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A note on extension type theorems in heterogeneous mediums

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Abstract

It is known for quite some time that the extension theorems play an important role in the homogenization of the periodic (heterogeneous) mediums. However, the construction of such extension operators depends on a reflection technique but for the functions in $H^{l,r}(\Omega_p^\varepsilon)$ $(l > 2)$ this reflection technique is not so straightforward, and would lead to a rather cumbersome anaylsis. In this work, we will give a short overview of some extension operators mapping from $L^r(S;H^{l,r}(\Omega_p^\varepsilon)) \cap H^{1,r}(S;H^{l,s}(\Omega_p^\varepsilon)^*) \to L^r(S;H^{l,r}(\Omega)) \cap H^{1,r}(S;H^{l,s}(\Omega)^*)$ using a much simpler approach. This note also generalizes the previously known results to Lipschitz domains and for any $r \in \mathbb{N}$ such that (s.t.) $\frac{1}{r} + \frac{1}{s} = 1$.

Keywords: periodic medium, extension theorems, scaling arguments, periodic homogenization.

MSC: 35B45, 35B27, 47N99.

1 Introduction

Several problems in the fields of physics, chemistry, biology and engineering sciences are governed by *partial differential equations* (PDEs). One of the most vital phenomena that can be explained with the help of these equations is the chemical transport in porous mediums (e.g. in soil, concrete, reservoir, rock etc)[2]. From literature it is known that a porous medium is a heterogeneous domain composed of solid parts (known as solid matrices) and a pore space (connected or disconnected) where usually transport processes take place[3]. Due to the heterogeneous structure of these mediums, it is difficult to perform numerical simulations and it is desirable to have

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[2]Mahato and Böhm, 2013, "Global existence and uniqueness for a system of nonlinear multi-species diffusion-reaction equations in an $H^{1,p}$ setting";

Rubin, 1983, "Transport of Reacting Solutes in Porous Media: Relation Between Mathematical Nature of Problem Formulation and Chemical Nature of Reactions".

[3]Bear and Bachmat, 1990, *Introduction to Modeling Phenomena of Transport in Porous Media*;

Hornung, 1997, *Homogenization and Porous Media*;

Knabner and Van Duijn, 1996, "Crystal dissolution in porous media flow";

Peter and Böhm, 2008, "Different choices of scaling in homogenization of diffusion and interfacial exchange in a porous medium";

an effective (upscaled) model which approximates the original model (original PDEs at the pore scale) and does not involve any heterogeneities. In other words, one would like to have the macroscopic description of a mathematical model which is microscopically heterogeneous, i.e. we obtain the effective (global) behaviour of the physical parameters involved in the micro-scale model. The effective model can be obtained via homogenization or some averaging method[4]. In this paper we mainly concern with the periodic homogenization. For periodic homogenization at first we obtain the *a priori estimates* of the solution at the micro-scale, i.e. for the domain $S\times\Omega_p^\varepsilon$ and then we extend these estimates in to all of $S\times\Omega$. Once the extended estimates are obtained, one can implement the concepts of *multi-scale convergence* to obtain the effective models. The homogenization is not our main concern in this work. Our main goal is to extend the *a priori* estimates from $S\times\Omega_p^\varepsilon$ in to all of $S\times\Omega$. Let us consider the following example of diffusion-reaction equations:

Assume that the heterogeneities inside a (perforated) porous medium are scaled by a parameter $\varepsilon>0$ and u^ε denotes the concentration of a chemical species present in the pore space Ω_p^ε of the medium Ω, then the diffusion-reaction equation is given as

$$\begin{aligned}\frac{\partial u^\varepsilon}{\partial t}-\nabla\cdot D^\varepsilon\nabla u^\varepsilon &= f && \text{in } S\times\Omega_p^\varepsilon, \\ -D^\varepsilon\nabla u^\varepsilon\cdot\vec{n} &= 0 && \text{on } \partial\Omega\cup\Gamma^\varepsilon, \\ u^\varepsilon(0,x) &= u_0(x) && \text{in } \Omega_p^\varepsilon,\end{aligned} \tag{1}$$

for notations and terminologies confer Section 1.1 on p. 108. Here D^ε is the diffusion-coefficient and f is the reaction rate or source term. In case Equation (1) is linear, very often in literature, the solution space $H^{1,2}(S;L^2(\Omega_p^\varepsilon))\cap L^2(S;H^{1,2}(\Omega_p^\varepsilon))$, i.e. $u^\varepsilon\in H^{1,2}(S;L^2(\Omega_p^\varepsilon))\cap L^2(S;H^{1,2}(\Omega_p^\varepsilon))$ and $u_0\in L^2(\Omega_p^\varepsilon)$. In Fatima et al. (2011), Hornung and Jäger (1991), Neuss-Radu (1992), and Peter and Böhm (2009) it is shown that there exists an extension operator $E_t^\varepsilon: H^{1,2}(S;L^2(\Omega_p^\varepsilon))\cap L^2(S;H^{1,2}(\Omega_p^\varepsilon))\to H^{1,2}(S;L^2(\Omega))\cap L^2(S;H^{1,2}(\Omega))$ which extends the estimates from $S\times\Omega_p^\varepsilon$ in to all of $S\times\Omega$. However, if Equation (1) is nonlinear, there are several situations where the solution space needs to be different than the trivial ones, i.e. the higher integrability exponent of the function spaces are required. One such example is sketched out below. Let $I\in\mathbb{N}$ number of mobile chemical species are present in the pore space Ω_p^ε of a porous medium Ω, cf. Figure 1 on the next page. These species diffuse and react with one another via the following reversible reactions:

$$\tau_{1j}X_1+\tau_{2j}X_2+\cdots+\tau_{Ij}X_I \rightleftharpoons \nu_{1j}X_1+\nu_{2j}X_2+\cdots+\nu_{Ij}X_I,$$

Van Duijn and Pop, 2004, "Crystal Dissolution and Precipitation in Porous Media: Pore Scale Analysis";

Whitaker, 1999, *The Method of Volume Averaging*.

[4] Allaire, 1992, "Homogenization and two scale convergence";

Neuss-Radu, 1992, "Homogenization techniques";

Whitaker, 1999, *The Method of Volume Averaging*.

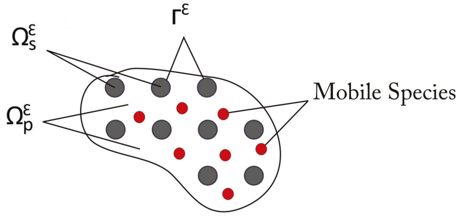

Figure 1 – Mobile species inside the pore space Ω_p^ε in Ω.

where $J \in \mathbb{N}$, the stoichiometric matrix $\tau_{ij}, \nu_{ij} \in \mathbb{N}$ and X_i are the species. Denote the concentration vector for each species as $u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon, \ldots, u_I^\varepsilon)$ and the matrix $\mathscr{S} := (s_{ij})_{1\le j\le J,\, 1\le i\le I}$, where $s_{ij} = \nu_{ij} - \tau_{ij}$. Then the system of diffusion-reaction equations for each species is given by

$$\frac{\partial u_i^\varepsilon}{\partial t} - \nabla\cdot(D^\varepsilon \nabla u^\varepsilon) = \mathscr{S}R(u^\varepsilon)_i \quad \text{in } \Omega_p^\varepsilon, \tag{2}$$

$$-D^\varepsilon \nabla u^\varepsilon \cdot \vec{n} = 0 \quad \text{on } \partial\Omega \cup \Gamma^\varepsilon, \tag{3}$$

$$u^\varepsilon(0,x) = u_0(x) \quad \text{in } \Omega_p^\varepsilon, \tag{4}$$

where the reaction rate for the i^{th} species is given by *mass-action kinetics* namely as

$$\mathscr{S}R(u^\varepsilon)_i := \sum_{j=1}^{J} s_{ij}\left(k_j^f \prod_{\substack{m=1\\ s_{mj}<0}}^{I} (u_m^\varepsilon)^{s_{mj}} - k_j^b \prod_{\substack{m=1\\ s_{mj}>0}}^{I} (u_m^\varepsilon)^{s_{mj}}\right)$$

and D^ε is the diffusive matrix. In Mahato and Böhm (2013) it is shown that the solution of Equations (2) to (4) exists in the vector-valued function space $[\Upsilon^\varepsilon]^I := \left\{\phi \in L^r(S;H^{1,r}(\Omega_p^\varepsilon))^I : \partial\phi/\partial t \in L^r(S;H^{1,s}(\Omega_p^\varepsilon)^*)^I\right\}$, i.e. $u^\varepsilon \in [\Upsilon^\varepsilon]^I$ and $u_0 \in (H^{1,s}(\Omega_p^\varepsilon)^*, H^{1,r}(\Omega_p^\varepsilon))^I_{1-1/r,r}$, where $r > n+2$ and $1/r + 1/s = 1$. For the definition of vector-valued function spaces we refer the interested reader to Kräutle (2011) and Mahato and Böhm (2013). The higher order nonlinear terms contribute to the difficulty in order to treat problems of type Equations (2) to (4). This type of problem has also been considered by Hoffmann (2010), Kräutle (2011), and Pierre (2010) and references therein. However, in the context of homogenization theory, not much work has been done for functions in Υ^ε and in $(H^{1,s}(\Omega_p^\varepsilon)^*, H^{1,r}(\Omega_p^\varepsilon))_{1-1/r,r}$ and their extensions from $S\times\Omega_p^\varepsilon$ in to all of $S\times\Omega$. This note generalizes the previous known extension theorems[5] to dual space setting to cover a more wider class of partial differential equations. In this paper, we shall prove that there exists extension operators $E_t^\varepsilon : L^r(S;H^{l,r}(\Omega_p^\varepsilon)) \cap H^{1,r}(S;H^{l,s}(\Omega_p^\varepsilon)^*) \to L^r(S;H^{l,r}(\Omega)) \cap H^{1,r}(S;H^{l,s}(\Omega)^*)$ and

[5] Cioranescu and Saint Jean Paulin, 1979, "Homogenization in open sets with holes"; Hornung and Jäger, 1991, "Diffusion, convection, adsorption and reaction of chemicals in porous media";

$F^\varepsilon : (H^{l,s}(\Omega_p^\varepsilon)^*, H^{l,r}(\Omega_p^\varepsilon))_{1-1/r,r} \to (H^{l,s}(\Omega)^*, H^{l,r}(\Omega))_{1-1/r,r}$, where $l \in \mathbb{N}_0$, $r \in \mathbb{N}$ and $1/r + 1/s = 1$.

1.1 Notations

Let $\Omega \subset \mathbb{R}^n$ ($n \geq 2$) be the perforated porous medium under consideration (see Figures 2 and 3 on the next page) with *Lipschitz boundary* and $Y := (0,1)^n \subset \mathbb{R}^n$ be a unit representative cell. To fix the ideas, let us make following assumptions:

- $Y = Y_s \cup Y_p$, where the solid part Y_s with boundary Γ and a pore part Y_p in Y are such that $\overline{Y}_s \subset Y$ and $\overline{Y}_s \cap Y_p = \emptyset$.
- Ω is composed of a pore space Ω_p and the union of disconnected solid parts Ω_s such that $\Omega := \Omega_p \cup \Omega_s$ and $\Omega_p \cap \overline{\Omega_s} = \emptyset$. Γ^* and $\partial\Omega$ are the union of boundaries of solid parts and the outer boundary of Ω. Ω is periodic (i.e. the solid parts in Ω are periodically distributed) and is covered by a finite union of the cells $Y_k := Y + k$, $k \in \mathbb{Z}^n$. $Y_{p_k} := Y_p + k$, $Y_{s_k} := Y_s + k$ and $\Gamma_k := \Gamma + k$.
- for a scale parameter $\varepsilon > 0$, we denote the pore space, solid parts and the union of the boundaries of solid matrices in Ω by Ω_p^ε, Ω_s^ε and Γ^ε and they are defined as: $\Omega_p^\varepsilon := \cup_{k\in\mathbb{Z}^n}\left\{\varepsilon Y_{p_k} : \varepsilon Y_{p_k} \subset \Omega\right\}$, $\Omega_s^\varepsilon := \cup_{k\in\mathbb{Z}^n}\left\{\varepsilon Y_{s_k} : \varepsilon Y_{s_k} \subset \Omega\right\}$ and $\Gamma^\varepsilon := \cup_{k\in\mathbb{Z}^n}\{\varepsilon\Gamma_k : \varepsilon\Gamma_k \subset \Omega\}$, see Figures 2 and 3 on the next page and on the next page.
- the boundaries $\Gamma, \Gamma^*, \Gamma^\varepsilon$ and $\partial\Omega$ are Lipschitz. We denote by $\mathrm{d}x$ and $\mathrm{d}y$ the volume elements in Ω and Y, and by $d\sigma_y$ and $d\sigma_x$ the surface elements on Γ and Γ^ε respectively.
- for a $T > 0$, $S := [0,T)$ is the time interval. Denote $\mathbb{R}_0^+ := \{x \in \mathbb{R} : x \geq 0\}$, $\mathbb{N} := \{1,2,3,\ldots\}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.
- for $\Xi \in \left\{\Omega, \Omega_p^\varepsilon\right\}$ and $l \in \mathbb{N}_0$, $r \in \mathbb{N}$ – $L^r(\Xi)$ and $H^{l,r}(\Xi)$ are the usual Lebesgue and Sobolev spaces with their usual norms and they are denoted by $\|\cdot\|_r$ and $\|.\|_{l,r}$. For sake of clarity, if $\phi \in H^{l,r}(\Xi)$, then

$$\|\phi\|_{l,r} := \|\phi\|_{H^{l,r}(\Xi)} := \begin{cases} \left[\displaystyle\sum_{|\alpha|\leq l} \int_\Xi \left|D^\alpha \phi\right|^r \mathrm{d}x\right]^{\frac{1}{r}} & \text{for } 1 \leq r < \infty \\ \displaystyle\sum_{|\alpha|\leq l} \operatorname*{ess\,sup}_{x\in\Xi} \left|D^\alpha \phi(x)\right| & \text{for } r = \infty, \end{cases}$$

Miller, 1992, "Extension theorems for homogenization on lattice structures";
Peter, 2003, "Modelling and homogenization of reaction interfacial exchange in porous media";
Tartar, 1980, "Incompressible fluid flow in a porous medium. Convergence of the homogenization process".

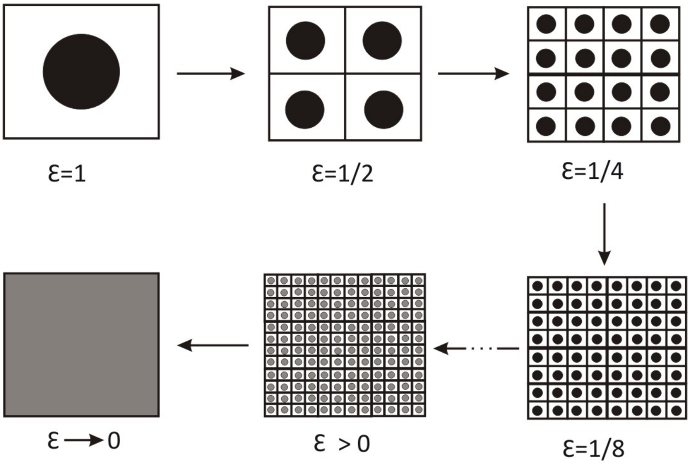

Figure 2 – A schematic representation of periodic homogenization.

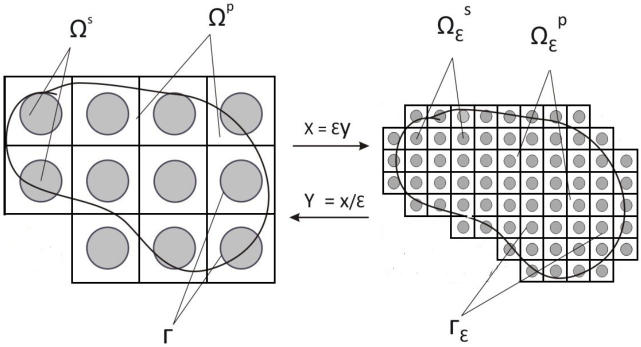

Figure 3 – This figure shows a perforated porous medium (left) under consideration which is assumed to be periodic and its homogeneous form is obtained via $x = \varepsilon y$, where $\varepsilon \ll 1$.

where $\alpha = (\alpha_1, \alpha_2, \dots \alpha_n) \in \mathbb{N}^n$ is a multi-index, $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$ and $D^\alpha = \frac{\partial^{|\alpha|}}{\partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \dots \partial_{x_n}^{\alpha_n}}$.

The extension and restriction operators are denoted by E and R; the symbols $\hookrightarrow$ and $\hookrightarrow\hookrightarrow$ denote the continuous and compact embeddings respectively. Moreover, we denote by X^* the dual space of X. Then for $\phi \in L^r(S; H^{l,r}(\Xi)) \cap H^{1,r}(S; H^{l,s}(\Xi)^*) =: \Upsilon_t(\Xi)$, we have

$$\|\phi\|_{\Upsilon_t(\Xi)} := \|\phi\|_{L^r(S;H^{l,r}(\Xi))} + \|\phi\|_{L^r(S;H^{l,s}(\Xi)^*)} + \left\|\frac{\partial \phi}{\partial t}\right\|_{L^r(S;H^{l,s}(\Xi)^*)}.$$

Suppose that X and Y are Banach spaces and $Y \hookrightarrow X$. The family of intermediate spaces between X and Y are called real-interpolation space, denoted by $(X,Y)_{\theta,p}$, $0 \le \theta \le 1$ and $1 \le p \le \infty$. We define this real-interpolation space with so called K-functional method. For every $x \in X$ and $t > 0$, we set

$$K(t,x,X,Y) = \inf_{\substack{x=a+b \\ a\in X,\ b\in Y}} (\|a\|_X + t\|b\|_Y)$$

then real-interpolation space is defined as

$$(X,Y)_{\theta,p} := \left\{ x \in X : t \mapsto t^{-\theta-\frac{1}{p}} K(t,x,X,Y) \in L^p(0,\infty) \right\}$$

and is endowed with the norm[6]

$$\|x\|_{(X,Y)_{\theta,p}} = \left\| t^{-\theta-\frac{1}{p}} K(t,x,X,Y) \right\|_{L^p(0,\infty)}.$$

The real-interpolation space between $H^{l,r}(\Xi)$ and $H^{l,s}(\Xi)^*$ is denoted by $(H^{l,r}(\Xi), H^{l,s}(\Xi)^*)_{1-1/r,r}$, where $\Xi \in \left\{\Omega, \Omega_p^\varepsilon\right\}$. Finally, C and C_l are the generic positive constants but independent of ε.

2 Main Results

We begin this section by a general case of *Second Poincaré inequality*[7].

Lemma 1 – *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$. Then, for all $\phi \in H^{l,r}(\Omega)$, $l = 1,2,\ldots$, we have the following inequality:*

$$\left\|\phi\right\|_{l,r}^r \le C\left(\sum_{|s|=l} \int_\Omega \left|D^s\phi\right|^r \mathrm{d}x + \sum_{|s|<l} \left| \int_\Omega D^s\phi \right|^r \mathrm{d}x \right), \tag{5}$$

where C is independent of ϕ.

Proof (sketch of proof). By *Kolmogorov compactness lemma*[8], it can be shown that $H_0^{\zeta_1,r}(\Omega) \hookrightarrow\hookrightarrow H_0^{\zeta_2,r}(\Omega)$, where $\zeta_1, \zeta_2 \in \mathbb{N}$, $\zeta_1 > \zeta_2$. By extension theorem on Lipschitz domains, $H^{\zeta_1,r}(\Omega) \xrightarrow{E} H_0^{\zeta_1,r}(\hat{\Omega}) \hookrightarrow H_0^{\zeta_2,r}(\hat{\Omega}) \xrightarrow{R} H^{\zeta_2,r}(\Omega)$, where $\hat{\Omega} := \{x \in \mathbb{R}^n : \mathrm{dist}(x,\Omega) < \eta \text{ for } \eta > 0\}$. Since E and R are continuous operators and the embedding is compact, hence the composition map is compact, i.e. $H^{\zeta_1,r}(\Omega) \hookrightarrow\hookrightarrow H^{\zeta_2,r}(\Omega)$. Now we assume that Equation (5) is false and there exists a sequence $\phi_n \in H^{l,r}(\Omega)$ s.t. $\left\|\phi_n\right\|_{l,r} = 1$. By Equation (5) and the compactness criteria, it can be shown easily that $\left\|\phi_n\right\|_{l,r} \to 0$ as $n \to \infty$ which is a contradiction to the assumption $\left\|\phi_n\right\|_{l,r} = 1$. Hence Equation (5) holds true. □

[6] Lunardi, 1995, *Analytic Semigroups and Optimal Regularity in Parabolic Problems*.
[7] For details see Wloka, 1987, *Partial Differential Equations*, theorem 7.7.

Lemma 2 (Miller[9]) – *For every multi-index α there exists a unique polynomial $p_\alpha(y)$ of degree $|\alpha|$ of type*

$$p_\alpha(y) = \frac{1}{\alpha!}y^\alpha + \sum_{|s|<|\alpha|} C_s^\alpha \frac{1}{s!} y^s \tag{6}$$

such that

$$\int_{Y_p} D^\beta p_\alpha(y)\,\mathrm{d}y = 0 \quad \textit{for all } |\beta| < |\alpha|. \tag{7}$$

Lemma 3 – *Let $Y \subset \mathbb{R}^n$ be a bounded domain and Y_s be an open set in Y such that $\overline{Y}_s \subset Y$, and the boundaries ∂Y_s and ∂Y are Lipschitz. Further assume that the domain $Y_p = Y \setminus \overline{Y}_s$ is also Lipschitz. Then, for $l = 1, 2, \ldots$, there exists an extension operator $E : H^{l,r}(Y_p) \to H^{l,r}(Y)$ such that for all $\phi \in H^{l,r}(Y_p)$, we have*

$$\sum_{|\alpha|=l} \left\| D^\alpha E\phi \right\|_{L^r(Y)}^r \le C_l \sum_{|\alpha|=l} \left\| D^\alpha \phi \right\|_{L^r(Y_p)}^r.$$

In particular, for $l = 0$, then

$$\left\| E\phi \right\|_{L^r(Y)} \le C_0 \left\| \phi \right\|_{L^r(Y_p)} \quad \textit{for } 1 \le r \le \infty.$$

Here the constants C_l and C_0 are independent of ϕ.

Before we prove Lemma 3 we note that the construction of the extension operator E depends on l, i.e. $E_l : H^{l,r}(Y_p) \to H^{l,r}(Y)$, but for the sake of notations we denote our operator by E and avoid the subscript l.

Proof (of Lemma 3). For $\phi \in H^{l,r}(Y_p)$, we set $\psi(y) = \phi(y) - \left\{\sum_{|s|<l} M_{Y_p}(D^s\phi) p_s(y)\right\}$, where $M_{Y_p}(\varphi) := \frac{1}{|Y_p|}\int_{Y_p} \varphi(y)\,\mathrm{d}y$ and $p_s(y)$ is the polynomial of type Equation (6) satisfying Equation (7). Since,

$$D^\alpha p_s(y) = \begin{cases} 1 & \text{if } |\alpha| = |s|, \\ 0 & \text{if } |\alpha| > |s| \text{ and } \alpha \ne s \end{cases}$$

and it can be shown easily, for $|\alpha| < l$,

$$\int_{Y_p} D^\alpha \psi(y)\,\mathrm{d}y = \int_{Y_p} D^\alpha \phi(y)\,\mathrm{d}y - \sum_{|s|<l} M_{Y_p}(D^s\phi) \int_{Y_p} D^\alpha p_s(y)\,\mathrm{d}y = 0.$$

[8] See Holden and Risebro, 2002, *Front Tracking for Hyperbolic Conservation Laws*, theorem A.5.
[9] Miller, 1992, "Extension theorems for homogenization on lattice structures", Lemma 1.

Then, by *Second Poincaré inequality*, we have

$$\begin{aligned}\|\psi\|^r_{H^{l,r}(Y_p)} &\le C_l \sum_{|\alpha|=l} \int_{Y_p} \left|D^\alpha \psi(y)\right|^r \mathrm{d}y \\ &\le 2^{r-1} C_l \sum_{|\alpha|=l} \left(\|D^\alpha \phi\|^r_{L^r(Y_p)} + \Big\| \sum_{|s|<l} M_{Y_p}(D^s\phi) D^\alpha p_s \Big\|^r_{L^r(Y_p)} \right) \\ &\le C_l \sum_{|\alpha|=l} \|D^\alpha \phi\|^r_{L^r(Y_p)},\end{aligned}$$

obviously the second term vanishes since the degree of the polynomial $p_s(y)$ is up to $l-1$ but the order of weak-differential operator D^α is l, so $D^l p_s(y) = 0$. By the extension theorem for Lipschitz domains[10], there exists a bounded linear extension operator $F : H^{l,r}(Y_p) \to H^{l,r}(Y)$ such that

$$\|F\psi\|^r_{H^{l,r}(Y)} \le C\|\psi\|^r_{H^{l,r}(Y_p)} \le C_l \sum_{|\alpha|=l} \|D^\alpha \phi\|^r_{L^r(Y_p)}. \tag{8}$$

Setting $E\phi(y) = F\psi(y) + \sum_{|s|<l} M_{Y_p}(D^s\phi)p_s(y)$. Then

$$\begin{aligned} D^\alpha E\phi(y) &= D^\alpha F\psi(y) + D^\alpha \left(\sum_{|s|<l} M_{Y_p}(D^s\phi) p_s(y) \right) \\ \implies \sum_{|\alpha|=l} \|D^\alpha E\phi\|^r_{L^r(Y)} &\le C \sum_{|\alpha|=l} \|D^\alpha F\psi\|^r_{L^r(Y)}, \text{ since } D^s p_s(y) = 0 \ \forall s = 0,1,\ldots,l-1 \\ \implies \sum_{|\alpha|=l} \|D^\alpha E\phi\|^r_{L^r(Y)} &\le C_l \sum_{|\alpha|=l} \|D^\alpha \phi\|^r_{L^r(Y_p)}. \end{aligned}$$

□

Theorem 1 – *Let Ω^ε_p, Ω be defined as in Section 1.1 on p. 108. There exists an extension operator $E^\varepsilon : H^{l,r}(\Omega^\varepsilon_p) \to H^{l,r}(\Omega)$ such that for any $\phi \in H^{l,r}(\Omega^\varepsilon_p)$ and $\beta \in \mathbb{R}^+_0$, it holds*

$$\sum_{|\alpha|=l} \|\varepsilon^\beta D^\alpha E^\varepsilon \phi\|^r_{L^r(\Omega)} \le C \sum_{|\alpha|=l} \|\varepsilon^\beta D^\alpha \phi\|^r_{L^r(\Omega^\varepsilon_p)}. \tag{9}$$

In particular,

$$\sum_{|\alpha|=l} \|D^\alpha E^\varepsilon \phi\|^r_{L^r(\Omega)} \le C \sum_{|\alpha|=l} \|D^\alpha \phi\|^r_{L^r(\Omega^\varepsilon_p)} \qquad \textit{for } \beta = 0 \textit{ and} \tag{10}$$

$$\|E^\varepsilon \phi\|_{L^r(\Omega)} \le C\|\phi\|_{L^r(\Omega^\varepsilon_p)} \qquad \textit{for } \beta = l = 0 \textit{ and } 1 \le r \le \infty. \tag{11}$$

[10] See Stein, 1970, *Singular Integrals and Differentiability Properties of Functions*.

Proof. Define $\overline{\phi}(y) = \frac{1}{\varepsilon^l}\phi(\varepsilon y)$ for $\phi \in H^{l,r}(\Omega_p^\varepsilon)$ and for each $k \in \mathbb{Z}^n$, $\overline{\phi_k}(y) := \overline{\phi}(y)|_{Y_{p_k}}$. Clearly, $\phi \in H^{l,r}(Y_{p_k})$ and by Lemma 3 on p. 111 E extends it to all Y_k. Since $\overline{Y}_s \subset Y$, we can define $E\overline{\phi}$ to all of Ω by $H^{l,r}(\Omega) \ni E^\varepsilon\phi(x) := \varepsilon^l E\overline{\phi}(\frac{x}{\varepsilon})$. Also, $x = \varepsilon y \implies D_x^\alpha = \frac{1}{\varepsilon^{|\alpha|}} D_y^\alpha$.

Using these terminologies and by a simple scaling argument and Lemma 3 on p. 111,

$$\begin{aligned}
\sum_{|\alpha|=l}\left\|\varepsilon^\beta D^\alpha E^\varepsilon\phi\right\|_{L^r(\Omega)}^r &= \varepsilon^{n+r\beta}\sum_{k\in\mathbb{Z}^n}\sum_{|\alpha|=l}\int_{Y_k}\left|D_y^\alpha E\overline{\phi}(y)\right|^r \mathrm{d}y \\
&\leq C\varepsilon^{n+r\beta}\sum_{k\in\mathbb{Z}^n}\sum_{|\alpha|=l}\int_{Y_{p_k}}\left|D_y^\alpha \phi(y)\right|^r \mathrm{d}y \\
&\leq C\varepsilon^{n+r\beta}\sum_{k\in\mathbb{Z}^n}\sum_{|\alpha|=l}\frac{1}{\varepsilon^n}\int_{\varepsilon Y_{p_k}}|D_x^\alpha\phi(x)|^r \mathrm{d}x \\
&\leq C\varepsilon^{r\beta}\sum_{|\alpha|=l}\int_{\Omega_p^\varepsilon}\left|D_x^\alpha\phi(x)\right|^r \mathrm{d}x.
\end{aligned}$$

Repeating the steps of the proof of Equation (9) on the preceding page, Equation (10) follows by setting $\beta = 0$ and Equation (11) follows by setting $\beta = l = 0$. □

Using the extension operator E^ε defined in previous section, one could define the time dependent extension operator $E_t^\varepsilon : L^r(S; H^{l,r}(\Omega_p^\varepsilon)) \to L^r(S; H^{l,r}(\Omega))$ s.t.

$$E_t^\varepsilon\phi(t,x) := [E^\varepsilon\phi(t,\cdot)](x) \quad \text{for } \phi \in L^r(S; H^{l,r}(\Omega_p^\varepsilon)). \tag{12}$$

Then by the linearity of E^ε,

$$\frac{\partial}{\partial t}(E_t^\varepsilon\phi(t,x)) := \frac{\partial}{\partial t}[E^\varepsilon\phi(t,\cdot)](x) = E^\varepsilon\left(\frac{\partial\phi}{\partial t}(t,\cdot)\right)(x) = E_t^\varepsilon\left(\frac{\partial\phi}{\partial t}\right)(t,x). \tag{13}$$

As the continuity of E_t^ε is straightforward, by a simple scaling argument we have:

Theorem 2 – *Let Ω_p^ε, Ω be defined as in Section 1.1 on p. 108. There exists an extension operator $E_t^\varepsilon : L^r(S; H^{l,r}(\Omega_p^\varepsilon)) \to L^r(S; H^{l,r}(\Omega))$ such that for any $\phi \in L^r(S; H^{l,r}(\Omega_p^\varepsilon))$ and $\beta \in \mathbb{R}_0^+$, it holds*

$$\sum_{|\alpha|=l}\left\|\varepsilon^\beta D^\alpha E_t^\varepsilon\phi\right\|_{L^r(S;L^r(\Omega))}^r \leq C\sum_{|\alpha|=l}\left\|\varepsilon^\beta D^\alpha\phi\right\|_{L^r(S;L^r(\Omega_p^\varepsilon))}^r. \tag{14}$$

In particular,

(Cont. next page)

$$\sum_{|\alpha|=l}\left\|\varepsilon^\beta D^\alpha E_t^\varepsilon\phi\right\|_{L^\infty(S;L^r(\Omega))} \leq C\sum_{|\alpha|=l}\left\|\varepsilon^\beta D^\alpha\phi\right\|_{L^\infty(S;L^r(\Omega_p^\varepsilon))}, \tag{15}$$

$$\sum_{|\alpha|=l}\left\|D^{\alpha}E_t^{\varepsilon}\phi\right\|^r_{L^r(S;L^r(\Omega))} \le C\sum_{|\alpha|=l}\left\|D^{\alpha}\phi\right\|^r_{L^r(S;L^r(\Omega_p^{\varepsilon}))} \quad \textit{for } \beta=0, \textit{ and} \tag{16}$$

$$\left\|E_t^{\varepsilon}\phi\right\|_{L^r(S;L^r(\Omega))} \le C\left\|\phi\right\|_{L^r(S;L^r(\Omega_p^{\varepsilon}))} \quad \textit{for } \beta=l=0 \textit{ and } 1\le r\le\infty. \tag{17}$$

Proof. We implement Equation (5) on p. 110 to obtain the Inequalities (14) to (17) on pp. 113–114. To begin with,

$$\begin{aligned}\sum_{|\alpha|=l}\left\|\varepsilon^{\beta}D^{\alpha}E_t^{\varepsilon}\phi\right\|^r_{L^r(S;L^r(\Omega))} &= \sum_{|\alpha|=l}\int_S\left\|\varepsilon^{\beta}D^{\alpha}(E_t^{\varepsilon}\phi(t))\right\|^r_{L^r(\Omega)}\,\mathrm{d}t\\ &= \sum_{|\alpha|=l}\int_S\left\|\varepsilon^{\beta}E_t^{\varepsilon}(D^{\alpha}\phi(t))\right\|^r_{L^r(\Omega)}\,\mathrm{d}t\\ &\le C\sum_{|\alpha|=l}\int_S\left\|\varepsilon^{\beta}D^{\alpha}\phi(t)\right\|^r_{L^r(\Omega_p^{\varepsilon})}\,\mathrm{d}t \quad \text{by (9)}\\ &= C\sum_{|\alpha|=l}\left\|\varepsilon^{\beta}D^{\alpha}\phi\right\|^r_{L^r(S;L^r(\Omega_p^{\varepsilon}))}.\end{aligned}$$

Also,

$$\begin{aligned}\sum_{|\alpha|=l}\left\|\varepsilon^{\beta}D^{\alpha}E_t^{\varepsilon}\phi\right\|_{L^\infty(S;L^r(\Omega))} &= \operatorname*{ess\,sup}_{t\in S}\sum_{|\alpha|=l}\left\|\varepsilon^{\beta}E_t^{\varepsilon}(D^{\alpha}\phi(t))\right\|_{L^r(\Omega)}\\ &\le C\operatorname*{ess\,sup}_{t\in S}\sum_{|\alpha|=l}\left\|\varepsilon^{\beta}D^{\alpha}\phi(t)\right\|_{L^r(\Omega_p^{\varepsilon})} \quad \text{by (9)}\\ &= C\sum_{|\alpha|=l}\left\|\varepsilon^{\beta}D^{\alpha}\phi\right\|_{L^\infty(S;L^r(\Omega_p^{\varepsilon}))}.\end{aligned}$$

The Inequalities (16) and (17) follow in the similar way as Inequalities (14) and (15) on the previous page. □

To define the extension operators on the dual spaces, we define the extension operator $F^{\varepsilon}: H^{l,s}(\Omega_p^{\varepsilon})^* \to H^{l,s}(\Omega)^*$ and the restriction operator $R^{\varepsilon}: H^{l,s}(\Omega)\to H^{l,s}(\Omega_p^{\varepsilon})$ as

$$\langle F^{\varepsilon}\Theta,\phi\rangle_{H^{l,s}(\Omega)^*\times H^{l,s}(\Omega)} = \langle\Theta,R^{\varepsilon}\phi\rangle_{H^{l,s}(\Omega_p^{\varepsilon})^*\times H^{l,s}(\Omega_p^{\varepsilon})}$$

for $\Theta\in H^{l,s}(\Omega_p^{\varepsilon})^*$ and $\phi\in H^{l,s}(\Omega)$. Therefore,

$$\|F^{\varepsilon}\Theta\|_{H^{l,s}(\Omega)^*} = \sup_{\|\phi\|_{H^{l,s}(\Omega)}\le 1}\left|\langle F^{\varepsilon}\Theta,\phi\rangle_{H^{l,s}(\Omega)^*\times H^{l,s}(\Omega)}\right| \le C\|\Theta\|_{H^{l,s}(\Omega_p^{\varepsilon})^*}.$$

We know that for a function $\phi\in L^r(S;H^{l,r}(\Omega_p^{\varepsilon}))\cap H^{1,r}(S;H^{l,s}(\Omega_p^{\varepsilon})^*)$, its trace at $t=0$ is in the real-interpolation space $(H^{l,s}(\Omega_p^{\varepsilon})^*,H^{l,r}(\Omega_p^{\varepsilon}))_{1-1/r,r}$[11]. This means that the

[11] Remark 1.2.11 in Lunardi, 1995, *Analytic Semigroups and Optimal Regularity in Parabolic Problems*.

initial condition from Equation (4) on p. 107 (to be precise in this case the i^{th} component of initial condition) is in $(H^{l,s}(\Omega_p^\varepsilon)^*, H^{l,r}(\Omega_p^\varepsilon))_{1-1/r,r}$. Thus we need an extension operator, say F^ε s.t. $F^\varepsilon : (H^{l,s}(\Omega_p^\varepsilon)^*, H^{l,r}(\Omega_p^\varepsilon))_{1-1/r,r} \to (H^{l,s}(\Omega)^*, H^{l,r}(\Omega))_{1-1/r,r}$. This is the aim of next result.

Theorem 3 – *Let $1 < r,s < \infty$ s.t. $\frac{1}{r} + \frac{1}{s} = 1$ and $\phi \in (H^{l,s}(\Omega_p^\varepsilon)^*, H^{l,r}(\Omega_p^\varepsilon))_{1-\frac{1}{r},r}$. Then there exists an extension $F^\varepsilon\phi$ of ϕ s.t. $F^\varepsilon\phi \in (H^{l,s}(\Omega)^*, H^{l,r}(\Omega))_{1-\frac{1}{r},r}$.*

Proof. Let $\theta = 1 - 1/r \in (0,1)$. Assume that $\phi \in (H^{l,s}(\Omega_p^\varepsilon)^*, H^{l,r}(\Omega_p^\varepsilon))_{1-\frac{1}{r},r}$ and its extension is denoted by $\bar{\phi} := F^\varepsilon\phi \in (H^{l,s}(\Omega)^*, H^{l,r}(\Omega))_{1-\frac{1}{r},r}$. We use the K-functional definition for real-interpolation space $(H^{l,s}(\Omega_p^\varepsilon)^*, H^{l,r}(\Omega_p^\varepsilon))_{\theta,p}$. To begin with, let $v \in H^{l,s}(\Omega)$, then there exists a restriction operator Q^ε such that

$$\begin{aligned} &Q^\varepsilon v := v|_{\Omega_p^\varepsilon} \\ \text{and} \quad &\|Q^\varepsilon v\|_{H^{l,s}(\Omega_p^\varepsilon)} \le C\|v\|_{H^{l,s}(\Omega)}, \end{aligned} \tag{18}$$

where C is independent of ε and v. Let $a_0 \in H^{l,s}(\Omega_p^\varepsilon)^*$, then we define the extension $\bar{a}_0$ of a_0 as

$$\langle \bar{a}_0, v\rangle_{H^{l,s}(\Omega)^*\times H^{l,s}(\Omega)} := \langle a_0, Q^\varepsilon v\rangle_{H^{l,s}(\Omega_p^\varepsilon)^*\times H^{l,s}(\Omega_p^\varepsilon)}. \tag{19}$$

This implies that

$$\begin{aligned} \|\bar{a}_0\|_{H^{l,s}(\Omega)^*} &= \sup_{\|v\|_{H^{l,s}(\Omega)\le 1}} \left|\langle \bar{a}_0, v\rangle_{H^{l,s}(\Omega)^*\times H^{l,s}(\Omega)}\right| \\ &= \sup_{\|Q^\varepsilon v\|_{H^{l,s}(\Omega_p^\varepsilon)}\le C} \left|\langle a_0, Q^\varepsilon v\rangle_{H^{l,s}(\Omega_p^\varepsilon)^*\times H^{l,s}(\Omega_p^\varepsilon)}\right| \quad \text{by (18) and (19)} \\ \implies \|\bar{a}_0\|_{H^{l,s}(\Omega)^*} &\le C\|\|a_0\|_{H^{l,s}(\Omega_p^\varepsilon)^*}. \end{aligned}$$

Again assume that $b_0 \in H^{l,r}(\Omega_p^\varepsilon)$. Let $\bar{b}_0 \in H^{l,r}(\Omega)$ denotes the extension of b_0 s.t.

$$\left\|\bar{b}_0\right\|_{H^{l,r}(\Omega)} \le C\|\|b_0\|_{H^{l,r}(\Omega_p^\varepsilon)} \text{ for } b_0 \in H^{l,r}(\Omega_p^\varepsilon),$$

where C is independent of ε and b_0. Let $t > 0$. Then

$$\begin{aligned} \|\bar{a}_0\|_{H^{l,s}(\Omega)^*} + t\left\|\bar{b}_0\right\|_{H^{l,r}(\Omega)} &\le C\left(\|a_0\|_{H^{l,s}(\Omega_p^\varepsilon)^*} + t\|b_0\|_{H^{l,r}(\Omega_p^\varepsilon)}\right) \\ &\le C\left(\|a_0\|_{H^{l,s}(\Omega_p^\varepsilon)^*} + t\|b_0\|_{H^{l,r}(\Omega_p^\varepsilon)}\right). \end{aligned}$$

Taking the infimum on both sides, we get successively

$$\inf_{\substack{\bar{\phi}=\bar{a}_0+\bar{b}_0 \\ \bar{a}_0\in H^{l,s}(\Omega)^*, \bar{b}_0\in H^{l,r}(\Omega)}} \left(\|\bar{a}_0\|_{H^{l,s}(\Omega)^*} + t\left\|\bar{b}_0\right\|_{H^{l,r}(\Omega)}\right) \le C \inf_{\substack{\phi=a_0+b_0 \\ a_0\in H^{l,s}(\Omega_p^\varepsilon)^*, b_0\in H^{l,r}(\Omega_p^\varepsilon)}} \left(\|a_0\|_{H^{l,s}(\Omega_p^\varepsilon)^*} + t\|b_0\|_{H^{l,r}(\Omega_p^\varepsilon)}\right),$$

$$\underbrace{t^{-\theta}\inf_{\substack{\bar\phi=\bar a_0+\bar b_0\\ \bar a_0\in H^{l,s}(\Omega)^*,\bar b_0\in H^{l,r}(\Omega)}}\Big(\|\bar a_0\|_{H^{l,s}(\Omega)^*}+t\big\|\bar b_0\big\|_{H^{l,r}(\Omega)}\Big)}_{\text{positive}}\le C\,\underbrace{t^{-\theta}\inf_{\substack{\phi=a_0+b_0\\ a_0\in H^{l,s}(\Omega_p^\varepsilon)^*,b_0\in H^{l,r}(\Omega_p^\varepsilon)}}\Big(\|a_0\|_{H^{l,s}(\Omega_p^\varepsilon)^*}+t\|b_0\|_{H^{l,r}(\Omega_p^\varepsilon)}\Big)}_{\text{positive}},$$

$$\bigg|t^{-\theta}\inf_{\substack{\bar\phi=\bar a_0+\bar b_0\\ \bar a_0\in H^{l,s}(\Omega)^*,\bar b_0\in H^{l,r}(\Omega)}}\Big(\|\bar a_0\|_{H^{l,s}(\Omega)^*}+t\big\|\bar b_0\big\|_{H^{l,r}(\Omega)}\Big)\bigg|^r\le C^r\bigg|t^{-\theta}\inf_{\substack{\phi=a_0+b_0\\ a_0\in H^{l,s}(\Omega_p^\varepsilon)^*,b_0\in H^{l,r}(\Omega_p^\varepsilon)}}\Big(\|a_0\|_{H^{l,s}(\Omega_p^\varepsilon)^*}+t\|b_0\|_{H^{l,r}(\Omega_p^\varepsilon)}\Big)\bigg|^r.$$

Thus

$$\begin{aligned}
&\int_0^\infty\frac{1}{t}\bigg|t^{-\theta}\inf_{\substack{\bar\phi=\bar a_0+\bar b_0\\ \bar a_0\in H^{l,s}(\Omega)^*,\bar b_0\in H^{l,r}(\Omega)}}\Big(\|\bar a_0\|_{H^{l,s}(\Omega)^*}+t\big\|\bar b_0\big\|_{H^{l,r}(\Omega)}\Big)\bigg|^r\mathrm{d}t\\
&\qquad\le C^r\int_0^\infty\frac{1}{t}\bigg|t^{-\theta}\inf_{\substack{\phi=a_0+b_0\\ a_0\in H^{l,s}(\Omega_p^\varepsilon)^*,b_0\in H^{l,r}(\Omega_p^\varepsilon)}}\Big(\|a_0\|_{H^{l,s}(\Omega_p^\varepsilon)^*}+t\|b_0\|_{H^{l,r}(\Omega_p^\varepsilon)}\Big)\bigg|^r\\
\Longrightarrow&\int_0^\infty\frac{1}{t}\Big|t^{-\theta}K(t,\bar\phi,H^{l,s}(\Omega)^*,H^{l,r}(\Omega))\Big|^r\mathrm{d}t\\
&\qquad\le C^r\int_0^\infty\frac{1}{t}\Big|t^{-\theta}K(t,\phi,H^{l,s}(\Omega_p^\varepsilon)^*,H^{l,r}(\Omega_p^\varepsilon))\Big|^r\mathrm{d}t\\
\Longrightarrow&\big\|\bar\phi\big\|_{(H^{l,s}(\Omega)^*,H^{l,r}(\Omega))_{1-\frac{1}{r},r}}\\
&\qquad\le C\big\|\phi\big\|_{(H^{l,s}(\Omega_p^\varepsilon)^*,H^{l,r}(\Omega_\varepsilon^p))_{1-\frac{1}{r},r}},
\end{aligned}$$

where the constant C is independent of ε and u. □

Next we will prove the last theorem of this work which is basically the extension theorem on *Bochner spaces*.

Theorem 4 – *For any function $\phi\in L^r(S;H^{l,r}(\Omega_p^\varepsilon))\cap H^{1,r}(S;H^{l,s}(\Omega_p^\varepsilon)^*)$, there exists an extension operator $E_t^\varepsilon: L^r(S;H^{l,r}(\Omega_p^\varepsilon))\cap H^{1,r}(S;H^{l,s}(\Omega_p^\varepsilon)^*)\to L^r(S;H^{l,r}(\Omega))\cap H^{1,r}(S;H^{l,s}(\Omega)^*)$ s.t.*

$$\begin{aligned}
&\big\|E_t^\varepsilon\phi\big\|_{L^r(S;H^{l,r}(\Omega))}+\bigg\|\frac{\partial}{\partial t}(E_t^\varepsilon\phi)\bigg\|_{L^r(S;H^{l,s}(\Omega)^*)}\\
&\qquad\le C\left(\big\|\phi\big\|_{L^r(S;H^{l,r}(\Omega_p^\varepsilon))}+\bigg\|\frac{\partial\phi}{\partial t}\bigg\|_{L^r(S;H^{l,s}(\Omega_p^\varepsilon)^*)}\right),
\end{aligned}$$

where C is independent of ε and ϕ.

Proof. Here:

$$\begin{aligned}\phi \in \Upsilon^\varepsilon &:= L^r(S;H^{l,r}(\Omega_p^\varepsilon)) \cap H^{1,r}(S;H^{l,s}(\Omega_p^\varepsilon)^*)\\ &:= \left\{\zeta \in L^r(S;H^{l,r}(\Omega_p^\varepsilon)) : \frac{\partial \zeta}{\partial t} \in L^r(S;H^{l,s}(\Omega_p^\varepsilon)^*)\right\}\end{aligned}$$

Then by Theorem 2 on p. 113 the operator $E_t^\varepsilon \phi \in L^r(S;H^{l,r}(\Omega))$. We claim that E_t^ε extends $\frac{\partial\phi}{\partial t}$ from $L^r(S;H^{l,s}(\Omega_p^\varepsilon)^*)$ to $L^r(S;H^{l,s}(\Omega)^*)$. Let $\frac{\partial\overline{\phi}}{\partial t}$ denotes the extension of $\frac{\partial\phi}{\partial t}$ in $L^r(S;H^{l,s}(\Omega)^*)$. To accomplish the claim we just need to show that $E_t^\varepsilon\left(\frac{\partial\phi}{\partial t}\right) = \frac{\partial}{\partial t}(E_t^\varepsilon\phi) = \frac{\partial\overline{\phi}}{\partial t}$ in $L^r(S;H^{l,s}(\Omega)^*)$, i.e.

$$\begin{aligned}&\left\|E_t^\varepsilon\left(\frac{\partial\phi}{\partial t}\right) - \frac{\partial}{\partial t}(E_t^\varepsilon\phi)\right\|^r_{L^r(S;H^{l,s}(\Omega)^*)}\\
&= \int_S \sup_{\|\theta\|_{H^{l,s}(\Omega)}\le 1}\left|\left\langle E_t^\varepsilon\left(\frac{\partial\phi}{\partial t}\right) - \frac{\partial}{\partial t}(E_t^\varepsilon\phi),\theta\right\rangle_{H^{l,s}(\Omega)^*\times H^{l,s}(\Omega)}\right|^r \mathrm{d}t\\
&= \int_S \sup_{\|\theta\|_{H^{l,s}(\Omega)}\le 1}\left|\left\langle E_t^\varepsilon\left(\frac{\partial\phi}{\partial t}\right),\theta\right\rangle_{H^{l,s}(\Omega)^*\times H^{l,s}(\Omega)} - \left\langle\frac{\partial}{\partial t}(E_t^\varepsilon\phi),\theta\right\rangle_{H^{l,s}(\Omega)^*\times H^{l,s}(\Omega)}\right|^r \mathrm{d}t\\
&= \int_S \sup_{\|R^\varepsilon\theta\|_{H^{l,s}(\Omega_p^\varepsilon)}\le C}\left|\left\langle\frac{\partial\phi}{\partial t},R^\varepsilon\theta\right\rangle_{H^{l,s}(\Omega_p^\varepsilon)^*\times H^{l,s}(\Omega_p^\varepsilon)} - \left\langle\frac{\partial\phi}{\partial t},R^\varepsilon\theta\right\rangle_{H^{l,s}(\Omega_p^\varepsilon)^*\times H^{l,s}(\Omega_p^\varepsilon)}\right|^r \mathrm{d}t.\end{aligned}$$

Hence, the operator E_t^ε extends the function ϕ in $L^r(S;H^{l,r}(\Omega_p^\varepsilon)) \cap H^{1,r}(S;H^{l,s}(\Omega_p^\varepsilon)^*)$ to all of $L^r(S;H^{l,r}(\Omega)) \cap H^{1,r}(S;H^{l,s}(\Omega)^*)$. The *a priori* estimate is straightforward from Theorem 2 on p. 113 and Definition 2 on p. 114. □

3 Conclusions

We saw in Section 2 on p. 110 how one could define the extension operator for Bochner spaces in dual space settings, which were to author's knowledge unknown results. We also extended the *a-priori* estimates in real-interpolation spaces. Both of these two extension results are very vital when one deals with general classes of parabolic equations in heterogeneous mediums. The results obtained by the author in this paper also generalize the previous known results from Cioranescu and Saint Jean Paulin (1979), Hornung and Jäger (1991), Miller (1992), Neuss-Radu (1992), Peter and Böhm (2008), and Tartar (1980) for L^2-space settings.

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Contents