\langle [p(\alpha(c)),c|_\alpha, |e,f|_\beta\rangle - \sum_{(k,\ell)\in \underline{g}^2} \mathcal{J}_{\ell k}\langle [p(\alpha(c)),c|_\alpha, \beta_\ell\rangle\langle \alpha_k, |e,f|_\beta\rangle = 0$$

since $\langle \alpha_k, |e,f|_\beta\rangle = 0$ for any k, and $\langle [p(\alpha(c)),c|_\alpha, |e,f|_\beta\rangle = 0$ for any $c\in\mathcal{C}$. Similarly, for any $d \in \mathcal{C}$, $\ell(e,d)=\ell(f,d)$ and

$$\ell(f-e,f-e)=\langle |e,f|_\alpha, |e,f|_\beta\rangle = 0.$$

Finally,

$$\ell_2(\mathcal{D}')-\ell_2(\mathcal{D}) = -\mathcal{J}_{ji}\sigma(f)(\ell(f,f)-\ell(e,e))$$

where

$$\ell(f,f)-\ell(e,e)=\langle [e,f|_\alpha,[e,f|_\beta\rangle - \langle [e,e|_\alpha,[e,e|_\beta\rangle = \sigma(e)+\frac{1}{4}\sigma(f)-\frac{1}{4}\sigma(e) = -\frac{1}{2}\sigma(f)$$

so that $\ell_2(\mathcal{D}')-\ell_2(\mathcal{D}) = \frac{1}{2}\mathcal{J}_{ji}$. Similarly, $s_\ell(\mathcal{D}',\mathfrak{m}) = s_\ell(\mathcal{D},\mathfrak{m})$. □

Lemma 19 – *We have*

$$e(\mathcal{D}',w,\mathfrak{m}) = e(\mathcal{D},w,\mathfrak{m}) + \mathcal{J}_{ji}/2.$$

Proof. Adding a bigon changes Figure 3 on p. 23 as in Figure 20.

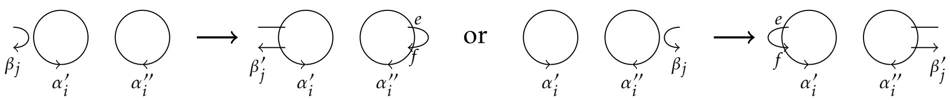

Figure 20 – Adding a bigon

In particular, the $d_e(\beta_s)$ of Section 1.5 on p. 23 are unchanged, and so are the $d_e(c)$, for $c \in \mathcal{C}$. Then $e(\mathcal{D}',w,\mathfrak{m}) - e(\mathcal{D},w,\mathfrak{m}) = \mathcal{J}_{ji}\sigma(f)d_e(|e,f|_\beta)$, which is $\frac{1}{2}\mathcal{J}_{ji}$, according to Figure 20. □

Remark 2 – If the two arcs of the bigon did not begin at the same vertex, then $\mathcal{J}_{ji}$ would be replaced by $-\mathcal{J}_{ji}$ in the results of Lemmas 18 and 19 on the previous page and on the current page.

6.3 Handle slides

This section is devoted to proving that $\tilde{\lambda}$ is invariant under handle slide. Since $\tilde{\lambda}$ depends neither on the orientations of the curves α_i and β_j, nor on their numberings, and since permuting the roles of the α_i and β_j does not change $\tilde{\lambda}$, it is sufficient to study a handle slide that transforms $\mathcal{D}$ to a diagram $\mathcal{D}'$ by changing β_1 to a band sum β_1' of β_1 and the parallel β_2^+ of β_2 (on its positive side) as in Figure 21. Up to the isotopies treated in the previous section, we may assume that the path γ from β_1 to β_2 does not meet the curves α_i, without loss, and we do. The first crossing on β_2^+ will be called e^+. It corresponds to a crossing $e \in \alpha_{i(e)} \cap \beta_2$ as in Figure 21.

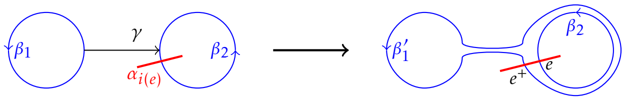

Figure 21 – The considered handle slide

Fix w outside a neighborhood of the path γ and β_2 so that it makes sense to say that w is the same for $\mathcal{D}$ and $\mathcal{D}'$. Fix a matching $\mathfrak{m}$ for $\mathcal{D}$. Assume $\mathfrak{m} = \{m_i\}_{i\in\underline{g}}$ and $m_i \in \alpha_i \cap \beta_i$ (by renumbering the α curves if necessary). The set $\mathcal{C}'$ of crossings of $\mathcal{D}'$ contains $\mathcal{C}$ so that $\mathfrak{m}$ is also a matching for $\mathcal{D}'$.

Under these assumptions, we are going to prove that $\tilde{\lambda}(\mathcal{D}') = \tilde{\lambda}(\mathcal{D})$ by proving the following lemmas.

Lemma 20 – *We have*

$$p_1(X(\mathcal{D}, w, \mathfrak{m})) = p_1(X(\mathcal{D}', w, \mathfrak{m})).$$

Lemma 21 – *We have*

$$\begin{aligned}\ell_2(\mathcal{D}') - \ell_2(\mathcal{D}) &= \sum_{c\in\beta_2, d\in[e,c|_\beta} \sigma(c)\sigma(d)\mathcal{J}_{1i(c)}\mathcal{J}_{2i(d)} \\ &\stackrel{def}{=} \sum_{c\in\beta_2, d\in[e,c[_\beta} \sigma(c)\sigma(d)\mathcal{J}_{1i(c)}\mathcal{J}_{2i(d)} + \frac{1}{2}\sum_{c\in\beta_2} \mathcal{J}_{1i(c)}\mathcal{J}_{2i(c)}.\end{aligned}$$

Lemma 22 – *We have*

$$s_\ell(\mathcal{D}', \mathfrak{m}) - s_\ell(\mathcal{D}, \mathfrak{m}) = \sum_{d\in\beta_2, c\in[e,d|_\beta} \sigma(c)\sigma(d)\mathcal{J}_{1i(c)}\mathcal{J}_{2i(d)} - \sum_{c\in[e,m_2|_\beta} \sigma(c)\mathcal{J}_{1i(c)}.$$

Lemma 23 – *We have*

$$e(\mathcal{D}', w, \mathfrak{m}) - e(\mathcal{D}, w, \mathfrak{m}) = \sum_{c\in]m_2,e[_\beta} \sigma(c)\mathcal{J}_{1i(c)}.$$

Since

$$\sum_{c\in]m_2,e[_\beta} \sigma(c)\mathcal{J}_{1i(c)} + \sum_{c\in [e,m_2]_\beta} \sigma(c)\mathcal{J}_{1i(c)} = \sum_{c\in\beta_2} \sigma(c)\mathcal{J}_{1i(c)} = \sum_{i=1}^{g} \mathcal{J}_{1i}\langle\alpha_i,\beta_2\rangle = 0$$

and the sum

$$\sum_{c\in\beta_2, d\in[e,c]_\beta} \sigma(c)\sigma(d)\mathcal{J}_{1i(c)}\mathcal{J}_{2i(d)} + \sum_{d\in\beta_2, c\in[e,d]_\beta} \sigma(c)\sigma(d)\mathcal{J}_{1i(c)}\mathcal{J}_{2i(d)}$$

is equal to

$$\sum_{(c,d)\in\beta_2^2} \sigma(c)\sigma(d)\mathcal{J}_{1i(c)}\mathcal{J}_{2i(d)} = 0,$$

these four lemmas imply that $\tilde{\lambda}(\mathcal{D}') = \tilde{\lambda}(\mathcal{D})$.

Proof (of Lemma 20 on the previous page). Let $X = X(\mathcal{D}, w, \mathfrak{m})$ and $X' = X(\mathcal{D}', w, \mathfrak{m})$. First note that X and X' coincide in $H_{\mathcal{A}}$. We describe a homotopy $(Y_t)_{t\in[0,1]}$ from $Y_0 = (-X)$ and $Y_1 = (-X')$ on $H_{\mathcal{B}}$.

See $(-X)$ in $H_{\mathcal{B}}$ as the upward vertical field in the first picture of Figure 23 on the next page. This field is an outward normal to $H_{\mathcal{B}}$ except around w, which is not shown in our figures, and around the crossings of $\mathfrak{m}$, more precisely on the gray disks D_i shown in Figure 22. Inside the disks D_i, $(-X)$ is an inward normal to $H_{\mathcal{B}}$. On the boundary of this disk, it is tangent to the surface. Our homotopy will fix $(-X)$ in the neighborhood of w where $(-X)$ is not an outward normal to $H_{\mathcal{B}}$, and the locus of $\partial H_{\mathcal{B}}$ where Y_t is a positive (resp. negative) normal to $H_{\mathcal{B}}$ will not depend on t. Thus this homotopy can be canonically modified (without changing the locus where Y_t is a positive (resp. negative) normal to $H_{\mathcal{B}}$) so that Y_t is fixed on $\partial H_{\mathcal{B}}$.

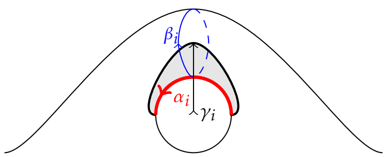

Figure 22 – The front part of the disk D_i where the field points inward the surface

Observe that there is no loss in assuming that the path γ from β_1 to β_2 that parametrizes the handle slide is as in the first picture of Figure 23 on the next page. The next pictures describe various positions of $H_{\mathcal{B}}$ under an ambient isotopy $(h_t)_{t\in[0,1]}$ of $\mathbb{R}^3$, which first moves the handle of β_2 upward (second picture), slides it over the handle of β_1 (fourth picture), moves the handle of β_1 upward (fifth picture) and replaces the slid foot of H_2 in its original position by letting it slide away from the handles (last picture). The isotopy $(h_t)_{t\in[0,1]}$ starts with h_0, which is the Identity, and finishes with a homeomorphism h_1 of $\mathbb{R}^3$ that maps $H_{\mathcal{B}}$ to itself. Let $\overrightarrow{N}$ be the upward vector field of $\mathbb{R}^3$. Then $(h_t)_*^{-1}(\overrightarrow{N}_{|h_t(H_{\mathcal{B}})})$ defines a homotopy of nowhere zero vector fields from $Y_0 = (-X)$ and $Y_1 = (-X')$ on $H_{\mathcal{B}}$ that behaves as wanted on the boundary. □

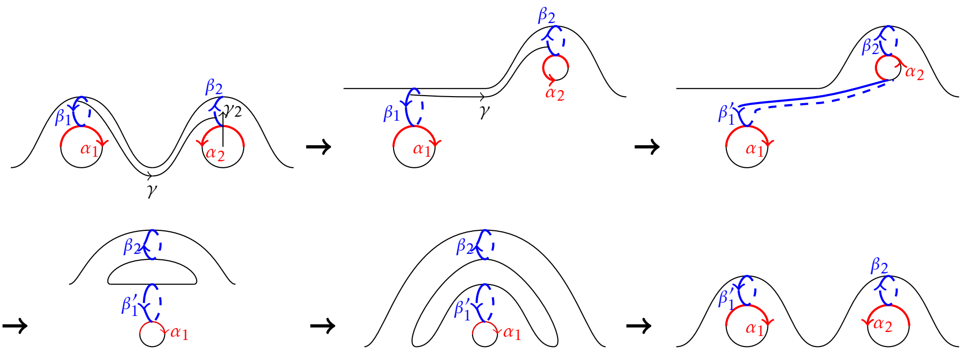

Figure 23 – Handle slide

Let us start with common preliminaries for the proofs of the remaining three lemmas.

Set $\mathcal{J}'_{2i} = \mathcal{J}_{2i} - \mathcal{J}_{1i}$. For any interval I of an α_r, $\langle I, \beta'_1\rangle = \langle I, \beta_1 + \beta_2^+\rangle$ and

$$\left\langle I, \mathcal{J}_{1i}\beta'_1 + \mathcal{J}'_{2i}\beta_2\right\rangle = \left\langle I, \mathcal{J}_{1i}\beta_1 + \mathcal{J}_{2i}\beta_2 + \mathcal{J}_{1i}(\beta_2^+ - \beta_2)\right\rangle.$$

Set $\mathcal{J}'_{ji} = \mathcal{J}_{ji}$ for any (j,i) such that $j \neq 2$. Every quantity associated with $\mathcal{D}'$ will have a prime superscript. Our definitions of the $\mathcal{J}'_{ji}$ ensure that

$$\left\langle \alpha_k, \sum_j \mathcal{J}'_{ji}\beta'_j\right\rangle = \left\langle \alpha_k, \sum_j \mathcal{J}_{ji}\beta_j\right\rangle = \delta_{ik},$$

for any i and k, as required.

Let $\mathcal{C}_2$ be the set of crossings of $\mathcal{D}$ on β_2, and let $\mathcal{C}_2^+$ be the set of crossings of $\mathcal{D}'$ on β_2^+, $\mathcal{C}_2^+$ is in natural one-to-one correspondence with $\mathcal{C}_2$ and the crossing of $\mathcal{C}_2^+$ that corresponds to a crossing c of $\mathcal{C}_2$ will be denoted by c^+.

$$\mathcal{C}' = \mathcal{C} \cup \mathcal{C}_2^+.$$

Proof (of Lemma 23 on p. 57). Without loss, assume that β_2 goes from right to left at the place of the band sum as in Figure 24 on the next page. Then β_1 is above β_2 and it goes from left to right. Thus after the band sum, the degree of β'_1 is increased by $(-1/2)$ before and after β_2^+ and by $(1/2)$ before and after m_2.

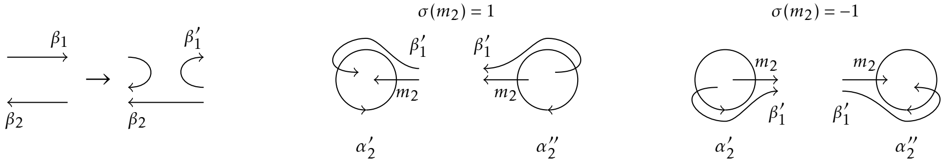

Figure 24 – Variation of d_e

Therefore $d'_e(\beta'_1) = d_e(\beta_1) + d_e(\beta_2)$, and, for any i,

$$\sum_{j=1}^{g} \mathcal{J}'_{ji} d'_e(\beta'_j) = \sum_{j=1}^{g} \mathcal{J}_{ji} d_e(\beta_j).$$

Then for any c that is not in $[e^+, m_1[_{\beta'_1}$, $d'_e(c) = d_e(c)$. Since

$$d_e(\beta_2) - \sum_{(r,s)\in \underline{g}^2} \mathcal{J}_{sr}\langle \alpha_r, \beta_2\rangle d_e(\beta_s) = 0,$$

for any $c \in [e^+, m_1[_{\beta'_1} \setminus \beta_2^+$, $d'_e(c) = d_e(c)$, too, so that

$$e(\mathcal{D}', w, \mathfrak{m}) - e(\mathcal{D}, w, \mathfrak{m}) = \sum_{c\in\beta_2} \mathcal{J}_{1i(c)}\sigma(c)(d'_e(c^+) - d_e(c)).$$

For $c \in \beta_2$,

$$d'_e(c^+) - d'_e(e^+) = \begin{cases} d_e(c) - d_e(e) & \text{if } c \in]e, m_2[\\ d_e(c) - d_e(e) + 1 & \text{if } c \in]m_2, e[\\ d_e(c) - d_e(e) + \frac{1}{2} & \text{if } c = m_2. \end{cases}$$

□

For the remaining two lemmas, for any 2-cycle $G = \sum_{(c,d)\in(\mathcal{C}')^2} g_{cd}(\gamma(c)\times\gamma(d)_{\|})$ of M^2, we compute $\ell^{(2)}(G)$ with Proposition 3 on p. 30 with

$$\ell(c,d) = \langle [p(\alpha(c)), c|_\alpha, [p(\beta(d)), d|_\beta\rangle - \sum_{(i,j)\in\underline{g}^2} \mathcal{J}'_{ji}\langle [p(\alpha(c)), c|_\alpha, \beta'_j\rangle\langle \alpha_i, [p(\beta(d)), d|_\beta\rangle$$

where $p(\beta_2) = e$ and $p(\beta'_1)$ is the first crossing of β'_1 after β_2^+ on β_1, the $p(\alpha_i)$ are not on β_2^+, and, if $p(\alpha_i) \in \beta_2$, then $\sigma(p(\alpha_i)) = 1$ (up to changing the orientation of α_i). This map $\ell^{(2)}$ may be used for any 2-cycle $G = \sum_{(c,d)\in\mathcal{C}^2} g_{cd}(\gamma(c)\times\gamma(d)_{\|})$ of M^2, as well, and we use it.

(The map contructed from ℓ by adding

$$\sum_{i\in\underline{g}} \mathcal{J}_{1i}\langle [p(\alpha(c)), c|_\alpha, \beta_2^+ - \beta_2\rangle\langle \alpha_i, [p(\beta(d)), d|_\beta\rangle$$

to $\ell(c,d)$ when $c \in \beta_2$ would clearly give rise to an appropriate map $\ell^{(2)}$ with respect to the diagram $\mathcal{D}$. Since $\langle [p(\alpha(c)), c|_\alpha, \beta_2^+ - \beta_2\rangle = -\frac{1}{2}$ for any $c \in \beta_2$, and since $\sum_{(c,d)\in\mathcal{C}^2; c\in\beta_2} g_{cd} = 0$ for any d as in the proof of Proposition 3 on p. 30, ℓ works as well.)

Lemma 24 – *Recall $\mathcal{C}' = \mathcal{C} \cup \mathcal{C}_2^+$. Let $(c,d) \in \mathcal{C}^2$.*

- *If $c \in \beta_2$, then*

$$\ell(c^+,d) - \ell(c,d) + \frac{1}{2}\sum_{i=1}^{g} \mathcal{J}_{2i}\langle \alpha_i, [p(\beta(d)), d|_\beta\rangle = \begin{cases} 0 & \text{if } d \notin \beta_2 \\ 0 & \text{if } d \in \beta_2 \text{ and } c \notin [e,d]_\beta \\ \frac{1}{2} & \text{if } d \in \beta_2 \text{ and } c \in [e,d[_\beta \\ \frac{1}{4} & \text{if } c = d. \end{cases}$$

- *If $d \in \beta_2$, then*

$$\ell(c,d^+) - \ell(c,d) = \begin{cases} \frac{1}{2}\delta_{j(c)2} - \frac{1}{2} & \text{if } c \in [e,d[_\beta \\ \frac{1}{2}\delta_{j(c)2} - \frac{1}{4} & \text{if } c = d \\ \frac{1}{2}\delta_{j(c)2} & \text{if } c \notin [e,d]_\beta. \end{cases}$$

- *If $(c,d) \in \mathcal{C}_2^2$, then*

$$\ell(c^+,d^+) - \ell(c^+,d) = \ell(c,d^+) - \ell(c,d).$$

Proof. Let $(c,d) \in \mathcal{C}^2$. Assume $c \in \beta_2$. If $\sigma(c) = 1$, then

$$\begin{aligned} \ell(c^+,d) - \ell(c,d) &= \langle |c,c^+|_\alpha, [p(\beta(d)), d|_\beta\rangle - \sum_{(i,j)\in\underline{g}^2} \mathcal{J}'_{ji}\langle |c,c^+|_\alpha, \beta'_j\rangle\langle \alpha_i, [p(\beta(d)), d|_\beta\rangle \\ &= \langle |c,c^+|_\alpha, [p(\beta(d)), d|_\beta\rangle - \frac{1}{2}\sum_{i=1}^{g}(\mathcal{J}'_{2i} + \mathcal{J}'_{1i})\langle \alpha_i, [p(\beta(d)), d|_\beta\rangle. \end{aligned}$$

If $\sigma(c) = -1$,

$$\ell(c^+,d) - \ell(c,d) = -\langle |c^+,c|_\alpha, [p(\beta(d)), d|_\beta\rangle - \frac{1}{2}\sum_{i=1}^{g} \mathcal{J}_{2i}\langle \alpha_i, [p(\beta(d)), d|_\beta\rangle.$$

Let $d \in \beta_2$. For any interval I of an α_i, $\langle I, [p(\beta'_1), d^+|_\beta\rangle = \langle I, \beta_1 + [e^+, d^+|_\beta\rangle$.

$$\begin{aligned} \ell(c,d^+) - \ell(c,d) = \langle [p(\alpha(c)), c|_\alpha, \beta_1 + [e^+,d^+|_\beta - [e,d|_\beta\rangle \\ - \sum_{(i,j)\in\underline{g}^2} \mathcal{J}'_{ji}\langle [p(\alpha(c)), c|_\alpha, \beta'_j\rangle\langle \alpha_i, \beta_1 + [e^+,d^+|_\beta - [e,d|_\beta\rangle \end{aligned}$$

where

$$\langle \alpha_i, \beta_1 + [e^+, d^+|_\beta - [e,d|_\beta\rangle = \langle \alpha_i, \beta_1\rangle, = \langle \alpha_i, \beta_1' - \beta_2'\rangle$$

$$\begin{aligned}\sum_{(i,j)\in\underline{g}^2} \mathcal{J}'_{ji}\langle [p(\alpha(c)), c|_\alpha, \beta'_j\rangle\langle \alpha_i, \beta'_1 - \beta'_2\rangle &= \sum_{j\in\underline{g}}(\delta_{j1} - \delta_{j2})\langle [p(\alpha(c)), c|_\alpha, \beta'_j\rangle \\ &= \langle [p(\alpha(c)), c|_\alpha, \beta'_1 - \beta'_2\rangle \\ &= \langle [p(\alpha(c)), c|_\alpha, \beta_1 + \beta_2^+ - \beta_2\rangle, \\ \text{and } \langle [p(\alpha(c)), c|_\alpha, \beta_2^+ - \beta_2\rangle &= -\frac{1}{2}\delta_{j(c)2}\end{aligned}$$

so that

$$\begin{aligned}\ell(c,d^+) - \ell(c,d) &= \frac{1}{2}\delta_{j(c)2} + \langle [p(\alpha(c)), c|_\alpha, [e^+, d^+|_\beta - [e,d|_\beta\rangle \\ &= \begin{cases} \frac{1}{2}\delta_{j(c)2} - \frac{1}{2} & \text{if } c \in [e,d[_\beta \\ \frac{1}{2}\delta_{j(c)2} - \frac{1}{4} & \text{if } c = d \\ \frac{1}{2}\delta_{j(c)2} & \text{if } c \notin [e,d]_\beta. \end{cases}\end{aligned}$$

When $c \in \beta_2$, we similarly get $\ell(c^+, d^+) - \ell(c^+, d) = \frac{1}{2} + \langle [p(\alpha(c)), c^+|_\alpha, [e^+, d^+|_\beta - [e,d|_\beta\rangle$ so that $\ell(c^+, d^+) - \ell(c^+, d) = \ell(c,d^+) - \ell(c,d)$. □

Proof (of Lemma 22 on p. 57). Set $L = L(\mathcal{D}, \mathfrak{m}) = \sum_{i=1}^{g} \gamma_i - \sum_{c\in\mathcal{C}} \mathcal{J}_{j(c)i(c)}\sigma(c)\gamma(c)$ and $L' = L(\mathcal{D}', \mathfrak{m})$. Then

$$L' - L = \sum_{c\in\mathcal{C}_2} \mathcal{J}_{1i(c)}\sigma(c)(\gamma(c) - \gamma(c^+))$$

is a cycle and

$$lk(L', L'_{||}) - lk(L, L_{||}) = \ell((L' - L) \times (L' - L)) + 2\ell((L' - L) \times L)$$

thanks to the symmetry of the linking number in Proposition 2 on p. 29.

The last assertion of Lemma 24 on the previous page guarantees that

$$\ell((L' - L) \times (L' - L)) = 0.$$

Now, $\ell((L' - L) \times L) = \ell_1 + \ell_2$ with

$$\ell_1 = \sum_{c\in\mathcal{C}_2, i\in\underline{g}} \mathcal{J}_{1i(c)}\sigma(c)(\ell(c, m_i) - \ell(c^+, m_i))$$

where $\mathfrak{m} = \{m_i\}_{i\in\underline{g}}$ and $m_i \in \alpha_i \cap \beta_i$ and

$$\ell_2 = \sum_{c\in\beta_2, d\in\mathcal{C}} \mathcal{J}_{1i(c)}\sigma(c)\mathcal{J}_{j(d)i(d)}\sigma(d)(\ell(c^+, d) - \ell(c,d)).$$

Since the part $\left(\frac{1}{2}\sum_{i=1}^{g}\mathcal{J}_{2i}\langle\alpha_i,[p(\beta(d)),d|_\beta\rangle\right)$ that occurs in the expressions of $(\ell(c^+,d)-\ell(c,d))$ in Lemma 24 on p. 61 is independent of c, the factor $\sum_{c\in\beta_2}\mathcal{J}_{1i(c)}\sigma(c)$, which vanishes, makes it disappear so that

$$\ell((L'-L)\times L)=\tilde{\ell}_1+\tilde{\ell}_2$$

where

$$\tilde{\ell}_1=-\frac{1}{2}\left(\sum_{c\in[e,m_2[_\beta}\sigma(c)\mathcal{J}_{1i(c)}+\frac{1}{2}\sigma(m_2)\mathcal{J}_{12}\right)=-\frac{1}{2}\sum_{c\in[e,m_2|_\beta}\sigma(c)\mathcal{J}_{1i(c)}$$

and

$$\tilde{\ell}_2=\frac{1}{2}\sum_{d\in\beta_2,c\in[e,d|_\beta}\sigma(c)\sigma(d)\mathcal{J}_{1i(c)}\mathcal{J}_{2i(d)}.$$

□

Proof (of Lemma 21 on p. 57). Recall

$$\ell_2(\mathcal{D})=\sum_{(c,d)\in\mathcal{C}^2}\mathcal{J}_{j(c)i(d)}\mathcal{J}_{j(d)i(c)}\sigma(c)\sigma(d)\ell(c,d)-\sum_{c\in\mathcal{C}}\mathcal{J}_{j(c)i(c)}\sigma(c)\ell(c,c).$$

Define the projection $q\colon\mathcal{C}'\to\mathcal{C}$ such that $q(c)=c$ if $c\in\mathcal{C}$ and $q(c^+)=c$ if $c\in\beta_2$. Since a crossing c of β_2 gives rise to two crossings c and c^+ of $\mathcal{C}'$ whose coefficients $\mathcal{J}'_{2r}$ and $\mathcal{J}'_{1r}$ add up to $\mathcal{J}_{2r}$,

$$\ell_2(\mathcal{D})=\sum_{(c,d)\in(\mathcal{C}')^2}\mathcal{J}'_{j(c)i(d)}\mathcal{J}'_{j(d)i(c)}\sigma(c)\sigma(d)\ell(q(c),q(d))-\sum_{c\in\mathcal{C}'}\mathcal{J}'_{j(c)i(c)}\sigma(c)\ell(q(c),q(c))$$

so that

$$\begin{aligned}\ell_2(\mathcal{D}')-\ell_2(\mathcal{D})=&\sum_{(c,d)\in(\mathcal{C}')^2}\mathcal{J}'_{j(c)i(d)}\mathcal{J}'_{j(d)i(c)}\sigma(c)\sigma(d)(\ell(c,d)-\ell(q(c),q(d)))\\&-\sum_{c\in\mathcal{C}_2}\mathcal{J}'_{1i(c)}\sigma(c)(\ell(c^+,c^+)-\ell(c,c)).\end{aligned}$$

Write $\ell(c,d)-\ell(q(c),q(d))=\ell(c,d)-\ell(c,q(d))+\ell(c,q(d))-\ell(q(c),q(d))$.

$$\ell(c,d^+)-\ell(c,d)=\ell(q(c),d^+)-\ell(q(c),d)=\begin{cases}\frac{1}{2}\delta_{j(c)2}-\frac{1}{2} & \text{if } q(c)\in[e,d[_\beta\\ \frac{1}{2}\delta_{j(c)2}-\frac{1}{4} & \text{if } q(c)=d\\ \frac{1}{2}\delta_{j(c)2} & \text{if } q(c)\notin[e,d]_\beta\end{cases}$$

for $d\in\mathcal{C}_2$ so that

(Cont. next page)

$$\ell_2(\mathcal{D}')-\ell_2(\mathcal{D})=\frac{1}{2}\sum_{d\in\mathcal{C}_2}\sum_{c\in\mathcal{C}_2}(\mathcal{J}'_{2i(d)}+\mathcal{J}'_{1i(d)})\mathcal{J}'_{1i(c)}\sigma(c)\sigma(d)$$

$$-\frac{1}{2}\sum_{d\in\mathcal{C}_2}\sum_{c\in[e,d|_\beta}(\mathcal{J}'_{2i(d)}+\mathcal{J}'_{1i(d)})\mathcal{J}'_{1i(c)}\sigma(c)\sigma(d)+A$$
$$-\frac{1}{4}\sum_{c\in\mathcal{C}_2}\mathcal{J}'_{1i(c)}\sigma(c)-\sum_{c\in\mathcal{C}_2}\mathcal{J}'_{1i(c)}\sigma(c)(\ell(c^+,c)-\ell(c,c))$$

where

$$\begin{aligned}
A &= \sum_{(c,d)\in(\mathcal{C}')^2}\mathcal{J}'_{j(c)i(d)}\mathcal{J}'_{j(d)i(c)}\sigma(c)\sigma(d)(\ell(c,q(d))-\ell(q(c),q(d)))\\
&= \sum_{(c,d)\in\mathcal{C}'\times\mathcal{C}}\mathcal{J}'_{j(c)i(d)}\mathcal{J}_{j(d)i(c)}\sigma(c)\sigma(d)(\ell(c,d)-\ell(q(c),d))\\
&= \sum_{(c,d)\in\mathcal{C}_2\times\mathcal{C}}\mathcal{J}'_{1i(d)}\mathcal{J}_{j(d)i(c)}\sigma(c)\sigma(d)(\ell(c^+,d)-\ell(c,d))\\
&= -\frac{1}{2}\sum_{(c,d)\in\mathcal{C}_2\times\mathcal{C}}\mathcal{J}'_{1i(d)}\mathcal{J}_{j(d)i(c)}\sigma(c)\sigma(d)\left(\sum_{i=1}^{g}\mathcal{J}_{2i}\langle\alpha_i,[p(\beta(d)),d|_\beta\rangle\right)\\
&\quad+\frac{1}{2}\sum_{d\in\mathcal{C}_2,c\in[e,d|_\beta}\mathcal{J}'_{1i(d)}\mathcal{J}_{2i(c)}\sigma(c)\sigma(d)\\
&= -\frac{1}{2}\sum_{i=1}^{g}\sum_{d\in\mathcal{C}_2}\mathcal{J}'_{1i(d)}\sigma(d)\mathcal{J}_{2i}\langle\alpha_i,[p(\beta(d)),d|_\beta\rangle\\
&\quad+\frac{1}{2}\sum_{d\in\mathcal{C}_2,c\in[e,d|_\beta}\mathcal{J}'_{1i(d)}\mathcal{J}_{2i(c)}\sigma(c)\sigma(d)\\
&= 0,
\end{aligned}$$

$$\sum_{d\in\mathcal{C}_2}\sum_{c\in\mathcal{C}_2}(\mathcal{J}'_{2i(d)}+\mathcal{J}'_{1i(d)})\mathcal{J}'_{1i(c)}\sigma(c)\sigma(d)=\sum_{d\in\mathcal{C}_2}(\mathcal{J}'_{2i(d)}+\mathcal{J}'_{1i(d)})\sigma(d)\left(\sum_{i=1}^{g}\mathcal{J}'_{1i}\langle\alpha_i,\beta_2\rangle\right)=0,$$

and

$$\begin{aligned}
\sum_{c\in\mathcal{C}_2}\mathcal{J}_{1i(c)}\sigma(c)(\ell(c^+,c)-\ell(c,c)) &= -\frac{1}{2}\sum_{c\in\mathcal{C}_2}\mathcal{J}_{1i(c)}\sigma(c)\left(\sum_{i=1}^{g}\mathcal{J}_{2i}\langle\alpha_i,[p(\beta(c)),c|_\beta\rangle\right)\\
&\quad+\frac{1}{4}\sum_{c\in\mathcal{C}_2}\mathcal{J}_{1i(c)}\sigma(c)\\
&= -\frac{1}{2}\sum_{(c,d)\in\mathcal{C}_2^2;d\in[e,c|_\beta}\mathcal{J}_{2i(d)}\mathcal{J}_{1i(c)}\sigma(c)\sigma(d).
\end{aligned}$$

We thus get

$$\ell_2(\mathcal{D}') - \ell_2(\mathcal{D}) = -\frac{1}{2}\sum_{d\in\mathcal{C}_2}\sum_{c\in[e,d|_\beta} \mathcal{J}_{2i(d)}\mathcal{J}_{1i(c)}\sigma(c)\sigma(d) + \frac{1}{2}\sum_{(c,d)\in\mathcal{C}_2^2; d\in[e,c|_\beta} \mathcal{J}_{2i(d)}\mathcal{J}_{1i(c)}\sigma(c)\sigma(d).$$

For $r, s \in \underline{g}$, set

$$V_{r,s} = \sum_{c\in\mathcal{C}_2}\sum_{d\in[e,c|_\beta} \mathcal{J}_{ri(d)}\mathcal{J}_{si(c)}\sigma(c)\sigma(d).$$

Note that $V_{r,s} + V_{s,r} = \delta_{r2}\delta_{s2}$ (recall the argument after the statement of Lemma 23 on p. 57). Thus $\ell_2(\mathcal{D}') - \ell_2(\mathcal{D}) = \frac{1}{2}(V_{2,1} - V_{1,2}) = V_{2,1}$. □

6.4 Connected sums and stabilizations

The previous subsections guarantee that $\tilde{\lambda}$ is an invariant of Heegaard decompositions.

Lemma 25 – *Let*

$$S^3 = T_A \cup_{\partial T_A \sim -\partial T_B} T_B$$

be the genus one decomposition of S^3 as a union of two solid tori T_A and T_B glued along their boundaries so that the meridian α_1 of T_A meets the meridian β_1 of T_B once.

$$\tilde{\lambda}\left(T_A \bigcup_{\partial T_A \sim -\partial T_B} T_B\right) = 0.$$

Proof. Orient α_1 and β_1 so that $\langle \alpha_1, \beta_1 \rangle_{\partial T_A} = 1$. Then $\mathcal{J}_{11} = 1$. Let $\mathfrak{m} = \{\alpha_1 \cap \beta_1\}$ be the unique matching. Let w be a point of the connected $\partial T_A \setminus (\alpha_1 \cup \beta_1)$. Then T_A can be assumed to intersect a cube $[-1,1]^3$ that contains the ball B_{S^3} as in Figure 7 on p. 33, so that T_B intersects this cube as the closure of the complement of Figure 7 on p. 33. In particular, $X(w, \mathfrak{m})$ is the vertical field of $\mathbb{R}^3$ and $p_1(X(w, \mathfrak{m})) = 0$. Figure 25 on the next page is a rectangular picture of the Heegaard diagram so that $e(\mathcal{D}, w, \mathfrak{m}) = 0$. Since $G(\mathcal{D}) = \emptyset$ and $L(\mathcal{D}, \mathfrak{m}) = \emptyset$, $\ell_2(\mathcal{D}) = 0$ and $s_\ell(\mathcal{D}, \mathfrak{m}) = 0$. □

The *connected sum* $M \sharp M'$ of two connected closed manifolds M and M' of dimension d is obtained by removing the interior of an open ball from M and from M' and by gluing the obtained manifolds along their spherical boundaries

$$M \sharp M' = \left(M \setminus \mathring{B}^d\right) \bigcup_{S^{d-1}} \left(M' \setminus \mathring{B}'^d\right).$$

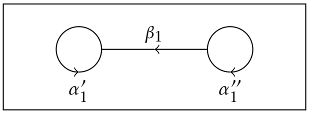

Figure 25 – Genus one Heegaard diagram of S^3

When the manifolds are 3-manifolds equipped with Heegaard decompositions $M = H_{\mathcal{A}} \cup_{\partial H_{\mathcal{A}}} H_{\mathcal{B}}$ and $M' = H'_{\mathcal{A}} \cup_{\partial H'_{\mathcal{A}}} H'_{\mathcal{B}}$, the *connected sum* of the Heegaard decompositions is the Heegaard decomposition

$$M\sharp M' = H_{\mathcal{A}}\sharp_{\partial} H'_{\mathcal{A}} \bigcup_{\partial H_{\mathcal{A}}\sharp \partial H'_{\mathcal{A}}} H_{\mathcal{B}}\sharp_{\partial} H'_{\mathcal{B}}$$

where the open ball B (resp. B') removed from M (resp. from M') intersects the Heegaard surface $\partial H_{\mathcal{A}}$ (resp. $\partial H'_{\mathcal{A}}$) as a properly embedded two dimensional disk that separates B into two half-balls $\mathring{H}_{\mathcal{A}} \cap B$ and $\mathring{H}_{\mathcal{B}} \cap B$ (resp. $\mathring{H}_{\mathcal{A}}{}' \cap B'$ and $\mathring{H}_{\mathcal{B}}{}' \cap B'$), the connected sum along the boundaries

$$H_{\mathcal{A}}\sharp_{\partial} H'_{\mathcal{A}} = \left(H_{\mathcal{A}} \setminus (H_{\mathcal{A}} \cap \mathring{B})\right) \bigcup_{H_{\mathcal{A}}\cap \partial B \sim (-H'_{\mathcal{A}} \cap \partial B')} \left(H'_{\mathcal{A}} \setminus (H'_{\mathcal{A}} \cap \mathring{B}')\right)$$

is homeomorphic to the manifold obtained by identifying $H_{\mathcal{A}}$ and $H'_{\mathcal{A}}$ along a two-dimensional disk of the boundary, and $H_{\mathcal{B}}\sharp_{\partial} H'_{\mathcal{B}}$ is defined similarly.

Proposition 14 – *Under the hypotheses above, if M and M' are rational homology 3-spheres, then*

$$\tilde{\lambda}\left(H_{\mathcal{A}}\sharp_{\partial} H'_{\mathcal{A}} \bigcup_{\partial H_{\mathcal{A}}\sharp \partial H'_{\mathcal{A}}} H_{\mathcal{B}}\sharp_{\partial} H'_{\mathcal{B}}\right) = \tilde{\lambda}\left(H_{\mathcal{A}} \bigcup_{\partial H_{\mathcal{A}}} H_{\mathcal{B}}\right) + \tilde{\lambda}\left(H'_{\mathcal{A}} \bigcup_{\partial H'_{\mathcal{A}}} H'_{\mathcal{B}}\right)$$

Proof. When performing such a connected sum on manifolds equipped with Heegaard diagrams $\mathcal{D} = (\partial H_{\mathcal{A}}, (\alpha_i)_{i\in \underline{g}}, (\beta_j)_{j\in\underline{g}})$ and $\mathcal{D}' = (\partial H'_{\mathcal{A}}, (\alpha'_i)_{i\in\underline{g'}}, (\beta'_j)_{j\in\underline{g'}})$ and with exterior points w and w' of $\mathcal{D}$ and $\mathcal{D}'$, we assume that the balls $\overline{D}$ and $\overline{D}'$ meet the Heegaard surfaces inside the connected component of w or w' outside the diagram curves, without loss, and we choose a basepoint w'' in the corresponding region of $\partial H_{\mathcal{A}}\sharp \partial H'_{\mathcal{A}}$. Then we obtain the obvious Heegaard diagram

$$\mathcal{D}'' = (\partial H_{\mathcal{A}}\sharp \partial H'_{\mathcal{A}}, (\alpha''_i)_{i\in\underline{g''}}, (\beta''_j)_{j\in\underline{g''}})$$

where $g'' = g+g'$, $\alpha''_i = \alpha_i$ and $\beta''_i = \beta_i$ when $i \leq g$, and, $\alpha''_i = \alpha'_{i-g}$ and $\beta''_i = \beta'_{i-g}$ when $i > g$, with the associated intersection matrix and its inverse, which are diagonal with respect to the two blocks corresponding to the former matrices associated with $\mathcal{D}$ and $\mathcal{D}'$.

When $\mathcal{D}$ and $\mathcal{D}'$ are furthermore equipped with matchings $\mathfrak{m}$ and $\mathfrak{m}'$, $\mathfrak{m}'' = \mathfrak{m} \cup \mathfrak{m}'$ is a matching for $\mathcal{D}''$ and a rectangular figure for $(\mathcal{D}'', w'', \mathfrak{m}'')$ similar to Figure 3 on p. 23 is obtained from the corresponding figures for $\mathcal{D}$ and $\mathcal{D}'$ by juxtapositions of the two rectangles of $\mathcal{D}$ and $\mathcal{D}'$. In particular,

$$e(\mathcal{D}'', w'', \mathfrak{m}'') = e(\mathcal{D}', w', \mathfrak{m}') + e(\mathcal{D}, w, \mathfrak{m}).$$

Furthermore, we can see $B_{M''}$ as the juxtaposition of two half-balls glued along a vertical disk equipped with the vertical field (over the intersection of the two rectangles above) such that the two half-balls are obtained from B_M and $B_{M'}$ by removing standard vertical half-balls equipped with the vertical field, so that the vector field $X(w'', \mathfrak{m}'')$ coincides with $X(w, \mathfrak{m})$ on the remaining part of B_M and with $X(w', \mathfrak{m}')$ on the remaining part of $B_{M'}$. This makes clear that

$$p_1(X(w'', \mathfrak{m}'')) = p_1(X(w, \mathfrak{m})) + p_1(X(w', \mathfrak{m}')).$$

Now it is easy to observe that $G(\mathcal{D}'') = G(\mathcal{D}) + G(\mathcal{D}')$, that

$$\ell_2(\mathcal{D}'') = \ell_2(\mathcal{D}) + \ell_2(\mathcal{D}'),$$

that $L(\mathcal{D}'', \mathfrak{m}'') = L(\mathcal{D}, \mathfrak{m}) + L(\mathcal{D}', \mathfrak{m}')$ and that

$$s_\ell(\mathcal{D}'', \mathfrak{m}'') = s_\ell(\mathcal{D}, \mathfrak{m}) + s_\ell(\mathcal{D}', \mathfrak{m}').$$ □

A connected sum of a Heegaard decomposition with the genus one decomposition of S^3 is called a *stabilization*. A well-known Reidemeister-Singer theorem[8], asserts that any two Heegaard decompositions of the same 3-manifold become isomorphic after some stabilizations. This Reidemeister-Singer theorem can also be proved using Cerf theory[9] as in Ozsváth and Szabó (2004, Proposition 2.2).

Together with Proposition 14 on the preceding page and Lemma 25 on p. 65, it implies that $\tilde{\lambda}$ does not depend on the Heegaard decomposition and allows us to prove the following theorem.

Theorem 4 – *There exists a unique invariant $\tilde{\lambda}$ of $\mathbb{Q}$-spheres such that for any Heegaard diagram $\mathcal{D}$ of a $\mathbb{Q}$-sphere M, equipped with a matching $\mathfrak{m}$ and with an exterior point w,*

$$24\tilde{\lambda}(M) = 4\ell_2(\mathcal{D}) + 4s_\ell(\mathcal{D}, \mathfrak{m}) - 4e(\mathcal{D}, w, \mathfrak{m}) - p_1(X(w, \mathfrak{m})).$$

Furthermore, $\tilde{\lambda}$ satisfies the following properties.

- *For any two rational homology 3-spheres M_1 and M_2,*

$$\tilde{\lambda}(M_1 \sharp M_2) = \tilde{\lambda}(M_1) + \tilde{\lambda}(M_2).$$

[8]Proved in Siebenmann, 1980, *Les bisections expliquent le théorème de Reidemeister-Singer*.

[9]Cerf, 1970, "La stratification naturelle des espaces de fonctions différentiables réelles et le théorème de la pseudo-isotopie".

- *For any rational homology* 3*-sphere* M*, if* $(-M)$ *denotes the manifold* M *equipped with the opposite orientation, then*

$$\tilde{\lambda}(-M) = -\tilde{\lambda}(M).$$

Proof. The invariance of $\tilde{\lambda}$ is already proved. Proposition 14 on p. 66 now implies that $\tilde{\lambda}$ is additive under connected sum. Reversing the orientation of M reverses the orientation of the surface that contains a diagram $\mathcal{D}$ of M. This changes the signs of the intersection points and reverses the sign of $\mathcal{J}$. Thus $L(\mathcal{D},\mathfrak{m})$, $G(\mathcal{D})$ and $X(w,\mathfrak{m})$ are unchanged, while the map ℓ of Proposition 3 on p. 30 is changed to its opposite. Changing the orientation of the ambient manifold reverses the sign of p_1. A rectangular diagram of $(-M)$ as in Figure 3 on p. 23 is obtained from the diagram of M by a orthogonal symmetry that fixes a vertical line so that the d_e are changed to their opposites. Thus all the terms of the formula are multiplied by (-1) when the orientation of M is reversed. □

7 The Casson surgery formula for $\tilde{\lambda}$

7.1 The statement and its consequences

In this section, we prove that $\tilde{\lambda}$ coincides with the Casson invariant for integer homology 3-spheres by proving that it satisfies the same surgery formula. More precisely, we prove the following theorem.

Theorem 5 – *Let* K *be a null-homologous knot in a* $\mathbb{Q}$*-sphere* M*.*

Let Σ *be an oriented connected surface of genus* $g(\Sigma)$ *in* M *bounded by* K *such that the closure of the complement of a collar*

$$H_{\mathcal{A}} = \Sigma \times [-1,1]$$

of $\Sigma = \Sigma \times \{0\}$ *in* M *is homeomorphic to a handlebody* $H_{\mathcal{B}}$*. This gives rise to the Heegaard decomposition*

$$M = H_{\mathcal{A}} \cup_{\Psi_M} H_{\mathcal{B}}$$

where Ψ_M *is an orientation reversing diffeomorphism from* $\partial H_{\mathcal{B}}$ *to* $\partial H_{\mathcal{A}}$*. Let* $M(K)$ *be the manifold obtained from* M *by surgery of coefficient* 1 *along* K*, which can be defined by its Heegaard decomposition*

$$M(K) = H_{\mathcal{A}} \cup_{\Psi_M \circ \mathfrak{t}_K} H_{\mathcal{B}}$$

where $\mathfrak{t}_K$ *is the right-handed Dehn twist of* $(-\partial H_{\mathcal{B}})$ *about* K*. Let* $g = 2g(\Sigma)$*. Let* $(z_i)_{i \in \underline{g}}$ *be closed curves of* $\Sigma = \Sigma \times \{0\}$ *that form a geometric symplectic basis of* $H_1(\Sigma)$ *as in Figure 26 on the next page, and let* $z_i^+ = z_i \times \{1\}$*. For any* $i \in \underline{g(\Sigma)}$*,* $\langle z_{2i-1}, z_{2i} \rangle = 1$*. Then*

$$\tilde{\lambda}(M(K)) - \tilde{\lambda}(M) = \sum_{(i,r) \in \underline{g(\Sigma)}^2} \Big(lk(z_{2i}^+, z_{2r}) lk(z_{2i-1}^+, z_{2r-1}) - lk(z_{2i}^+, z_{2r-1}) lk(z_{2i-1}^+, z_{2r}) \Big).$$

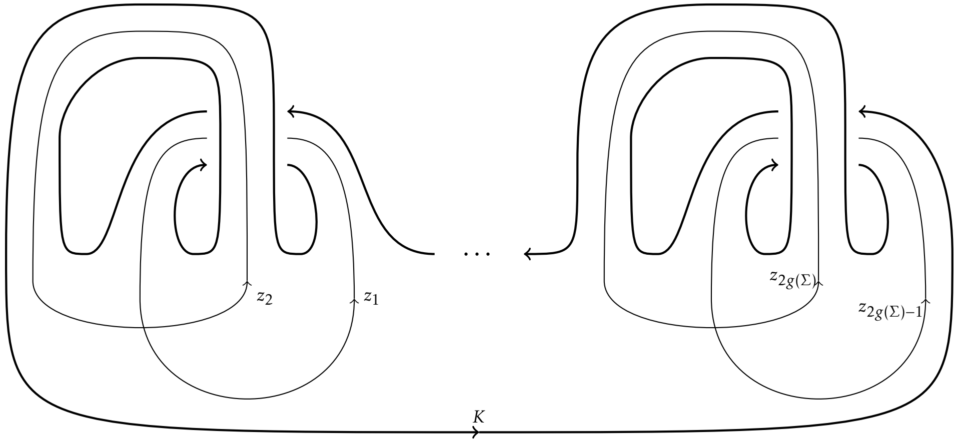

Figure 26 – Curves on the surface Σ

We will prove the theorem exactly as it is stated. A *Seifert surface* Σ of K as in the statement is said to be *unknotted*. It is well-known that any Seifert surface can be transformed to an unknotted one by adding some tubes (to remove unwanted 2-handles from its exterior). (See Marin (1988, Lemme 5.1), Akbulut and McCarthy (1990, p. 84) or Guillou and Marin (1992, Lemme 4.1) in the original surveys[10] of the Casson invariant, for example.) Thus any null-homologous knot bounds an unknotted surface as in the statement. The manifold $M(K;\frac{p}{q})$ obtained from M by *Dehn surgery* with coefficient p/q along K, for two coprime integers p and q, is usually defined as

$$M(K;\frac{p}{q}) = \left(M \setminus \mathring{N}(K)\right) \bigcup_{\partial N(K)\sim \partial D^2\times S^1} \left(D^2 \times S^1\right)$$

where $N(K)$ is a tubular neighborhood of K, and the gluing homeomorphism from $\partial D^2 \times S^1$ to $\partial N(K)$ identifies the meridian $\partial D^2 \times \{x\}$ of $D^2 \times S^1$ with a curve homologous to $pm(K)+q\ell(K)$ where $m(K)$ is the meridian of K such that $lk(m(K),K) = 1$ and $\ell(K)$ is the curve parallel to K such that $lk(\ell(K),K) = 0$. In our case, for $n \in \mathbb{Z}\setminus\{0\}$, the manifold $M(K;\frac{1}{n})$ obtained from M by surgery of coefficient $\frac{1}{n}$ along K can also be defined by its Heegaard decomposition

$$M(K;\frac{1}{n}) = H_{\mathcal{A}} \bigcup_{\Psi_M\circ t_K^n} H_{\mathcal{B}},$$

[10] Marin, 1988, "Un nouvel invariant pour les sphères d'homologie de dimension trois (d'après Casson)";
Akbulut and McCarthy, 1990, *Casson's invariant for oriented homology 3-spheres*;
Guillou and Marin, 1992, "Notes sur l'invariant de Casson des sphères d'homologie de dimension trois".

and it is easy to observe that the variation $(\tilde{\lambda}(M(K;\frac{1}{n}))-\tilde{\lambda}(M))$ can be deduced from the general knowledge of $(\tilde{\lambda}(M(K))-\tilde{\lambda}(M))$. In our case, Theorem 5 on p. 68 implies that

$$\tilde{\lambda}(M(K;\frac{1}{n}))-\tilde{\lambda}(M)=n(\tilde{\lambda}(M(K))-\tilde{\lambda}(M)).$$

In our proof, we will obtain the variation $\tilde{\lambda}(M(K))-\tilde{\lambda}(M)$ as it is stated, directly, so that our proof also directly shows that

$$\lambda'=\sum_{(i,r)\in\underline{g(\Sigma)}^2}\Big(lk(z_{2i}^+,z_{2r})lk(z_{2i-1}^+,z_{2r-1})-lk(z_{2i}^+,z_{2r-1})lk(z_{2i-1}^+,z_{2r})\Big)$$

is a knot invariant. In Lemma 41 on p. 84, we will identify λ' with $\frac{1}{2}\Delta_K''(1)$ where Δ_K denotes the Alexander polynomial of K so that the surgery formula of Theorem 5 on p. 68 coincides with the Casson surgery formula of Marin (1988, Theorem 1.1 (v)), Akbulut and McCarthy (1990, p. xii) or Guillou and Marin (1992, Theorem 1.5). Since any integer homology 3-sphere can be obtained from S^3 by a finite sequence of surgeries with coefficients ± 1[11], it follows that $\tilde{\lambda}$ coincides with the Casson invariant for integer homology 3-spheres.

Our proof will also yield the following theorem. Recall that the *Euler class* of a nowhere zero vector field of a 3-manifold M is the Euler class of its orthogonal plane bundle in M.

Theorem 6 – *Let F be a genus $g(F)$ oriented compact surface with connected boundary embedded in an oriented compact 3-manifold M whose boundary ∂M is either empty or identified with $\partial B(1)$. Let $[-2,2]\times F$ be a neighborhood of $\mathring{F}=\{0\}\times\mathring{F}$ in M, and let X be a nowhere zero vector field of M whose Euler class is a torsion element of $H^2(M;\mathbb{Z})$, which is tangent to $[-2,2]\times\{x\}$ at any point (u,x) of $[-2,2]\times F$, and which is constant on $\partial B(1)$ when $\partial M=\partial B(1)$. Let K be a parallel of ∂F inside F, and let $([-2,2]\times F)(K)$ be obtained from $[-2,2]\times F$ by $+1$-Dehn surgery along K. Let t_K denote the right-handed Dehn twist about K. Then*

$$([-2,2]\times F)(K)=[-2,0]\times F\bigcup_{\{0\}\times F\overset{\mathrm{t}_K}{\longleftarrow}\{0\}\times F^+}[0,2]\times F^+$$

where F^+ is a copy of F and $(0,x)\in\{0\}\times F^+$ is identified with $(0,\mathrm{t}_K(x))\in\{0\}\times F$. Define the diffeomorphism

(Cont. next page)

$$\psi_F\colon([-2,2]\times F)(K)\to[-2,2]\times F$$

[11] See Marin, 1988, “Un nouvel invariant pour les sphères d’homologie de dimension trois (d’après Casson)”, Section 4; or Guillou and Marin, 1992, “Notes sur l’invariant de Casson des sphères d’homologie de dimension trois”, Lemme 2.1, for example.

$$(t,x) \mapsto \begin{cases} (t,x) & \text{if } (t,x) \in [-2,0] \times F \\ (t, \mathrm{t}_K(x)) & \text{if } (t,x) \in [0,2] \times F^+ \end{cases}$$

and let Y be a nowhere zero vector field of $M(K)$ that coincides with X outside $]-1,1[\times \mathring{F}$ and that is normal to $\psi_F^{-1}(\{t\} \times F)$ on $\psi_F^{-1}(\{t\} \times F)$ for any $t \in [-2,2]$. Then

$$p_1(Y) - p_1(X) = (4g(F) - 1)g(F).$$

7.2 A preliminary lemma on Pontrjagin numbers

Lemma 26 – *Under the assumptions of Theorem 6 on the preceding page, the variation $(p_1(Y) - p_1(X))$ does not depend on M, K and F. This variation only depends on $g(F)$. It will be denoted by $p_1(g(F))$.*

Proof. Let $\tau_F \colon F \times \mathbb{R}^2 \to TF$ be a parallelization of F such that the parallelization $X \oplus \tau_F$ of $[-2,2] \times F$ extends to a trivialization τ of M — which is standard on ∂M if $\partial M = S^2$. (Since M is parallelizable and since $\pi_1(SO(3))$ is generated by a loop of rotations with arbitrary fixed axis, there exists a parallelization of M that has this prescribed form on $[-2,2] \times F$.) Observe that the degree of the tangent map to K is $(1-2g)$ with respect to τ_F. (This degree does not depend on τ_F and can be computed in Figure 26 on p. 69.) Let $K \times [-1,1]$ be a tubular neighborhood of K in F such that $K \times \{-1\} = \partial F$. Then $[-1,1] \times K \times [-1,1]$ is a neighborhood $N_{\square}(K)$ of K in M that has a standard parallelization $\tau_\nu = (X, TK, \nu)$ where TK stands for the unit tangent vector to K and ν is tangent to $\{(h,x)\} \times [-1,1]$. Without loss, assume that

$$\tau_\nu^{-1} \tau\left((t, k = \exp(2i\pi\theta), u), v \in \mathbb{R}^3\right) = \left((t,k,u), \rho_{(2g-1)\theta}(v)\right)$$

where $\rho_{(2g-1)\theta}$ is the rotation whose axis is directed by the first basis vector e_1 of $\mathbb{R}^3$ with angle $(2g-1)\theta$.

Let $\hat{K}$ be the image of K (which is fixed by t_K) in $M(K)$. The neighborhood $N_{\square}(\hat{K}) = \psi_F^{-1}(N_{\square}(K))$ of $\hat{K}$ in $([-2,2] \times F)(K)$ is also equipped with a standard parallelization $\hat{\tau}_\nu = (Y, T\hat{K}, \hat{\nu}) = \psi_{F*}^{-1} \circ \tau_\nu$.

Define the parallelization τ' of $M(K)$ that coincides with τ outside $]-1,1[\times \mathring{F}$ and that is the following stabilization of the positive normal Y to F on $[-1,1] \times F$. Let $\check{F} = F \setminus (K \times [-1,1[)$. On $[-1,1] \times \check{F}$,

$$\tau'\left(t, x, v \in \mathbb{R}^3\right) = \tau\left(t, x, \rho_{(1-2g)\pi(t+1)}(v)\right).$$

This parallelization extends to $N_{\square}(\hat{K})$ as a stabilization of Y because it extends to a square bounded by the following square meridian μ_K of $\hat{K}$

$$\mu_K = \{-1\} \times \{k\} \times [-1,1] + ([-1,1] \times (k,1)) - \{1\} \times \mathrm{t}_K^{-1}(\{k\} \times [-1,1]) - ([-1,1] \times (k,-1))$$

written with respect to coordinates of $\partial N_{\square}(K)$.

Write a (round) tubular neighborhood $N(K)$ in $N_{\square}(K)$ as $S^1 \times D^2 = \partial D^2 \times D^2$ so that μ_K induces the same parallelization of K as the longitude $(\{x\} \times D^2)$. Let

$$W_F = \left(\left([0,1] \times [-1,1] \times F\right) \bigcup_{\{1\}\times N(K) \sim \partial D^2 \times D^2} D^2 \times D^2 \right) \sharp(-\mathbb{C}P^2)$$

be a cobordism from $[-1,1] \times F$ to $([-1,1] \times F)(K)$ obtained from $[0,1] \times [-1,1] \times F$ by gluing a 2-handle $D^2 \times D^2$ along $N(K)$ using the identification of $N(K)$ with $\partial D^2 \times D^2$ above, by smoothing in a standard way, and by next performing a connected sum with a copy of $(-\mathbb{C}P^2)$ in the interior of the 2-handle. We compute $(p_1(\tau') - p_1(\tau))$ by using the cobordism W_F completed to a signature 0 cobordism by the product $[0,1] \times (M \setminus \mathrm{Int}([-1,1] \times F))$ where $T[0,1] \oplus \tau$ extends both τ and τ'. Since $\pi_1(SU(2))$ is trivial, the induced complex parallelization over $\partial([0,1] \times [-1,1]) \times \check{F}$ extends as a stabilization of $T[0,1] \oplus X$ whose restriction to $[0,1] \times [-1,1] \times \partial\check{F}$ only depends on the genus of F. Thus $(p_1(\tau') - p_1(\tau))$ is the obstruction to extending this extension to $\left([0,1] \times N_{\square}(K) \cup_{\{1\}\times N(K) \sim \partial D^2 \times D^2} D^2 \times D^2\right) \sharp(-\mathbb{C}P^2)$ and it only depends on $g(F)$. Call it $p_1(g(F))$.

Now compose τ and τ' by a small rotation whose axis is the second basis vector e_2 of $\mathbb{R}^3$ around $[-2,2] \times F$, so that $X \neq \pm\tau(e_1)$ on $[-1,1] \times F$, and X and $\tau(e_1)$ are transverse. Then $L_{X=\tau(e_1)} = L_{Y=\tau'(e_1)}$, $L_{X=-\tau(e_1)} = L_{Y=-\tau'(e_1)}$. Furthermore, since $L_{X=\tau(e_1)}$ does not meet $[-1,1] \times F$, and since it is rationally null-homologous (because the Euler class of X is a torsion element of $H^2(M;\mathbb{Z})$[12]) $L_{X=\tau(e_1)}$ bounds a Seifert surface disjoint from $N_{\square}(K)$ and $L_{Y=\tau'(e_1)}$ bounds the same Seifert surface in $M(K) \setminus N_{\square}(\hat{K})$ so that

$$lk(L_{X=\tau(e_1)}, L_{X=-\tau(e_1)}) = lk(L_{Y=\tau'(e_1)}, L_{Y=-\tau'(e_1)})$$

and

$$p_1(X) - p_1(\tau) = p_1(Y) - p_1(\tau')$$

according to Theorem 1 on p. 34, if $H_1(M;\mathbb{Q}) = 0$, and according to Lescop (2015b, Theorem 1.2), more generally. □

7.3 Introduction to the proof of the surgery formula

Let us now begin our proof of Theorem 5 on p. 68 by fixing the Heegaard diagrams that we are going to use.

Let u_i be non-intersecting curves of Σ as in Figure 27 on the next page with boundaries in $\partial\Sigma$ such that u_i is homologous to z_i in $H_1(\Sigma, \partial\Sigma)$. Then the $u_i \times [-1,1]$

[12]For details, see Lescop, 2015b, "On homotopy invariants of combings of three-manifolds", Theorem 1.1.

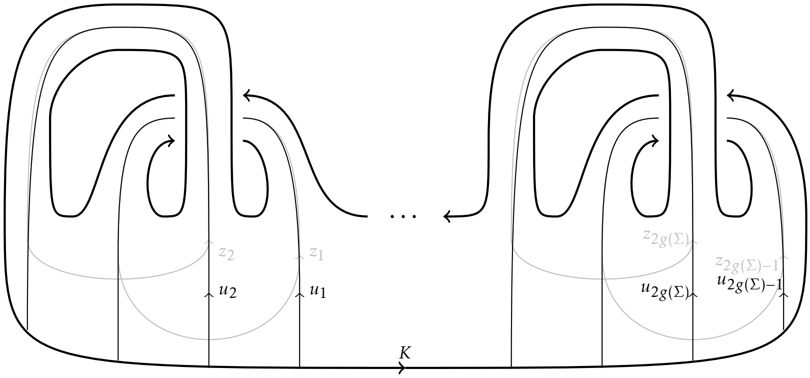

Figure 27 – The curves u_i on the surface Σ

form a system of (topological) meridian disks for the handlebody $H_{\mathcal{A}}$. Set $\alpha_i = -\partial(u_i \times [-1,1])$. Fix a system of meridians $(\beta_j)_{j\in\underline{g}}$ that meet the α curves transversally and that meet $K \times [-1,1]$ as a product by $[-1,1]$. Set $\Sigma^+ = \Sigma \times \{1\}$ and $\Sigma^- = \Sigma \times \{-1\}$. Assume that the Heegaard diagram $\mathcal{D} = ((\alpha_i)_{i\in\underline{g}}, (\beta_j)_{j\in\underline{g}})$ has a matching $\mathfrak{m} = \{m_i\}_{i\in\underline{g}}$ where $m_i \in \alpha_i \cap \beta_i$ and $m_i \in \Sigma^-$ (up to isotopies of the curves β). The invariant $\tilde{\lambda}(M)$ will be computed with the diagram $\mathcal{D}$, and the invariant $\tilde{\lambda}(M(K))$ will be computed with the diagram

$$\mathcal{D}' = \Big((\alpha_i)_{i\in\underline{g}}, (\beta'_j = \mathfrak{t}_K(\beta_j))_{j\in\underline{g}}\Big).$$

We fix a common exterior point w for $\mathcal{D}$ and $\mathcal{D}'$ in Σ^-.

Lemma 27 – *The variation $(p_1(X(\mathcal{D}', w, \mathfrak{m})) - p_1(X(\mathcal{D}, w, \mathfrak{m})))$ is equal to $p_1(g(\Sigma))$ where $p_1(g(\Sigma))$ is defined in Lemma 26 on p. 71.*

Proof. Apply Lemma 26 on p. 71 to

$$F = \Sigma^+ \cup_{K\times\{1\}} (K \times [-1,1]) \subset \partial H_{\mathcal{A}},$$

$X = X(\mathcal{D}, w, \mathfrak{m})$ and $Y = X(\mathcal{D}', w, \mathfrak{m})$. □

Let u_i also denote $u_i \times \{1\} = \alpha_i \cap \Sigma^+$.

Assume that along K, from some basepoint of K, we first meet all the intersection points of K with the β_j and next the intersection points of K with the α_i, which correspond to the endpoints of the u_i, as in Figure 28 on the next page.

Recall $\lambda' = \sum_{(i,r)\in\underline{g(\Sigma)}^2} \Big(lk(z^+_{2i}, z_{2r})lk(z^+_{2i-1}, z_{2r-1}) - lk(z^+_{2i}, z_{2r-1})lk(z^+_{2i-1}, z_{2r})\Big)$.

We are going to prove the following lemmas.

Lemma 28 – *We have*

$$\ell_2(\mathcal{D}') - \ell_2(\mathcal{D}) = 8\lambda'.$$

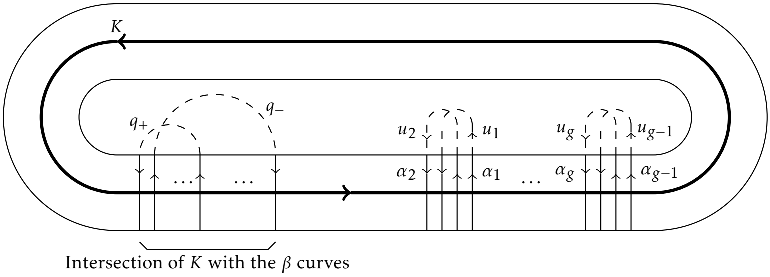

Figure 28 – The intersections of K with the curves of $\mathcal{D}$

Lemma 29 – *We have*

$$s_\ell(\mathcal{D}',\mathfrak{m}) - s_\ell(\mathcal{D},\mathfrak{m}) = -g(\Sigma)^2 - 2\lambda'.$$

Lemma 30 – *We have*

$$e(\mathcal{D}',w,\mathfrak{m}) - e(\mathcal{D},w,\mathfrak{m}) = (1-2g(\Sigma))g(\Sigma).$$

It follows from these lemmas that

$$24\tilde{\lambda}(M(K)) - 24\tilde{\lambda}(M) = 24\lambda' + 4g(\Sigma)(g(\Sigma)-1) - p_1(g(\Sigma)).$$

Applying this formula to a trivial knot U seen as the boundary of a genus $g(\Sigma)$ surface Σ_U for which $\lambda' = 0$ shows that

$$p_1(g(\Sigma)) = 4g(\Sigma)(g(\Sigma)-1)$$

since $M(U)$ is diffeomorphic to M.

Thus Lemmas 28 to 30 on pp. 73–74 imply Theorems 5 and 6 on p. 68 and on p. 70, and we are left with their proofs that occupy most of the end of this section.

7.4 Preliminaries for the proofs of the remaining three lemmas

Set $\overline{2r} = 2r-1$ and $\overline{2r-1} = 2r$.

Lemma 31 – *For any* $(i,r) \in \underline{g}^2$,

$$\sum_{j=1}^{g} \mathcal{J}_{ji}\langle u_r, \beta_j\rangle = \langle z_i, z_{\overline{i}}\rangle lk(z_r^+, z_{\overline{i}}).$$

Proof. Think of $H_{\mathcal{A}}$ as a thickening of a wedge of the z_i. Let $m(z_i)$ denote a meridian of z_i on $\partial H_{\mathcal{A}}$. Then $z_r^+ = \sum_{k=1}^{g} lk(z_r^+, z_k)m(z_k)$ in $H_1(H_{\mathcal{B}};\mathbb{Q})$. Since $m(z_k)$ is homologous to $\langle z_{\overline{k}}, z_k\rangle(z_{\overline{k}}^+ - z_{\overline{k}}^-)$ in $\partial H_{\mathcal{A}}$,

(Cont. next page)

$$\langle m(z_k), \beta_j\rangle = \langle z_{\overline{k}}, z_k\rangle\langle u_{\overline{k}} - u_{\overline{k}}^-, \beta_j\rangle$$

$$= \langle z_{\overline{k}}^-, z_k \rangle \langle \alpha_{\overline{k}}, \beta_j \rangle,$$

$$\langle u_r, \beta_j \rangle = \langle z_r^+, \beta_j \rangle = \sum_{k=1}^{g} lk(z_r^+, z_k) \langle z_{\overline{k}}^-, z_k \rangle \langle \alpha_{\overline{k}}, \beta_j \rangle,$$

and

$$\sum_{j=1}^{g} \mathcal{J}_{ji} \langle u_r, \beta_j \rangle = \langle z_i, z_{\overline{i}}^- \rangle lk(z_r^+, z_{\overline{i}}^-).$$ □

Lemma 32 – *We have*

$$\sum_{(i,j)\in \underline{g}^2} \mathcal{J}_{ji} \langle u_i, \beta_j \rangle = g(\Sigma).$$

Proof. $\sum_{i=1}^{g} \langle z_i, z_{\overline{i}}^- \rangle lk(z_i^+, z_{\overline{i}}^-) = \sum_{r=1}^{g(\Sigma)} \Big(lk(z_{2r-1}^+, z_{2r}) - lk(z_{2r}^+, z_{2r-1}) \Big).$ □

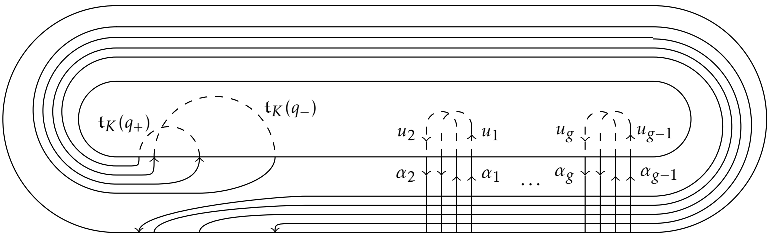

Figure 29 – The diagram $\mathcal{D}'$ in a neighborhood of K on $\partial H_{\mathcal{A}}$

For $j \in \underline{g}$, let Q_j denote the set of connected components of $\beta_j' \cap (\Sigma^+ \cup (\partial\Sigma \times [-1,1]))$. Let $Q = \cup_{j=1}^{g} Q_j$. For an arc q of Q_j, set $j(q) = j$. The intersection of an arc q of Q with $\Sigma^+ \times \{1\}$ will be denoted by q^+. Let $\mathcal{C}$ and $\mathcal{C}'$ denote the set of crossings of $\mathcal{D}$ and $\mathcal{D}'$, respectively.

For each $(q,i) \in Q \times \underline{g}$, there is a set $\mathcal{C}(q,i) = \alpha_i \cap (q \setminus q^+)$ of 4 crossings. Then

$$\mathcal{C}' = \mathcal{C} \coprod \left(\coprod_{(q,i)\in Q\times \underline{g}} \mathcal{C}(q,i) \right).$$

Denote $\mathcal{C}(q,i) = \{d_1(q,i), d_2(q,i), d_3(q,i), d_4(q,i)\}$ where following α_i from m_i, $d_1(q,i)$, $d_2(q,i)$, $d_3(q,i)$ and $d_4(q,i)$ are met in this order. Set $\sigma(q,i) = \sigma(d_2(q,i))$. Then

$$\sigma(q,i) = \sigma(d_2(q,i)) = \sigma(d_4(q,i)) = -\sigma(d_1(q,i)) = -\sigma(d_3(q,i)).$$

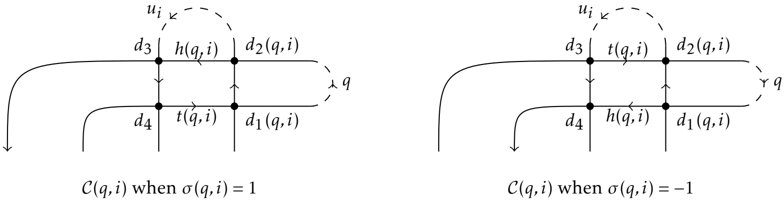

Figure 30 – The new crossings of $\mathcal{D}'$

Let $t(q,i)$ denote the (tail) arc of q before q^+ with its ends in $\mathcal{C}(q,i)$ and let $h(q,i)$ denote the (head) arc of q after q^+ with its ends in $\mathcal{C}(q,i)$.

If $\sigma(q,i)=-1$, then q goes from left to right as q_- in Figures 28 and 29 on p. 74 and on the previous page, following q we meet $\mathcal{C}(q,i)$ in the order d_3, d_2, d_1, d_4, $t(q,i)=|d_3,d_2|_\beta$ and $h(q,i)=|d_1,d_4|_\beta$.

If $\sigma(q,i)=1$, then q goes from right to left as q_+ in Figures 28 and 29 on p. 74 and on the previous page, following q we meet $\mathcal{C}(q,i)$ in the reversed order d_4, d_1, d_2, d_3, $t(q,i)=|d_4,d_1|_\beta$ and $h(q,i)=|d_2,d_3|_\beta$. Thus $t(q,i)$ begins at $d_{b(q,i)}(q,i)$ where $b(q,i)=3$ if $\sigma(q,i)=-1$ and $b(q,i)=4$ if $\sigma(q,i)=1$.

Note that for any $(i,j)\in\underline{g}^2$, $\langle\alpha_i,\beta_j\rangle=\langle\alpha_i,\beta'_j\rangle$ so that the coefficients $\mathcal{J}_{ji}$ are the same for $\mathcal{D}$ and $\mathcal{D}'$.

The set of crossings of $\mathcal{D}$ on Σ^+ (resp. on Σ^-) will be denoted by $\mathcal{C}^+$ (resp. by $\mathcal{C}^-$).

Proof (of Lemma 30 on p. 74). On the rectangle $R_\mathcal{D}$ of Figure 3 on p. 23 for $(\mathcal{D},w,\mathfrak{m})$, let p'_i (resp. p''_i) denote the other end of the diameter of α'_i (resp. α''_i) that contains the crossing m_i of $\mathfrak{m}$. Draw the knot K on a picture of the Heegaard diagram as in Figure 3 on p. 23 so that K meets the curves α' and α'' as the β_j do, away from the points of $\mathfrak{m}$, with horizontal tangent vectors near the p'_i and the p''_i. Let $N(\mathfrak{m})$ denote an open tubular neighborhood of $\mathfrak{m}$ in $\partial H_\mathcal{A}$ made of $2g(\Sigma)$ open disks. See $\partial H_\mathcal{A}\setminus\mathring{N}(\mathfrak{m})$ as obtained from the rectangle $R_\mathcal{D}$ with holes bounded by the α'_i and the α''_i, by gluing horizontal thin rectangles D_i along their two vertical small sides, which are neighborhoods of p'_i or p''_i in α'_i or α''_i. The standard parallelization of this picture equips $\partial H_\mathcal{A}\setminus\mathring{N}(\mathfrak{m})$ with a parallelization so that the degree $d_e(K)$ of the tangent to K is $1-2g(\Sigma)$ in this figure. A similar picture for $(\mathcal{D}',w,\mathfrak{m})$ is obtained by performing the Dehn twist about K on the β-curves in this figure. Since these curves do not intersect K algebraically, the $d_e(\beta_j)$ are unchanged by this operation. Similarly, for any crossing c of $\mathcal{C}^-$, $d_e(|m_{j(c)},c|_\beta)$ is unchanged and so is $d_e(c)$. For any crossing c of $\mathcal{C}^+$, we have $d'_e(c)=d_e(c)+1-2g(\Sigma)$ since $\langle K,|m_{j(c)},c|_\beta\rangle=1$. Now, let $(q,i)\in Q\times\underline{g}$. The contribution of $\mathcal{C}(q,i)$ to $(e(\mathcal{D}',w,\mathfrak{m})-e(\mathcal{D},w,\mathfrak{m}))$ is

$$\pm\mathcal{J}_{j(q)i}\left(d_e(h(q,i))+d_e(t(q,i))-\sum_{(r,s)\in\underline{g}^2}\mathcal{J}_{sr}\langle\alpha_r,h(q,i)+t(q,i)\rangle d_e(\beta_s)\right),$$

which is zero. Finally, according to Lemma 32 on p. 75,

$$\begin{aligned} e(\mathcal{D}',w,\mathfrak{m})-e(\mathcal{D},w,\mathfrak{m}) &= (1-2g(\Sigma))\sum_{c\in\mathcal{C}^+}\mathcal{J}_{j(c)i(c)}\sigma(c) \\ &= (1-2g(\Sigma))\sum_{(i,j)\in\underline{g}^2}\mathcal{J}_{ji}\langle u_i,\beta_j\rangle \\ &= (1-2g(\Sigma))g(\Sigma) \end{aligned}$$

□

7.5 Study of ℓ

Let ℓ and ℓ' be the maps of Proposition 3 on p. 30 associated with $\mathcal{D}$ and $\mathcal{D}'$, respectively, with respect to the basepoints m_i of Σ^-.

$$\ell(c,d)=\langle [m_{i(c)},c|_\alpha,[m_{j(d)},d|_\beta\rangle-\sum_{(i,j)\in\underline{g}^2}\mathcal{J}_{ji}\langle [m_{i(c)},c|_\alpha,\beta_j\rangle\langle\alpha_i,[m_{j(d)},d|_\beta\rangle$$

Lemma 33 – *Let $(c,d)\in\mathcal{C}^2$. If $(c,d)\in(\mathcal{C}^+)^2$, then*

$$\ell'(c,d)=\ell(c,d)-1.$$

Otherwise,

$$\ell'(c,d)=\ell(c,d).$$

Proof. Recall that the m_i are in Σ^-. Note that $\mathrm{t}_K([m_{j(d)},d|_\beta)$ is obtained from $[m_{j(d)},d|_\beta$ by adding some multiple of K located in $K\times[-1,1]$, algebraically, so that

$$\langle\alpha_i,\mathrm{t}_K([m_{j(d)},d|_\beta)\rangle=\langle\alpha_i,[m_{j(d)},d|_\beta\rangle$$

for any $i\in\underline{g}$. Since $\mathrm{t}_K(\beta_j)$ differs from β_j by an algebraically null sum of copies of K in $K\times[-1,1]$,

$$\langle [m_{i(c)},c|_\alpha,\beta'_j\rangle=\langle [m_{i(c)},c|_\alpha,\beta_j\rangle$$

for any $j\in\underline{g}$. Thus in any case,

$$\ell'(c,d)-\ell(c,d)=\langle [m_{i(c)},c|_\alpha,\mathrm{t}_K([m_{j(d)},d|_\beta)-[m_{j(d)},d|_\beta\rangle.$$

If $d\in\Sigma^-$, then $\mathrm{t}_K([m_{j(d)},d|_\beta)$ differs from $[m_{j(d)},d|_\beta$ by an algebraically null sum of copies of K in $K\times[-1,1]$ so that $\ell'(c,d)=\ell(c,d)$. If $d\in\Sigma^+$, then $\ell'(c,d)-\ell(c,d)=\langle [m_{i(c)},c|_\alpha,K\rangle$. If $c\in\Sigma^-$, then the arc $[m_{i(c)},c|_\alpha$ meets $K\times[-1,1]$ as the empty set or as two parallel arcs with opposite direction and $\ell'(c,d)=\ell(c,d)$. If $c\in\Sigma^+$, then the arc $[m_{i(c)},c|_\alpha$ meets $K\times[-1,1]$ as an arc that crosses K once with a negative sign. □

Lemma 34 – *Let $c \in \mathcal{C}$ and let $(q,i) \in Q \times \underline{g}$.*

$$\sum_{d \in \mathcal{C}(q,i)} \sigma(d)\ell'(c,d) = \sum_{d \in \mathcal{C}(q,i)} \sigma(d)\ell'(d,c) = 0.$$

Proof. For any interval I of a β'-curve,

$$\sum_{d \in \mathcal{C}(q,i)} \sigma(d)\langle [m_i, d|_\alpha, I\rangle = \sigma(q,i)\langle |d_1(q,i), d_2(q,i)|_\alpha + |d_3(q,i), d_4(q,i)|_\alpha, I\rangle,$$

which is zero if I has no end points in $K \times [-1,1]$. This shows that $\sum_{d \in \mathcal{C}(q,i)} \sigma(d)\ell'(d,c) = 0$. For any interval I of an α-curve,

$$\sum_{d \in \mathcal{C}(q,i)} \sigma(d)\langle I, [m_{j(q)}, d|_\beta\rangle = -\langle I, t(q,i) + h(q,i)\rangle.$$

Again, this is zero if I has no end points in $K \times [-1,1]$. □

Lemma 35 – *Let (q,i) and (q',r) belong to $Q \times \underline{g}$. If $q \neq q'$ and $i \neq r$, then*

$$\sum_{(c,d) \in \mathcal{C}(q,i) \times \mathcal{C}(q',r)} \sigma(c)\sigma(d)\ell'(c,d) = -lk_{K\times\{1\}}(\partial q^+, \partial q'^+)lk_{K\times\{1\}}(\partial u_i, \partial u_r).$$

If $q = q'$ or $i = r$, then $\sum_{(c,d) \in \mathcal{C}(q,i) \times \mathcal{C}(q',r)} \sigma(c)\sigma(d)\ell'(c,d) = 0$.

Proof. Set $A = \sum_{(c,d) \in \mathcal{C}(q,i) \times \mathcal{C}(q',r)} \sigma(c)\sigma(d)\ell'(c,d)$. As in the proof of Lemma 34,

$$\begin{aligned} A &= -\sum_{c \in \mathcal{C}(q,i)} \sigma(c)\langle [m_i, c|_\alpha, t(q',r) + h(q',r)\rangle \\ &= -\sigma(q,i)\langle |d_1(q,i), d_2(q,i)|_\alpha + |d_3(q,i), d_4(q,i)|_\alpha, t(q',r) + h(q',r)\rangle \end{aligned}$$

This is zero unless $q \neq q'$, $i \neq r$ and $lk(\partial q, \partial q')lk(\partial u_i, \partial u_r) \neq 0$. When the sign of q' changes, so does the result. Furthermore, the result is symmetric when (q,i) and (q',r) are exchanged, thanks to the symmetry of the linking number (see Proposition 2 on p. 29).

Therefore, it suffices to prove the lemma when $\sigma(q,i) = \sigma(q',r) = 1$ and $(i,r) = (2k-1, 2k)$. When we have the order $h(q',r)h(q,i)t(q',r)t(q,i)$ on K, which coincides with $h(u_r)h(u_i)t(u_r)t(u_i)$, we get $A = -1$ as in Figure 31 on the next page. For the order $h(q,i)h(q',r)t(q,i)t(q',r)$, we get $A = 1$. □

Lemma 36 – *When $i \neq r$, $lk_{K\times\{1\}}(\partial u_i, \partial u_r) = -\langle z_i, z_r\rangle$.*

When $q \neq q'$,

$$lk_{K\times\{1\}}(\partial q^+, \partial q'^+) = -\sum_{k=1}^{g} \langle z_k, z_{\overline{k}}\rangle\langle u_k, q\rangle\langle u_{\overline{k}}, q'\rangle$$

and, for any $q \in Q$, $\sum_{k=1}^{g} \langle z_k, z_{\overline{k}}\rangle\langle u_k, q\rangle\langle u_{\overline{k}}, q'\rangle = 0$.

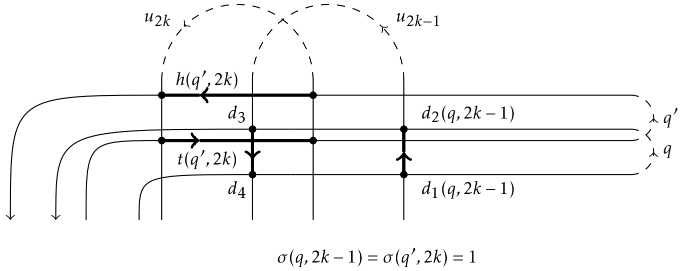

Figure 31 – Computation of $lk(\partial q, \partial q')lk(\partial u_{2k-1}, \partial u_{2k})$

Proof. Let $\gamma(q')$ be a curve on $K \times \{1\}$ that does not meet the α-curves, such that $\partial\gamma(q') = \partial q'^{+}$. Then $lk_{K\times\{1\}}(\partial q^{+}, \partial q'^{+}) = \langle \partial q^{+}, \gamma(q')\rangle_K$. Since q and q' do not intersect, this also reads $lk_{K\times\{1\}}(\partial q^{+}, \partial q'^{+}) = -\langle q, q'^{+} - \gamma(q')\rangle_{\partial H_A}$ where $(q'^{+} - \gamma(q'))$ is a closed curve of Σ^{+} whose homology class reads

$$(q'^{+} - \gamma(q')) = \sum_{k=1}^{g} \langle z_k, z_{\overline{k}}\rangle \langle q'^{+} - \gamma(q'), z_{\overline{k}}\rangle_{\Sigma^{+}} z_k = \sum_{k=1}^{g} \langle z_k, z_{\overline{k}}\rangle \langle q', u_{\overline{k}}\rangle_{\partial H_A} z_k.$$

□

Lemma 37 – *We have*

$$\sum_{(q,q')\in Q_j\times Q_s}\ \sum_{(c,d)\in\mathcal{C}(q,i)\times\mathcal{C}(q',r)} \sigma(c)\sigma(d)\ell'(c,d) = -\langle z_i, z_r\rangle \sum_{k=1}^{g} \langle z_k, z_{\overline{k}}\rangle \langle u_k, \beta_j\rangle \langle u_{\overline{k}}, \beta_s\rangle.$$

Proof. According to Lemmas 35 and 36 on the preceding page,

$$\sum_{(c,d)\in\mathcal{C}(q,i)\times\mathcal{C}(q',r)} \sigma(c)\sigma(d)\ell'(c,d) = -\langle z_i, z_r\rangle \sum_{k=1}^{g} \langle z_k, z_{\overline{k}}\rangle \langle u_k, q\rangle \langle u_{\overline{k}}, q'\rangle.$$

□

Lemma 38 – *We have*

$$\begin{aligned}\sum_{c\in\mathcal{C}'\setminus\mathcal{C}} \mathcal{J}_{j(c)i(c)}\sigma(c)\ell'(c,c) = &\sum_{(i,j,r,s)\in\underline{g}^4} \mathcal{J}_{ji}\mathcal{J}_{sr}\langle u_i, \beta_s\rangle\langle u_r, \beta_j\rangle - g(\Sigma)\\ &- \sum_{(i,j,k,s)\in\underline{g}^4} \mathcal{J}_{ji}\mathcal{J}_{s\overline{i}}\langle z_i, z_{\overline{i}}\rangle\langle z_k, z_{\overline{k}}\rangle\langle u_k, \beta_j\rangle\langle u_{\overline{k}}, \beta_s\rangle.\end{aligned}$$

Proof. Let us fix $(q,i) \in Q \times \underline{g}$ and compute $\sum_{c\in\mathcal{C}(q,i)} \sigma(c)\ell'(c,c)$.

Since the arc $[m_i, d_1(q,i)[_\alpha$ does not intersect the arcs $[d_{b(q,i)}(q,i), c]_\beta$,

(Cont. next page)

$$\sum_{c\in\mathcal{C}(q,i)} \sigma(c)\langle [m_i, c|_\alpha, [m_{j(q)}, c|_\beta\rangle = \sum_{c\in\mathcal{C}(q,i)} \sigma(c)\langle [d_1(q,i), c|_\alpha, [d_{b(q,i)}(q,i), c|_\beta\rangle$$

$$+\sum_{c\in\mathcal{C}(q,i)}\sigma(c)\langle [d_1(q,i),c|_\alpha,[m_{j(q)},d_{b(q,i)}(q,i)[_\beta\rangle$$

where $\sum_{c\in\mathcal{C}(q,i)}\sigma(c)\langle [d_1(q,i),c|_\alpha,[m_{j(q)},d_{b(q,i)}(q,i)[_\beta\rangle$ equals

$$\sigma(q,i)\langle |d_1(q,i),d_2(q,i)|_\alpha+|d_3(q,i),d_4(q,i)|_\alpha,[m_{j(q)},d_{b(q,i)}(q,i)[_\beta\rangle=0$$

since the arc $[m_{j(q)},d_{b(q,i)}(q,i)[_\beta$ intersects the arcs $[d_1(q,i),c]_\alpha$ as whole arcs q' of $Q_{j(c)}$.

Thus

$$\sum_{c\in\mathcal{C}(q,i)}\sigma(c)\langle [m_i,c|_\alpha,[m_{j(q)},c|_\beta\rangle=1+\sum_{c\in\mathcal{C}(q,i)}\sigma(c)\langle [d_1(q,i),c[_\alpha,[d_{b(q,i)}(q,i),c[_\beta\rangle.$$

Note that neither $d_1(q,i)$ nor $d_{b(q,i)}$ contributes to the new sum.

$$\langle [d_1(q,i),d_2(q,i)[_\alpha,[d_{b(q,i)}(q,i),d_2(q,i)[_\beta\rangle=\begin{cases}\sigma(d_1(q,i))=-1 & \text{if } \sigma(q,i)=1\\ 0 & \text{if } \sigma(q,i)=-1.\end{cases}$$

If $\sigma(q,i)=1$, then we are left with the computation of

$$\langle [d_1(q,i),d_3(q,i)[_\alpha,[d_{b(q,i)}(q,i),d_3(q,i)[_\beta\rangle=\langle u_i,q\rangle.$$

If $\sigma(q,i)=-1$, then we are left with the computation of

$$\langle [d_1(q,i),d_4(q,i)[_\alpha,[d_{b(q,i)}(q,i),d_4(q,i)[_\beta\rangle\langle u_i,q\rangle+\sigma(d_3(q,i)).$$

In any case,

$$\sum_{c\in\mathcal{C}(q,i)}\sigma(c)\langle [m_i,c|_\alpha,[m_{j(q)},c|_\beta\rangle=-\langle u_i,q\rangle.$$

Let us fix $(r,s)\in\underline{g}^2$ and compute $A=\sum_{c\in\mathcal{C}(q,i)}\sigma(c)\langle [m_i,c|_\alpha,\beta'_s\rangle\langle\alpha_r,[m_{j(q)},c|_\beta\rangle$. Observe

$$\langle [m_i,d_4(q,i)|_\alpha,\beta'_s\rangle=\langle [m_i,d_1(q,i)|_\alpha,\beta'_s\rangle+\langle u_i,\beta_s\rangle$$

and

$$\langle [m_i,d_3(q,i)|_\alpha,\beta'_s\rangle=\langle [m_i,d_2(q,i)|_\alpha,\beta'_s\rangle+\langle u_i,\beta_s\rangle.$$

Let

$$B=\sigma(q,i)\langle u_i,\beta_s\rangle\Big(\langle\alpha_r,[m_{j(q)},d_4(q,i)|_\beta\rangle-\langle\alpha_r,[m_{j(q)},d_3(q,i)|_\beta\rangle\Big)=-\langle u_i,\beta_s\rangle\langle\alpha_r,q^+\rangle.$$

(Cont. next page)

$$A-B=\sigma(q,i)\langle [m_i,d_1(q,i)|_\alpha,\beta'_s\rangle\Big(\langle\alpha_r,[m_{j(q)},d_4(q,i)|_\beta\rangle-\langle\alpha_r,[m_{j(q)},d_1(q,i)|_\beta\rangle\Big)$$

$$
\begin{aligned}
&+\sigma(q,i)\langle [m_i, d_2(q,i)|_{\alpha}, \beta'_s\rangle \Big(\langle \alpha_r, [m_{j(q)}, d_2(q,i)|_{\beta}\rangle - \langle \alpha_r, [m_{j(q)}, d_3(q,i)|_{\beta}\rangle\Big) \\
&= \sigma(q,i)\left(\langle [m_i, d_1(q,i)|_{\alpha}, \beta'_s\rangle - \langle [m_i, d_2(q,i)|_{\alpha}, \beta'_s\rangle\right)\langle \alpha_r, h(q,i)\rangle \\
&= -\sigma(q,i)\langle |d_1(q,i), d_2(q,i)|_{\alpha}, \beta'_s\rangle\langle \alpha_r, h(q,i)\rangle
\end{aligned}
$$

where $\langle \alpha_r, h(q,i)\rangle = lk_{K\times\{1\}}(\partial u_r, \partial u_i)$ when $r \neq i$, so that $\langle \alpha_r, h(q,i)\rangle = \langle z_i, z_r\rangle$ in any case. Summarizing, $\sum_{c\in\mathcal{C}(q,i)} \sigma(c)\ell'(c,c)$ is equal to

$$
\sum_{(r,s)\in\underline{g}^2} \mathcal{J}_{sr}\left(\langle u_i, \beta_s\rangle\langle u_r, q\rangle + \sigma(q,i)\langle |d_1(q,i), d_2(q,i)|_{\alpha}, \beta'_s\rangle\langle z_i, z_r\rangle\right) - \langle u_i, q\rangle.
$$

where

$$
\begin{aligned}
\sigma(q,i)\langle |d_1(q,i), d_2(q,i)|_{\alpha}, \beta'_s\rangle &= -\sum_{q'\in Q_s; q'\neq q} lk_{K\times\{1\}}(\partial q', \partial q) \\
&= \sum_{q'\in Q_s; q'\neq q} lk_{K\times\{1\}}(\partial q, \partial q') \\
&= -\sum_{k=1}^{g} \langle z_k, z_{\overline{k}}\rangle\langle u_k, q\rangle\langle u_{\overline{k}}, \beta_s\rangle
\end{aligned}
$$

according to Lemma 36 on p. 78.

Now, let us fix $j \in \underline{g}$ and compute

$$
\begin{aligned}
\sum_{q\in Q_j}\sum_{c\in\mathcal{C}(q,i)} \sigma(c)\ell'(c,c) &= -\langle u_i, \beta_j\rangle + \sum_{(r,s)\in\underline{g}^2} \mathcal{J}_{sr}\langle u_i, \beta_s\rangle\langle u_r, \beta_j\rangle \\
&\quad - \sum_{(r,s)\in\underline{g}^2} \mathcal{J}_{sr}\langle z_i, z_r\rangle \sum_{k=1}^{g} \langle z_k, z_{\overline{k}}\rangle\langle u_k, \beta_j\rangle\langle u_{\overline{k}}, \beta_s\rangle \\
&= -\langle u_i, \beta_j\rangle + \sum_{(r,s)\in\underline{g}^2} \mathcal{J}_{sr}\langle u_i, \beta_s\rangle\langle u_r, \beta_j\rangle \\
&\quad - \sum_{(k,s)\in\underline{g}^2} \mathcal{J}_{s\overline{i}}\langle z_i, z_{\overline{i}}\rangle\langle z_k, z_{\overline{k}}\rangle\langle u_k, \beta_j\rangle\langle u_{\overline{k}}, \beta_s\rangle
\end{aligned}
$$

$$
\begin{aligned}
\sum_{c\in\mathcal{C}'\setminus\mathcal{C}} \mathcal{J}_{j(c)i(c)}\sigma(c)\ell'(c,c) &= \sum_{(i,j)\in\underline{g}^2} \mathcal{J}_{ji}\left(\sum_{(r,s)\in\underline{g}^2} \mathcal{J}_{sr}\langle u_i, \beta_s\rangle\langle u_r, \beta_j\rangle - \langle u_i, \beta_j\rangle\right) \\
&\quad - \sum_{(i,j,k,s)\in\underline{g}^4} \mathcal{J}_{ji}\mathcal{J}_{s\overline{i}}\langle z_i, z_{\overline{i}}\rangle\langle z_k, z_{\overline{k}}\rangle\langle u_k, \beta_j\rangle\langle u_{\overline{k}}, \beta_s\rangle.
\end{aligned}
$$

Conclude with Lemma 32 on p. 75. □

7.6 Proofs of the remaining two lemmas

Lemma 39 – *We have*

$$2\lambda' = \sum_{(i,r)\in \underline{g}^2} lk(z_r^+, z_i) lk(z_{\overline{r}}^+, z_{\overline{i}}) \langle z_i, z_{\overline{i}}\rangle \langle z_r, z_{\overline{r}}\rangle$$
$$= \sum_{(i,j,k,s)\in \underline{g}^4} \mathcal{J}_{ji}\mathcal{J}_{s\overline{i}} \langle z_i, z_{\overline{i}}\rangle \langle z_k, z_{\overline{k}}\rangle \langle u_k, \beta_j\rangle \langle u_{\overline{k}}, \beta_s\rangle$$

Proof. Let C be the expression of the second line. Computing C with Lemma 31 on p. 74 yields

$$C = \sum_{(i,k)\in \underline{g}^2} lk(z_k^+, z_{\overline{i}}) \langle z_i, z_{\overline{i}}\rangle^2 lk(z_{\overline{k}}^+, z_i) \langle z_{\overline{i}}, z_i\rangle \langle z_k, z_{\overline{k}}\rangle.$$

□

Proof (of Lemma 29 on p. 74). According to Propositions 1 and 3 on p. 27 and on p. 30, and to Remark 1 on p. 31,

$$s_\ell(\mathcal{D}', \mathfrak{m}) = \sum_{(c,d)\in(\mathcal{C}')^2} \mathcal{J}_{j(c)i(c)}\sigma(c)\mathcal{J}_{j(d)i(d)}\sigma(d)\ell'(c,d)$$
$$+ \sum_{(i,j)\in \underline{g}^2} \ell'(m_i, m_j) - \sum_{(i,c)\in \underline{g}\times\mathcal{C}'} \mathcal{J}_{j(c)i(c)}\sigma(c)\left(\ell'(m_i,c) + \ell'(c,m_i)\right),$$

so that Lemmas 33, 34 and 37 on p. 77, on p. 78 and on p. 79 imply

$$s_\ell(\mathcal{D}', \mathfrak{m}) - s_\ell(\mathcal{D}, \mathfrak{m}) = -\sum_{(c,d)\in(\mathcal{C}^+)^2} \mathcal{J}_{j(c)i(c)}\sigma(c)\mathcal{J}_{j(d)i(d)}\sigma(d)$$
$$- \sum_{(i,j,k,s)\in \underline{g}^4} \mathcal{J}_{ji}\mathcal{J}_{s\overline{i}} \langle z_i, z_{\overline{i}}\rangle \langle z_k, z_{\overline{k}}\rangle \langle u_k, \beta_j\rangle \langle u_{\overline{k}}, \beta_s\rangle.$$

Therefore, according to Lemma 39,

$$s_\ell(\mathcal{D}', \mathfrak{m}) - s_\ell(\mathcal{D}, \mathfrak{m}) = -\left(\sum_{(i,j)\in \underline{g}^2} \mathcal{J}_{ji}\langle u_i, \beta_j\rangle\right)^2 - 2\lambda',$$

which equals $\left(-g(\Sigma)^2 - 2\lambda'\right)$, according to Lemma 32 on p. 75. □

Lemma 40 – *Set*

$$\lambda'_+(\Sigma) = \sum_{(i,r)\in \underline{g}^2} lk(z_r^+, z_i) lk(z_{\overline{i}}^+, z_{\overline{r}}) \langle z_i, z_{\overline{i}}\rangle \langle z_r, z_{\overline{r}}\rangle.$$

Then

$$\lambda'_+(\Sigma) = -\sum_{(i,j,r,s)\in\underline{g}^4} \mathcal{J}_{ji}\mathcal{J}_{sr}\langle u_i,\beta_s\rangle\langle u_r,\beta_j\rangle = 2\lambda' - g(\Sigma).$$

Proof. Using Lemma 31 on p. 74, we get

$$\begin{aligned}\sum_{(i,j,r,s)\in\underline{g}^4}\left(-\mathcal{J}_{ji}\mathcal{J}_{sr}\langle u_i,\beta_s\rangle\langle u_r,\beta_j\rangle\right) &= -\sum_{(i,r)\in\underline{g}^2} lk(z_r^+,z_{\bar{i}})\langle z_i,z_{\bar{i}}\rangle lk(z_i^+,z_{\bar{r}})\langle z_r,z_{\bar{r}}\rangle\\ &= \sum_{(i,r)\in\underline{g}^2} lk(z_r^+,z_i)lk(z_{\bar{i}}^+,z_{\bar{r}})\langle z_i,z_{\bar{i}}\rangle\langle z_r,z_{\bar{r}}\rangle\\ &= \sum_{(i,r)\in\underline{g}^2} lk(z_i^+,z_r)lk(z_{\bar{i}}^+,z_{\bar{r}})\langle z_i,z_{\bar{i}}\rangle\langle z_r,z_{\bar{r}}\rangle\\ &\quad -\sum_{(i,r)\in\underline{g}^2}\langle z_i,z_r\rangle lk(z_{\bar{i}}^+,z_{\bar{r}})\langle z_i,z_{\bar{i}}\rangle\langle z_r,z_{\bar{r}}\rangle\\ &= 2\lambda' + \sum_{i\in\underline{g}} lk(z_{\bar{i}}^+,z_i)\langle z_i,z_{\bar{i}}\rangle = 2\lambda' - g(\Sigma).\end{aligned}$$

□

Proof (of Lemma 28 on p. 73). According to Propositions 3 and 4 on p. 30 and on p. 31,

$$\ell_2(\mathcal{D}') = \sum_{(c,d)\in(\mathcal{C}')^2}\mathcal{J}_{j(c)i(d)}\sigma(c)\mathcal{J}_{j(d)i(c)}\sigma(d)\ell'(c,d) - \sum_{c\in\mathcal{C}'}\mathcal{J}_{j(c)i(c)}\sigma(c)\ell'(c,c).$$

According to Lemmas 38 to 40 on p. 79 and on the preceding page,

$$\sum_{c\in\mathcal{C}'\setminus\mathcal{C}}\mathcal{J}_{j(c)i(c)}\sigma(c)\ell'(c,c) = -\lambda'_+(\Sigma) - g(\Sigma) - 2\lambda' = -4\lambda'.$$

Therefore

$$\sum_{c\in\mathcal{C}}\mathcal{J}_{j(c)i(c)}\sigma(c)\ell(c,c) - \sum_{c\in\mathcal{C}'}\mathcal{J}_{j(c)i(c)}\sigma(c)\ell'(c,c) = 4\lambda' + \sum_{(i,j)\in\underline{g}^2}\mathcal{J}_{ji}\langle u_i,\beta_j\rangle = 4\lambda' + g(\Sigma)$$

according to Lemmas 32 and 33 on p. 75 and on p. 77. Using Lemmas 33, 34 and 37 on p. 77, on p. 78 and on p. 79 again, we get

$$\begin{aligned}\ell_2(\mathcal{D}') - \ell_2(\mathcal{D}) = &-\sum_{(i,j,k,s)\in\underline{g}^4}\mathcal{J}_{j\bar{i}}\mathcal{J}_{si}\langle z_i,z_{\bar{i}}\rangle\langle z_k,z_{\bar{k}}\rangle\langle u_k,\beta_j\rangle\langle u_{\bar{k}},\beta_s\rangle\\ &-\sum_{(i,j,k,s)\in\underline{g}^4}\mathcal{J}_{jr}\mathcal{J}_{si}\langle u_i,\beta_j\rangle\langle u_r,\beta_s\rangle + 4\lambda' + g(\Sigma)\end{aligned}$$

(Cont. next page)

$$= 2\lambda' + \lambda'_+(\Sigma) + 4\lambda' + g(\Sigma) = 8\lambda'$$

thanks to Lemma 40 on p. 82. □

Finally, we identify λ' to $\frac{1}{2}\Delta''_K(1)$ where

$$\Delta_K(t) = t^{-g(\Sigma)} \det\left([tlk(z_r^+, z_s) - lk(z_s^+, z_r)]_{(r,s)\in\underline{g}^2}\right)$$

denotes the Alexander polynomial of K.

Lemma 41 – *We have*

$$\frac{1}{2}\Delta''_K(1) = \lambda'.$$

Proof. Note $tlk(z_r^+, z_s) - lk(z_s^+, z_r) = (t-1)lk(z_r^+, z_s) + \langle z_r, z_s\rangle$.

$$\Delta_K(t) = t^{-g(\Sigma)} + t^{-g(\Sigma)}(t-1)\sum_{i\in\underline{g}} lk(z_i^+, z_{\bar{i}})\langle z_i, z_{\bar{i}}\rangle + t^{-g(\Sigma)}(t-1)^2 A + B(t-1)^3,$$

for some polynomial B, where $\sum_{i\in\underline{g}} lk(z_i^+, z_{\bar{i}})\langle z_i, z_{\bar{i}}\rangle = g(\Sigma)$ (see Lemma 32 on p. 75) and, thanks to Lemma 40 on p. 82,

$$\begin{aligned}
A &= \sum_{\{i,r\}\subset\underline{g}} \langle z_i, z_{\bar{i}}\rangle\langle z_r, z_{\bar{r}}\rangle \left(lk(z_i^+, z_{\bar{i}})lk(z_r^+, z_{\bar{r}}) - lk(z_i^+, z_{\bar{r}})lk(z_r^+, z_{\bar{i}})\right)\\
&= \frac{1}{2}\sum_{(i,r)\in\underline{g}^2} \langle z_i, z_{\bar{i}}\rangle\langle z_r, z_{\bar{r}}\rangle \left(lk(z_i^+, z_{\bar{i}})lk(z_r^+, z_{\bar{r}}) - lk(z_i^+, z_{\bar{r}})lk(z_r^+, z_{\bar{i}})\right)\\
&= \frac{g(\Sigma)^2}{2} + \frac{1}{2}\sum_{(i,r)\in\underline{g}^2} lk(z_r^+, z_i)lk(z_{\bar{i}}^+, z_{\bar{r}})\langle z_i, z_{\bar{i}}\rangle\langle z_r, z_{\bar{r}}\rangle\\
&= \frac{1}{2}\left(g(\Sigma)^2 + \lambda'_+(\Sigma)\right),
\end{aligned}$$

For some polynomial C

$$\Delta'_K(t) = -g(\Sigma)t^{-g(\Sigma)-1} + g(\Sigma)(t^{-g(\Sigma)} - g(\Sigma)t^{-g(\Sigma)-1}(t-1)) + 2t^{-g(\Sigma)}(t-1)A + C(t-1)^2$$

Therefore, according to Lemma 40 on p. 82,

$$\Delta''_K(1) = g(\Sigma)(g(\Sigma) + 1 - 2g(\Sigma)) + 2A = g(\Sigma) + \lambda'_+(\Sigma) = 2\lambda'$$ □

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