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The mixing property of STIT tessellations revisited

Christian Deuß[1] Christoph Thäle[1]

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Abstract

In this paper iteration stable (STIT) tessellations of the d-dimensional Euclidean space are considered. By a careful analysis of the capacity functional an alternative proof is given for the fact that STIT tessellations are mixing.

Keywords: Capacity functional, mixing property, STIT tessellation, stochastic geometry.

MSC: 60D05, 37A25.

1 Introduction and result

Random tessellations are used in a variety of practical applications such as for cellular or polycrystalline materials, plant cells or in the modelling of telecommunication networks[2]. The model of iteration stable random tessellations (or STIT tessellations, for short) has been introduced by Nagel and Weiss[3] and has quickly attracted considerable interest in stochastic geometry because of its analytical tractability[4]. Since STIT tessellations have the feature of not being side-to-side, they have the potential

[1] Ruhr University Bochum, Germany.

[2] Cf. Beil et al., 2006, "Fitting of random tessellation models to keratin filament networks";
Lautensack, 2008, "Fitting three-dimensional Laguerre tessellations to foam structures".

[3] Nagel and Weiss, 2005, "Crack STIT tessellations: characterization of stationary random tessellations stable with respect to iteration".

[4] As demonstrated in Martínez and Nagel, 2014, "STIT tessellations have trivial tail σ-algebra";
Mecke, Nagel, and Weiss, 2011, "Some distributions for I-segments of planar random homogeneous STIT tessellations";
Schreiber and Thäle, 2010, "Second-order properties and central limit theory for the vertex process of iteration infinitely divisible and iteration stable random tessellations in the plane";
Schreiber and Thäle, 2011, "Intrinsic volumes of the maximal polytope process in higher dimensional STIT tessellations";
Schreiber and Thäle, 2012, "Second-order theory for iteration stable tessellations";
Schreiber and Thäle, 2013a, "Geometry of iteration stable tessellations: connection with Poisson hyperplanes";
Schreiber and Thäle, 2013b, "Limit theorems for iteration stable tessellations";
Thäle and Weiss, 2013, "The combinatorial structure of spatial STIT tessellations";

to serve as reference models for crack structures in geology or as an ingredient for stochastic models for multi-hierarchical networks[5].

The present paper deals with STIT tessellations in the d-dimensional Euclidean space. Informally, their continuous time construction can be described as follows. Given a translation-invariant measure Λ on the space of hyperplanes in $\mathbb{R}^d$ with the property that Λ is not concentrated on less than d families of parallel hyperplanes with linearly independent directions and a set $B \subset \mathbb{R}^d$, we write $\Lambda([B])$ for the set of all hyperplanes that have non-empty intersection with B. At time $t = 0$ the construction starts with an 'empty' polytope W, which is supplied with an exponential random life-time, whose parameter is $\Lambda([W])$. Now, the time is running and when the life-time of W runs out, a random hyperplane is picked with respect to the normalized measure $\Lambda(\cdot \cap [W])/\Lambda([W])$, which splits W into two sub-polytopes W^+ and W^-. Now, W^+ and W^- are supplied again with exponential random life-times, whose parameters are $\Lambda([W^+])$ and $\Lambda([W^-])$, respectively, and the construction continues recursively, see Figure 2 on p. 7. It is stopped at a deterministically prescribed time threshold $t > 0$. The union of all $(d-1)$-dimensional random polytopes (i.e., hyperplane pieces) constructed within W until time t is denoted by $Y_W(t)$. In other words, this means that $Y_W(t)$ is a random closed subset of W, see Section 2 on p. 4 for formal definitions and Figure 1 on the next page for a simulation.

By consistency, $Y_W(t)$ can be extended to a whole-space random tessellation $Y(t)$ with the property that for any polytope W as above, $Y(t) \cap W$ coincides in distribution with $Y_W(t)$. In other words, $Y(t)$ can be regarded as the canonical random variable on the probability space

$$(\mathcal{F}^d, \mathcal{B}(\mathcal{F}^d), \mathbf{P}_{Y(t)}),$$

where $\mathcal{F}^d$ is the family of closed subsets of $\mathbb{R}^d$, $\mathcal{B}(\mathcal{F}^d)$ is the Borel σ-field generated by the Fell topology (see Section 2 on p. 4) and $\mathbf{P}_{Y(t)}$ stands for the distribution of $Y(t)$. It is known that $Y(t)$ is stationary, meaning that its distribution is invariant under all deterministic shifts in $\mathbb{R}^d$.

We recall[6] that a general stationary random closed set Z with distribution $\mathbf{P}_Z$ on $(\mathcal{F}^d, \mathcal{B}(\mathcal{F}^d))$ is said to be mixing if

$$\lim_{\|x\|\to\infty} \mathbf{P}_Z(F_1 \cap \vartheta_x F_2) = \mathbf{P}_Z(F_1)\mathbf{P}_Z(F_2) \quad \text{for all} \quad F_1, F_2 \in \mathcal{B}(\mathcal{F}^d), \tag{1}$$

where ϑ_x stands for the shift $\vartheta_x F = F + x$, $x \in \mathbb{R}^d$. We further denote by

$$\mathcal{I}_Z := \left\{ F \in \mathcal{B}(\mathcal{F}^d) : \mathbf{P}_Z(\vartheta_x F \Delta F) = 0 \quad \text{for all} \quad x \in \mathbb{R}^d \right\}$$

Thäle, Weiss, and Nagel, 2012, "Spatial STIT tessellations: distributional results for I-segments";
Weiss, Ohser, and Nagel, 2010, "Second moment measure and K-function for planar STIT tessellations".

[5]See Neuhäuser et al., 2016, "A stochastic model for multi-hierarchical networks";
Nguyen, Weiss, and Cowan, 2015, "Column tessellations".

[6]Schneider and Weil, 2008, *Stochastic and integral geometry*, Chapter 9.3.

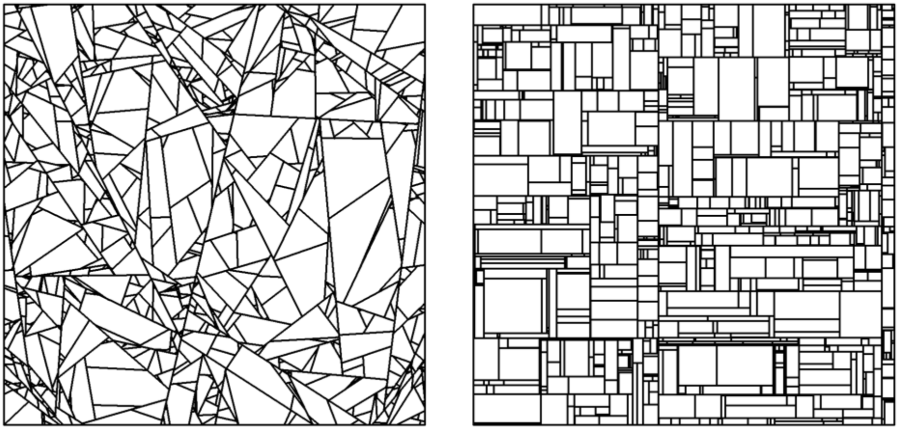

Figure 1 – Realization of two planar STIT tessellations in a square (by courtesy of Joachim Ohser).

the σ-field of all invariant events, where Δ denotes the symmetric difference operation. We recall[7] that a stationary random closed set Z is ergodic if the σ-field $\mathcal{I}_Z$ is $\mathbf{P}_Z$-trivial, i.e., if $\mathbf{P}_Z(F) \in \{0,1\}$ for all $F \in \mathcal{I}_Z$. It is not difficult to verify that ergodicity of Z is implied by the mixing property of Z.

Mixing or ergodicity properties of random sets play an important role in stochastic geometry. They are particularly useful when pathwise functional densities of random sets are defined or if one wants to apply an ergodic theorem[8], for example. Moreover, it can be useful when typical tessellation objects such as the typical cell should be defined without resorting to Palm calculus for point processes. The mixing property has been verified for various fundamental models arising in stochastic geometry including the Boolean model and the Poisson-Voronoi or Poisson hyperplane tessellation[9].

We are now prepared to present the main result of this paper.

Theorem 1 – *Let $t > 0$. The STIT tessellation $Y(t)$ is mixing and hence ergodic.*

We remark that our proof yields more than Theorem 1, it also delivers in representative situations an upper bound for the difference between the left and the right hand side in (1) of order $\mathcal{O}(\|x\|^{-1})$, see Lemma 2 on p. 11. Both, Theorem 1 as well as an upper bound for the rate of convergence in (1) are known from Lachièze-Rey

[7] Again from Schneider and Weil, 2008, *Stochastic and integral geometry*, Chapter 9.3.
[8] As in Schreiber and Thäle, 2013b, “Limit theorems for iteration stable tessellations”.
[9] See Schneider and Weil, 2008, *Stochastic and integral geometry*, Chapters 9.3 and 10.5.

(2011)[10]. However, the main goal of the paper is to present a different and elementary approach to the mixing property of STIT tessellations that is based on a careful analysis of the capacity functional of a STIT tessellation. For a general random closed set Z in $\mathbb{R}^d$ with distribution $\mathbf{P}_Z$ the capacity functional is defined as

$$T_Z(C) := \mathbf{P}_Z(\mathcal{F}_C) \quad \text{with} \quad \mathcal{F}_C = \{F \in \mathcal{F}^d : F \cap C \neq \emptyset\}, \qquad C \subset \mathbb{R}^d \text{ compact.}$$

It can be considered as a generalization to random sets of the concept of a distribution function of a real-valued random variable and is one of the most fundamental quantities associated with a random set. It is thus natural to ask whether the mixing property of a STIT tessellation can be characterized by means of its capacity functional. Our proof of Theorem 1 on the previous page confirms that this is in fact the case and relies on a characterization of the mixing property of a general random closed set in terms of its associated capacity functional taken from Schneider and Weil (2008)[11] as well as the recursive representation of the capacity functional of $Y(t)$ from Nagel and Weiss (2005)[12]. In addition, we also present a new way, relying on a simple martingale argument, to compute $T_{Y(t)}(C)$ in the case that the compact set C is connected.

The rest of this paper is structured as follows. In Section 2 we recall some necessary background material and formally introduce STIT tessellations. The capacity functional of a STIT tessellation is considered in Section 3 on p. 8, while the final Section 4 on p. 10 contains the proof of Theorem 1 on the previous page.

2 Preliminaries

2.1 Random sets

Fix a locally compact topological space E with countable base and let $\mathcal{F}(E)$ be the collection of closed subsets of E. By $\mathcal{C}(E)$ we denote the family of all compact subsets of E and by $\mathcal{G}(E)$ that of all open subsets of E. For a set $A \subset E$ define

$$\mathcal{F}^A := \{F \in \mathcal{F}(E) : F \cap A = \emptyset\} \quad \text{and} \quad \mathcal{F}_A := \{F \in \mathcal{F}(E) : F \cap A \neq \emptyset\}.$$

The Fell topology on $\mathcal{F}(E)$ is generated by the set system[13]

$$\{\mathcal{F}^C : C \in \mathcal{C}(E)\} \cup \{\mathcal{F}_G : G \in \mathcal{G}(E)\},$$

We denote by $\mathcal{B}(\mathcal{F}(E))$ the Borel σ-field on $\mathcal{F}(E)$ generated by this topology. To simplify our notation, we shall write $\mathcal{F}^d$ and $\mathcal{C}^d$ instead of $\mathcal{F}(\mathbb{R}^d)$ and $\mathcal{C}(\mathbb{R}^d)$ if

[10] Lachièze-Rey, 2011, "Mixing properties for STIT tessellations".

[11] Schneider and Weil, 2008, *Stochastic and integral geometry*.

[12] Nagel and Weiss, 2005, "Crack STIT tessellations: characterization of stationary random tessellations stable with respect to iteration".

[13] Cf. Schneider and Weil, 2008, *Stochastic and integral geometry*, Chapter 12.2.

$E = \mathbb{R}^d$ for some space dimension $d \geq 1$. Let us also introduce the symbol $\mathcal{C}_0^d$ for the subspace of $\mathcal{C}^d$ that consists of all compact subsets of $\mathbb{R}^d$ with finitely many connected components.

Definition 1 – Let $(\Omega, \mathcal{A}, \mathbf{P})$ be a probability space. By a random closed set in E we understand a $(\mathcal{A} - \mathcal{B}(\mathcal{F}(E)))$-measurable mapping $Z : \Omega \to \mathcal{F}(E)$. Its distribution is denoted by $\mathbf{P}_Z$ and the capacity functional of Z is defined as

$$T_Z(C) := \mathbf{P}_Z(\mathcal{F}_C) = \mathbf{P}(Z \cap C \neq \emptyset), \qquad C \in \mathcal{C}(E).$$

It is well known that the capacity functional of a random closed set uniquely determines its distribution[14].

As indicated in the introduction it is often convenient to identify a random closed set Z with the canonical random variable on the probability space $(\mathcal{F}(E), \mathcal{B}(\mathcal{F}(E)), \mathbf{P}_Z)$ and we also make use of this convention.

A random closed set Z in $\mathbb{R}^d$ with distribution $\mathbf{P}_Z$ is called stationary provided that $\mathbf{P}_{Z+x} = \mathbf{P}_Z$ for all $x \in \mathbb{R}^d$. Equivalently, Z is stationary if its capacity functional T_Z is translation invariant[15].

2.2 Tessellations

We work in the Euclidean space $\mathbb{R}^d$ with $d \geq 2$ and denote by $\mathcal{P}^d$ the collection of all (closed) polytopes in $\mathbb{R}^d$ that have non-empty interior. We call the elements of $\mathcal{P}^d$ cells in the sequel and write ∂c for the boundary of a cell $c \in \mathcal{P}^d$.

Definition 2 – Let $\widehat{T}$ be a countable subset of $\mathcal{P}^d$ such that

(i) $\widehat{T}$ is locally finite, i.e., any bounded subset of $\mathbb{R}^d$ has non-empty intersection with only a finite number of cells of $\widehat{T}$,

(ii) any two different cells from $\widehat{T}$ have disjoint interiors,

(iii) the cells of $\widehat{T}$ cover the space in that $\bigcup_{c \in \widehat{T}} c = \mathbb{R}^d$.

Then $T := \bigcup_{c \in \widehat{T}} \partial c \in \mathcal{F}^d$ is a tessellation of $\mathbb{R}^d$ and we call $\widehat{T}$ its associated cell set.

We denote by $\mathcal{T}^d \subset \mathcal{F}^d$ the space of tessellations of $\mathbb{R}^d$ and equip $\mathcal{T}^d$ with the trace σ-field $\mathcal{B}(\mathcal{T}^d)$ of $\mathcal{B}(\mathcal{F}^d)$. Let $(\Omega, \mathcal{A}, \mathbf{P})$ be some abstract probability space. Then, a random tessellation of $\mathbb{R}^d$ can now be defined as a $(\mathcal{A} - \mathcal{B}(\mathcal{T}^d))$-measurable mapping from Ω into $\mathcal{T}^d$.

For a tessellation $T \in \mathcal{T}^d$ and a polytope $W \in \mathcal{P}^d$ (often referred to as window) we denote by $T_W = T \cap W$ the restriction of T to W. By $\mathcal{T}_W^d$ we indicate the space

[14] See Schneider and Weil, 2008, *Stochastic and integral geometry*, Theorem 2.1.3.

[15] See ibid., Theorem 2.4.5.

of all tessellations of W. That is, $T_W = (\bigcup_{c\in\widehat{T}_W} \partial c) \setminus \partial W \in \mathcal{T}_W^d$, where $\widehat{T}_W$ is a finite subset of the space $\mathcal{P}_W^d$ of cells $c \in \mathcal{P}^d$ satisfying $c \subseteq W$, which have pairwise disjoint interiors and cover W. Similarly as outlined above, $\mathcal{T}_W^d$ is supplied with a Borel σ-field $\mathcal{B}(\mathcal{T}_W^d)$.

In the sequel we often apply translations to a tessellation. For $T \in \mathcal{T}^d$ or $T \in \mathcal{T}_W^d$ for some $W \in \mathcal{P}^d$ and $z \in \mathbb{R}^d$ we define $\vartheta_z T := \{x + z : x \in T\}$. We will also apply the shift operator ϑ_z to arbitrary sets $B \subset \mathbb{R}^d$ which are not necessarily tessellations.

2.3 STIT tessellations and STIT tessellation processes

We denote by $\mathcal{H}$ the space of hyperplanes (i.e., $(d-1)$-dimensional affine subspaces) in $\mathbb{R}^d$, $d \geq 2$. For a Borel set $B \subset \mathbb{R}^d$ we define $[B] \subseteq \mathcal{H}$ as

$$[B] := \{H \in \mathcal{H} : B \cap H \neq \emptyset\}.$$

By $\mathcal{H}_0 := [0]$ we indicate the collection of those hyperplanes in $\mathcal{H}$ which contain the origin. We let Λ_0 be a probability measure on the space $\mathcal{H}_0$. The measure Λ_0 induces a translation invariant σ-finite measure Λ on $\mathcal{H}$ by the relation

$$\Lambda(\cdot) := \int_{\mathcal{H}_0} \int_{H^\perp} \mathbf{1}(H + x \in \cdot)\ell_{H^\perp}(\mathrm{d}x)\Lambda_0(\mathrm{d}H), \tag{2}$$

where $\ell_{H^\perp}$ stands for the Lebesgue measure on the orthocomplement $H^\perp$ of H. In what follows, it is always assumed that Λ is non-degenerate in that the support of Λ_0 contains d hyperplanes with linearly independent unit normal vectors.

For $W \in \mathcal{P}^d$, $T \in \mathcal{T}_W^d$, $c \in \widehat{T}$ and $H \in [c]$ let us define the tessellation $\oslash_{c,H}(T) \in \mathcal{T}_W^d$ by

$$\oslash_{c,H}(T) := T \cup (c \cap H),$$

where H^+ and H^- are the two closed half-spaces determined by H and $\mathrm{cl}(\cdot)$ stands for the closure of a set. In other words, $\oslash_{c,H}(T)$ is the tessellation which arises from T when the cell c is split by the hyperplane H. Clearly, this definition also applies to tessellations $T \in \mathcal{T}^d$, in which case the subtraction of ∂W is superfluous and has to be omitted in the definition of $\oslash_{c,H}(T)$.

We are now prepared to introduce the main object of the present paper[16].

Definition 3 – By the STIT tessellation process in $W \in \mathcal{P}^d$ with initial tessellation $Y_W(0) := \emptyset$ we understand the continuous time Markov process $(Y_W(t))_{t\geq 0}$ on the space $\mathcal{T}_W^d$ whose infinitesimal generator $\mathbb{L}$ is given by

$$\mathbb{L}f(T) := \sum_{c\in\widehat{T}} \int_{[c]} \Big[f(\oslash_{c,H}(T)) - f(T)\Big]\Lambda(\mathrm{d}H), \qquad T \in \mathcal{T}_W^d, \tag{3}$$

where $f : \mathcal{T}_W^d \to \mathbb{R}$ is bounded and measurable.

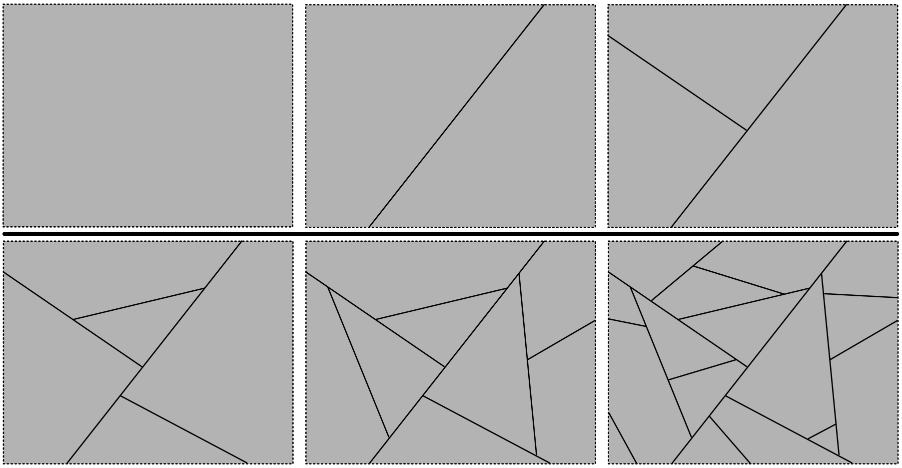

Figure 2 – Illustration of the STIT tessellation process in a rectangle.

Informally, the dynamic of the process $(Y_W(t))_{t\geq 0}$ can be described as follows[17]. The tessellation evolves according to a branching process on $\mathcal{P}_W^d$ in continuous time. The cells c behave independently of each other, live for an exponential random time with parameter $\Lambda([c])$ and are then split into two sub-cells by means of a random hyperplane, which is chosen according to the probability measure $\Lambda(\cdot\cap[c])/\Lambda([c])$. This in particular ensures that 'smaller' cells live stochastically for a longer time. Moreover, the process starts with the trivial initial tessellation $Y_W(0)=\emptyset$ that consists of the single cell W. An illustration of the STIT tessellation process is provided in Figure 2.

Definition 4 – By a STIT tessellation of $W\in\mathcal{P}^d$ with time parameter $t>0$ we understand the random tessellation $Y_W(t)\in\mathcal{T}_W^d$.

A simulation of two STIT tessellations in a square with different hyperplane (line) measures Λ is shown in Figure 1 on p. 3.

2.4 Two key properties

After having formally introduced STIT tessellations within polyhedral windows $W\in\mathcal{P}^d$, we collect here two of their key properties that are needed throughout this paper.

[16]For the necessary background material on Markov process theory we refer the reader to Kallenberg, 2002, *Foundations of Modern Probability*, Chapter 8.

[17]See Schreiber and Thäle, 2013a, "Geometry of iteration stable tessellations: connection with Poisson hyperplanes".

Global STIT tessellation. For any $t > 0$ there exists a global STIT tessellation $Y(t) \in \mathcal{T}^d$ with the property that $Y(t) \cap W$ has the same distribution as $Y_W(t)$ for all $W \in \mathcal{P}^d$. We call $Y(t)$ the global STIT tessellation with time parameter $t > 0$. It will often be convenient for us to suppress the adjective 'global' whenever it is clear from the context that we are dealing with a tessellation of the whole space.

Stationarity. The global STIT tessellations $Y(t)$ are stationary, meaning that $\vartheta_z Y(t)$ has the same distribution as $Y(t)$ for all $z \in \mathbb{R}^d$ and $t > 0$.

The existence of a global STIT tessellation can be concluded from the form of the capacity functional given in Propositions 1 and 2 on the current page and on the next page together with the consistency theorem for random closed sets[18]. A direct global construction of a STIT tessellation can be found in Mecke, Nagel, and Weiss (2011). Moreover, the stationarity of $Y(t)$ follows from the translation invariance of the capacity functional in Propositions 1 and 2 on the current page and on the next page.

3 Capacity functional for STIT tessellations

Fix $t > 0$, $W \in \mathcal{P}^d$ and let $Y_W(t)$ be a STIT tessellation in W with time parameter t. The goal of this section is to present formulas for the capacity functional $T_{Y_W(t)}$ of $Y_W(t)$. Our first result deals with $T_{Y_W(t)}(C)$ in the case that $C \in \mathcal{C}(W)$ is connected. The formula is known from Nagel and Weiss (2005), but we give an independent proof using the representation (3) for the infinitesimal generator $\mathbb{L}$ of the STIT tessellation process in W as well as a martingale argument.

Proposition 1 – *If $C \in \mathcal{C}(W)$ is connected, then*

$$T_{Y_W(t)}(C) = 1 - e^{-t\Lambda([C])}.$$

Proof. We recall from the standard theory of Markov processes that the stochastic process

$$\left(f(Y_W(t)) - f(Y_W(0)) - \int_0^t \mathbb{L} f(Y_W(t))\,\mathrm{d}s\right)_{t\geq 0} \tag{4}$$

is a martingale with respect to the canonical filtration induced by the STIT tessellation process in W for all bounded measurable functions f on $\mathcal{T}_W^d$. We use the martingale property of (4) for the special choice $f(Y_W(t)) = \mathbf{1}(Y_W(t) \cap C = \emptyset)$ for connected $C \in \mathcal{C}(W)$. Since $f(Y_W(0)) = 1$ we conclude from (4) by taking expectations that

(Cont. next page)

$$\mathbb{P}(Y_W(t) \cap C = \emptyset) = 1 + \mathbb{E}\left[\int_0^t \mathbb{L} f(Y_W(s))\,\mathrm{d}s\right]$$

[18] Schneider and Weil, 2008, *Stochastic and integral geometry*, Theorem 2.3.1.

$$= 1 + \int_0^t \mathbb{E}\Big[\sum_{c\in\hat{Y}_W(s)} \int_{[c]} f(\oslash_{c,H}(Y_W(s))) - f(Y_W(s))\Lambda(\mathrm{d}H)\Big]\mathrm{d}s$$

$$= 1 + \int_0^t \mathbb{E}\Big[-\sum_{c\in\hat{Y}_W(s)} \int_{[c]} \mathbf{1}(C\subset c)\mathbf{1}(C\cap H\neq\varnothing)\Lambda(\mathrm{d}H)\Big]\mathrm{d}s$$

$$= 1 - \int_0^t \mathbb{E}\Big[\sum_{c\in\hat{Y}_W(s)} \mathbf{1}(C\subset c)\int_{[c]} \mathbf{1}(C\cap H\neq\varnothing)\Lambda(\mathrm{d}H)\Big]\mathrm{d}s$$

$$= 1 - \Lambda([C])\int_0^t \mathbb{E}\Big[\sum_{c\in\hat{Y}_W(s)} \mathbf{1}(C\subset c)\Big]\mathrm{d}s$$

$$= 1 - \Lambda([C])\int_0^t \mathbb{P}(Y_W(s)\cap C=\varnothing)\,\mathrm{d}s.$$

This leads immediately to the integral equation

$$y(t) = 1 - \Lambda([C])\int_0^t y(s)\,\mathrm{d}s \quad \text{with} \quad y(0)=1$$

for $y(t)=\mathbb{P}(Y_W(t)\cap C=\varnothing)$. Its unique solution is $y(t)=e^{-t\Lambda([C])}$ and we have thus completed the proof. □

Unfortunately, the above method does not continue to work for $C\in\mathcal{C}(W)$ that have more than one connected component. However, the recursion technique from Nagel and Weiss (2005) can be used to compute $T_{Y_W(t)}(C)$ for all $C\in\mathcal{C}_0^d$ with $C\subset W$. To present the formula we denote by $\operatorname{conv} C$ the convex hull of C. Since $C\in\mathcal{C}_0^d$, it can be represented as disjoint union of $k\geq 2$ connected sets $C_1,\dots,C_k\in\mathcal{C}(W)$. We define

$$\{Z_1,Z_2\} = \Big\{\bigcup_{i\in I} C_i, \bigcup_{j\in\{1,\dots,k\}\setminus I} C_j\Big\}$$

for a non-empty subset $I\subset\{1,\dots,k\}$ with less than k elements (the dependence of $\{Z_1,Z_2\}$ on I is suppressed in our notation) and we indicated by $\sum_{Z_1,Z_2}$ a sum taken over all such sets. Finally, we use the symbol $[Z_1|Z_2]$ for the set of all hyperplanes separating Z_1 and Z_2. We are now in the position to recall a recursion formula for $T_{Y_W(t)}(C)$ from Nagel and Weiss (2005, Lemma 5) where we remark that the proof given makes essential use of the result of Proposition 1 on the preceding page.

Proposition 2 – *If $C\in\mathcal{C}_0^d$ with $C\subset W$ as above, then*

(Cont. next page)

$$1 - T_{Y_W(t)}(C) = e^{-t\Lambda([\operatorname{conv} C])} + t\sum_{Z_1,Z_2} \Lambda([Z_1|Z_2])$$

$$\times \int_0^1 e^{-t\Lambda([\mathrm{conv}\, C])s}(1 - T_{Y_W(t(1-s))}(Z_1))(1 - T_{Y_W(t(1-s))}(Z_2))\, \mathrm{d}s.$$

Finally, we notice that the distribution of the STIT tessellation $Y_W(t)$ in W is uniquely determined by its capacity functional on sets $C \in \mathcal{C}_0^d$ with $C \subset W$[19] and the subsequent discussion. The same is true also for a global STIT tessellation $Y(t)$.

4 Proof of Theorem 1 on p. 3

Let Z be a stationary random closed set in $\mathbb{R}^d$ and let T_Z be its capacity functional. It is known from Schneider and Weil (2008, Theorem 9.3.2) that the mixing property of Z can be characterized in terms of T_Z. Namely, one has that Z is mixing if and only if

$$\lim_{\|x\|\to\infty} (1 - T_Z(C_1 \cup \vartheta_x C_2)) = (1 - T_Z(C_1))(1 - T_Z(C_2)) \quad \text{for all} \quad C_1, C_2 \in \mathcal{C}^d. \quad (5)$$

This means that in order to prove Theorem 1 on p. 3 it remains to check condition (5) for the random closed set $Y(t)$ for all $C_1, C_2 \in \mathcal{C}_0^d$, as discussed at the end of the previous section, see again the discussion after Theorem 2.4 in Molchanov (1993). To do so, we fix $x \in \mathbb{R}^d$ and define

$$C(x) = C_1 \cup \vartheta_x C_2, \quad C_1, C_2 \in \mathcal{C}_0^d.$$

Lemma 1 – *For $i \in \{1,2\}$ let $C_i \in \mathcal{C}_0^d$ with $\Lambda(C_i) > 0$. Then there are constants $c_1, c_2 \in (0,\infty)$ such that*

$$c_1\|x\| \le \Lambda([\mathrm{conv}\, C(x)]) \le c_2\|x\|$$

for sufficiently large $\|x\|$.

Proof. For the upper bound we notice that

$$\begin{aligned}\sup_{z_1\in C_1, z_2\in\vartheta_x C_2} \|z_1 - z_2\| &= \sup_{z_1\in C_1, z_2\in C_2} \|z_1 - z_2 + x\| \le \sup_{z_1\in C_1, z_2\in C_2} \|z_1 - z_2\| + \|x\| \\ &= c_0 + \|x\|\end{aligned}$$

with a finite constant $c_0 := \sup_{z_1\in C_1, z_2\in C_2} \|z_1 - z_2\|$. Now, we choose $\|x\| \ge 1$, define $r(x) := (c_0 + 1)\|x\|$ and observe that

$$\sup_{z_1\in C_1, z_2\in\vartheta_x C_2} \|z_1 - z_2\| \le r(x).$$

[19] See Molchanov, 1993, *Limit Theorems for Unions of Random Closed Sets*, Theorem 2.4.

Let $B(x)$ be a ball with radius $r(x)$ that contains $\operatorname{conv} C(x)$. In view of (2) we conclude that

$$\Lambda([\operatorname{conv} C(x)]) \le \Lambda([B(x)]) = \int_{\mathcal{H}_0}\int_{H^\perp} \mathbf{1}((H+x)\cap B(x) \neq \emptyset)\ell_{H^\perp}(\mathrm{d}x)\Lambda_0(\mathrm{d}H)$$
$$= 2r(x)\int_{\mathcal{H}_0}\Lambda_0(\mathrm{d}H) = 2r(x) = c_2\|x\|$$

with a constant $c_2 := 2(c_0+1) \in (0,\infty)$.

To obtain the lower bound, we choose two arbitrary, but different points $z_1 \in C_1$ and $z_2 \in \vartheta_x C_2$ and denote by $\overline{z_1 z_2}$ the line segment determined by z_1 and z_2. One clearly has that $\Lambda([\operatorname{conv} C(x)]) \ge \Lambda([\overline{z_1 z_2}])$. We choose $\|x\|$ large enough to ensure that $\|z_1 - z_2\| \ge \|x\|/2$, say. To bound the quantity $\Lambda([\overline{z_1 z_2}])$ from below we use (2) and the fact that Λ_0 contains d hyperplanes with linearly independent unit normal vectors. This ensures the existence of a constant $c_1 \in (0,\infty)$ such that

$$\Lambda([\overline{z_1 z_2}]) = \int_{\mathcal{H}_0}\int_{H^\perp} \mathbf{1}((H+x)\cap \overline{z_1 z_2} \neq \emptyset)\ell_{H^\perp}(\mathrm{d}x)\Lambda_0(\mathrm{d}H)$$
$$= \int_{\mathcal{H}_0} \ell(\overline{z_1 z_2}|H^\perp)\Lambda_0(\mathrm{d}H) \ge \frac{\|x\|}{2}\int_{\mathcal{H}_0} |\cos\angle(\overline{z_1 z_2}, H^\perp)|\Lambda_0(\mathrm{d}H)$$
$$=: c_1\|x\|,$$

where $\ell(\overline{z_1 z_2}|H^\perp)$ stands for the length of the orthogonal projection of $\overline{z_1 z_2}$ to $H^\perp$ and $\angle(\overline{z_1 z_2}, H^\perp)$ denotes the angle between $H^\perp$ and the line through z_1 and z_2. Thus, $\Lambda([\operatorname{conv} C(x)]) \ge c_1\|x\|$ and this completes the proof. □

In what follows, let us write $f \in \mathcal{O}(g)$ for two functions $f, g : \mathbb{R} \to \mathbb{R}$ whenever $\limsup_{y\to\infty} \frac{f(y)}{g(y)} \in (-\infty,\infty)$. By slight abuse of notation we will also use the symbol $\mathcal{O}(g)$ to indicate a quantity f satisfying $f \in \mathcal{O}(g)$.

Lemma 2 – *As $\|x\| \to \infty$, one has that*

$$|1 - T_{Y(t)}(C_1 \cup \vartheta_x C_2) - (1 - T_{Y(t)}(C_1))(1 - T_{Y(t)}(C_2))| = \mathcal{O}\left(\|x\|^{-1}\right) \tag{6}$$

for all $C_1, C_2 \in \mathcal{C}_0^d$.

Proof. Without loss of generality we can assume that $\Lambda([C_1]), \Lambda([C_2]) > 0$, since otherwise (6) is trivially satisfied.

We write $C_1 = \bigcup_{k=1}^n C_{1,k}$ and $C_2 = \bigcup_{\ell=1}^m C_{2,\ell}$ with $C_{1,k} \in \mathcal{C}^d$ and $C_{2,\ell} \in \mathcal{C}^d$ connected. It is easily verified that

$$\Lambda([\operatorname{conv} C(x)]) = \Lambda([C_1]) + \Lambda([C_2]) + \Lambda([C_1|\vartheta_x C_2]) - \Lambda([C_1 \cap \vartheta_x C_2]) \tag{7}$$

for all $x \in \mathbb{R}^d$, since Λ is translation invariant. From Lemma 1 on p. 10 we conclude that there are constants $c_1', c_2' \in (0,\infty)$ such that

$$c_1'\|x\| \le \Lambda([C_1|\vartheta_x C_2]) \le c_2'\|x\|$$

for sufficiently large x. Furthermore, from Proposition 2 on p. 9 we have that, for all $x \in \mathbb{R}^d$,

$$\begin{aligned}1 - T_{Y(t)}(C(x)) = e^{-t\Lambda([\operatorname{conv} C(x)])} + t \sum_{Z_1,Z_2} \Lambda([Z_1|Z_2]) \\ \times \int_0^1 e^{-t\Lambda([\operatorname{conv} C(x)])s}(1 - T_{Y(t(1-s))}(Z_1))(1 - T_{Y(t(1-s))}(Z_2))\,\mathrm{d}s;\end{aligned}$$

note that for each $x \in \mathbb{R}^d$ one can find a polytopal window $W_x \in \mathcal{P}^d$ such that $C(x) \subset W_x$, which allows to replace $Y(t)$ by $Y_{W_x}(t)$ in the above argument. We also remark that Z_1 and Z_2 are not independent of $x \in \mathbb{R}^d$, although this is not visible in our notation.

We may now assume that $C_1 \cap \vartheta_x C_2 = \emptyset$. In what follows, we argue that in the above representation for $1 - T_{Y(t)}(C(x))$, all summands except of $Z_1 = C_1$ and $Z_2 = \vartheta_x C_2$ are negligible, as $\|x\| \to \infty$. For this, we consider the following three cases. In case that $Z_1 \cap C_1 \ne \emptyset$, $Z_1 \cap \vartheta_x C_2 \ne \emptyset$, $Z_2 \cap C_1 \ne \emptyset$ and $Z_2 \cap \vartheta_x C_2 \ne \emptyset$ one immediately has that $\Lambda([Z_1|Z_2]) = \mathcal{O}(1)$. If $Z_1 \cap C_1 \ne \emptyset$, $Z_1 \cap \vartheta_x C_2 \ne \emptyset$ and $Z_2 \cap C_1 = \emptyset$, we can choose $\ell_1, \ell_2 \in \{1,\dots,m\}$ with $\ell_1 \ne \ell_2$ such that

$$Z_1 \cap \vartheta_x C_{2,\ell_1} = \emptyset \quad \text{and} \quad Z_2 \cap \vartheta_x C_{2,\ell_2} = \emptyset.$$

It follows that

$$\Lambda([Z_1|Z_2]) \le \Lambda([\vartheta_x C_{2,\ell_1}|\vartheta_x C_{2,\ell_2}]) = \Lambda([C_{2,\ell_1}|C_{2,\ell_2}]) = \mathcal{O}(1).$$

In the remaining case that $Z_1 \cap C_1 \ne \emptyset$, $Z_1 \cap \vartheta_x C_2 \ne \emptyset$ and $Z_2 \cap \vartheta_x C_2 = \emptyset$ we can similarly conclude that $\Lambda([Z_1|Z_2]) = \mathcal{O}(1)$ and we have thus shown that $\Lambda([Z_1|Z_2]) = \mathcal{O}(1)$ for all Z_1, Z_2 with $Z_1 \ne C_1$. We can now apply Lemma 1 on p. 10 and conclude that

$$\begin{aligned}&t\Lambda([Z_1|Z_2]) \int_0^1 e^{-t\Lambda([\operatorname{conv} C(x)])s}(1 - T_{Y(t(1-s))}(Z_1))(1 - T_{Y(t(1-s))}(Z_2))\,\mathrm{d}s \\ &\quad \le t\Lambda([Z_1|Z_2]) \int_0^1 e^{-t\Lambda([\operatorname{conv} C(x)])s}\,\mathrm{d}s \\ &\quad = \Lambda([Z_1|Z_2]) \frac{1 - e^{-t\Lambda([\operatorname{conv} C(x)])}}{\Lambda([\operatorname{conv} C(x)])} = \mathcal{O}\Big(\|x\|^{-1}\Big), \quad \text{as } \|x\| \to \infty.\end{aligned}$$

Thus, it remains to consider the case that $Z_1 = C_1$ and $Z_2 = \vartheta_x C_2$. To handle it, we notice that $T_{Y(t)}(C)$ is continuously differentiable in t for all $C = \bigcup_{j=1}^p C_j \in \mathcal{C}_0^d$ and satisfies

$$\sup_{s\in(0,1)} \frac{\partial(1-T_{Y(t(1-s))}(C_2))}{\partial s} \leq c_p$$

for a constant $c_p \in (0,\infty)$. We can thus apply integration-by-parts to see that

$$\begin{aligned}
&t\Lambda([C_1|\vartheta_x C_2]) \int_0^1 e^{-t\Lambda([\operatorname{conv} C(x)])s}(1-T_{Y(t(1-s))}(C_1))(1-T_{Y(t(1-s))}(\vartheta_x C_2))\,\mathrm{d}s \\
&= \frac{\Lambda([C_1|\vartheta_x C_2])}{\Lambda([\operatorname{conv} C(x)])}\Big[-e^{-t\Lambda([\operatorname{conv} C(x)])s}(1-T_{Y(t(1-s))}(C_1))(1-T_{Y(t(1-s))}(C_2))\Big]_{s=0}^1 \\
&\qquad + \frac{\Lambda([C_1|\vartheta_x C_2])}{\Lambda([\operatorname{conv} C(x)])}\int_0^1 e^{-t\Lambda([\operatorname{conv} C(x)])s} \\
&\qquad\qquad \times \frac{\partial((1-T_{Y(t(1-s))}(C_1))(1-T_{Y(t(1-s))}(C_2)))}{\partial s}\,\mathrm{d}s \\
&= \frac{\Lambda([C_1|\vartheta_x C_2])}{\Lambda([\operatorname{conv} C(x)])}\Big((1-T_{Y(t)}(C_1))(1-T_{Y(t)}(C_2)) + \mathcal{O}(\|x\|^{-1})\Big), \quad \text{as } \|x\|\to\infty.
\end{aligned}$$

In combination with (7), we deduce that the last expression equals

$$(1-T_{Y(t)}(C_1))(1-T_{Y(t)}(C_2)) + \mathcal{O}\Big(\|x\|^{-1}\Big), \quad \text{as } \|x\|\to\infty.$$

Together with (3), the proof is thus complete. □

Proof (of Theorem 1 on p. 3). The result follows by combining (5) with Lemma 2 on p. 11. □

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Contents