



Cohomologies of deformations of solvmanifolds and closedness of some properties

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Abstract

We provide further techniques to study the Dolbeault and Bott-Chern cohomologies of deformations of solvmanifolds by means of finite-dimensional complexes. By these techniques, we can compute the Dolbeault and Bott-Chern cohomologies of some complex solvmanifolds, and we also get explicit examples, showing in particular that either the $\partial\bar{\partial}$ -Lemma or the property that the Hodge and Frölicher spectral sequence degenerates at the first level are not closed under deformations.

Keywords: Dolbeault cohomology, Bott-Chern cohomology, solvmanifold, deformation, $\partial\bar{\partial}$ -Lemma.

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Introduction

Among other techniques, the theory of *small deformations of holomorphic structures*, initiated and developed by Kodaira and Spencer, Nirenberg, and Kuranishi, provides a large source of examples of compact complex manifolds.

As a natural problem, the behaviour of special metrics or cohomological properties under deformations deserves special interests in order to better understand the geometry of complex manifolds. In such a context, the stability results for Kähler structures plays a guiding role: in fact, Kodaira and Spencer proved³ that any small deformations of a compact Kähler manifold still admits a Kähler metric. On the other hand, the result holds no more true when replacing the Kähler condition with weaker metric conditions, such as, for example, the existence of balanced metrics in

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³Kodaira and Spencer, 1960, "On deformations of complex analytic structures. III. Stability theorems for complex structures", Theorem 15.

the sense of Michelsohn⁴, or the existence of pluri-closed metrics⁵ (, nor also in the non-elliptic context of \mathbf{D} -complex geometry in the sense of Harvey and Lawson⁶, nor in the non-integrable case of almost-Kähler geometry). As regards cohomological properties, the stability of the $\partial\bar{\partial}$ -Lemma under deformations has been proved in several ways⁷. (We recall that a compact complex manifold is said to satisfy the $\partial\bar{\partial}$ -Lemma if every ∂ -closed, $\bar{\partial}$ -closed, d -exact form is also $\partial\bar{\partial}$ -exact⁸.) Kodaira and Spencer's result, for example, can be phrased by saying that, for any family of compact complex manifolds parametrized over the manifold \mathcal{B} , the set of parameters of \mathcal{B} for which the corresponding complex manifold admits a Kähler metric is open in the topology of \mathcal{B} . In Angella and Kasuya (2017, Theorem 2.20), in studying the cohomologies of the completely-solvable Nakamura manifold, the authors provided an example of a curve $\{J_t\}_{t \in B}$ of complex structures and of a sequence $\{t_k\}_{k \in \mathbb{N}} \subset B$ converging to t_∞ in the topology of B such that (X, J_{t_k}) satisfies the $\partial\bar{\partial}$ -Lemma for any $k \in \mathbb{N}$ but (X, J_{t_∞}) does not; in other words, the set of parameters for which the $\partial\bar{\partial}$ -Lemma holds is not closed in the (strong) topology of the base space. Actually, as Ugarte pointed out to us, in studying the behaviour under limits of compact complex manifolds, it is common to consider Zariski topology instead of strong topology: in fact, e.g., Moisëzon property⁹ is supposed to be closed with respect to the Zariski topology¹⁰ for motivations and results, while it is not closed in the strong topology. With such a notion of (Zariski) closedness, we provide here an example to prove the following result. Note that the non-closedness of E_1 -degeneration of the Hodge and Frölicher spectral sequences was already proven by Eastwood and Singer¹¹ by using twistor spaces.

Theorem (see Corollary 3 on p. 88) – *The property of E_1 -degeneration of the Hodge and Frölicher spectral sequences and the property of satisfying the $\partial\bar{\partial}$ -Lemma are not closed under holomorphic deformations.*

In order to provide such an example, we continue in investigating the class of nilmanifolds and solvmanifolds from the point of view of cohomologies computa-

⁴Alessandrini and Bassanelli, 1990, "Small deformations of a class of compact non-Kähler manifolds", Proposition 4.1.

⁵Fino and Tomassini, 2009, "Blow-ups and resolutions of strong Kähler with torsion metrics", Theorem 2.2.

⁶Angella and Rossi, 2012, "Cohomology of \mathbf{D} -complex manifolds", Theorem 4.2.

⁷See Voisin, 2002, *Hodge Theory and Complex Algebraic Geometry I*, Proposition 9.21;

Wu, 2006, "On the geometry of superstrings with torsion", Theorem 5.12;

Tomasiello, 2008, "Reformulating supersymmetry with a generalized Dolbeault operator", § B;

Angella and Tomassini, 2013, "On the $\partial\bar{\partial}$ -lemma and Bott-Chern cohomology", Corollary 2.7.

⁸See, e.g., Deligne et al., 1975, "Real homotopy theory of Kähler manifolds".

⁹Moisëzon, 1966, "On n -dimensional compact complex manifolds having n algebraically independent meromorphic functions. I, II, III".

¹⁰See Popovici, 2013, "Deformation limits of projective manifolds: Hodge numbers and strongly Gauduchon metrics".

¹¹Eastwood and Singer, 1993, "The Frölicher spectral sequence on a twistor space", Theorem 5.4.

Introduction

tions. More precisely, we would enlarge the class of solvmanifolds for which the de Rham, Dolbeault, and Bott-Chern cohomologies can be computed by means of just a finite-dimensional sub-complex of the double-complex of differential forms, by carrying over the results in Angella (2013), Angella and Kasuya (2017), Console and Fino (2001), Console, Fino, and Kasuya (2016), Console and Fino (2011), Cordero et al. (2000), Guan (2007), Hattori (1960), Kasuya (2013a,b, 2014a), Mostow (1954, 1957), Nomizu (1954), Rollenske (2009, 2011a), and Sakane (1976). We recall that, given a double-complex $(A^{\bullet,\bullet}, \partial, \bar{\partial})$, the Dolbeault cohomology is

$$H_{\bar{\partial}}^{\bullet,\bullet}(A^{\bullet,\bullet}) := \frac{\ker \bar{\partial}}{\text{im } \bar{\partial}}$$

and the *Bott-Chern cohomology* is

$$H_{BC}^{\bullet,\bullet}(A^{\bullet,\bullet}) := \frac{\ker \partial \cap \ker \bar{\partial}}{\text{im } \partial \bar{\partial}};$$

one can also consider the *Aeppli cohomology*,

$$H_A^{\bullet,\bullet}(A^{\bullet,\bullet}) := \frac{\ker \partial \bar{\partial}}{\text{im } \partial + \text{im } \bar{\partial}},$$

which is, in a sense, the dual of the Bott-Chern cohomology; finally, in considering a complex manifold, the Dolbeault and Bott-Chern cohomology are defined by means of the double-complex $(\wedge^{\bullet,\bullet} X, \partial, \bar{\partial})$ of complex-valued differential forms¹²¹³. More precisely, we provide the following stability results for cohomology computations of deformations of solvmanifolds, in the vein of the results proven in Console and Fino (2001, Theorem 1) and Angella (2013, Theorem 3.9) for nilmanifolds; (we refer to Theorems 1 and 2 for the precise statement).

Theorem (see Theorems 1 and 2 on the next page and on p. 77) – *Given a solvmanifold $X = \Gamma \backslash G$ endowed with a left-invariant complex structure J , for which there exists a finite-dimensional sub-complex $C^{\bullet,\bullet} \subset \wedge^{\bullet,\bullet} X$ computing the Dolbeault cohomology, we provide conditions in order that suitable deformations of $C^{\bullet,\bullet}$, still allow to compute Dolbeault and Bott-Chern cohomologies of some small deformations of J .*

The proof of this theorem is inspired by the proof of Kodaira and Spencer’s theorem on the upper-semi-continuity of the dimensions of the Dolbeault cohomology groups¹⁴. Considering downers of cohomologies, differing from upper-semi-continuity, by this theorem we can observe “nose-diving” phenomena, as in the

¹²See Aeppli, 1965, “On the cohomology structure of Stein manifolds”;
Bott and Chern, 1965, “Hermitian vector bundles and the equidistribution of the zeroes of their holomorphic sections”.

¹³See also Demailly, 2012, *Complex Analytic and Differential Geometry*;
Schweitzer, 2007, “Autour de la cohomologie de Bott-Chern”.

¹⁴Kodaira and Spencer, 1960, “On deformations of complex analytic structures. III. Stability theorems for complex structures”.

following examples, which are generalizations of the three-dimensional examples found by Kodaira and Nakamura¹⁵.

Example (see Section 5 on p. 85) – Let N be a complex nilpotent Lie group. Suppose that the Lie algebra of N has a $(\mathbb{Z} + \sqrt{-1}\mathbb{Z})$ -basis. Then, for certain semidirect product $G = \mathbb{C} \ltimes_{\phi} (N \times N)$, we have a lattice Γ of G by the results of Sawai and Yamada¹⁶, and there exists a deformation $\{J_t\}_t$ of the holomorphically parallelizable solvmanifold $\Gamma \backslash G$ such that $\dim H_{\partial_t}^{1,0}(\Gamma \backslash G) = 0$, where $H_{\partial_t}^{\bullet,\bullet}(\Gamma \backslash G)$ is the Dolbeault cohomology of a deformed complex solvmanifold.

In particular, it follows that the above examples provide a new class of “Dolbeault-cohomologically-computable” complex solvmanifolds, since they are not holomorphic fibre bundles over complex tori as in Console, Fino, and Kasuya (2016) and Kasuya (2013a, 2014a).

1 Deformations and cohomology

Let (M, J) be a compact complex manifold and $(\wedge_J^{\bullet,\bullet} M, \partial, \bar{\partial})$ be the double-complex of complex-valued differential forms on M associated with the complex structure J . We consider deformations $\{J_t\}_{t \in B}$ over a ball $B \subset \mathbb{C}^m$ such that $J_0 = J$. We also consider the double-complex $(\wedge_{J_t}^{\bullet,\bullet} M, \partial_t, \bar{\partial}_t)$ associated with the deformed complex structure J_t .

We are interested in manifolds whose cohomologies can be computed by means of just a finite-dimensional sub-double-complex of $(\wedge_{J_t}^{\bullet,\bullet} M, \partial, \bar{\partial})$. In particular, in this section, we are concerned in studying the behaviour of such a property under small deformations of the complex structure.

Inspired by Kodaira and Spencer (1960), we prove the following result.

Theorem 1 – *Let (M, J) be a compact complex manifold, and consider deformations $\{J_t\}_{t \in B}$ such that $J_0 = J$. We suppose that we have a family $\{C_t^{\bullet,\bullet} = \mathbb{C}\langle \phi_i^{\bullet,\bullet}(t) \rangle_i\}_{t \in B}$ of sub-vector spaces of $(\wedge_{J_t}^{\bullet,\bullet} M, \partial_t, \bar{\partial}_t)$ parametrized by $t \in B$ and spanned by linearly-independent vectors $\phi_i^{\bullet,\bullet}(t)$ so that:*

(A₁) *for each $t \in B$, it holds that $(C_t^{\bullet,\bullet}, \bar{\partial}_t)$ is a sub-complex of $(\wedge_{J_t}^{\bullet,\bullet} M, \bar{\partial}_t)$;*

(A₂) *$\phi_i^{\bullet,\bullet}(t)$ is smooth on $M \times B$, for any i ;*

(A₃) *the inclusion $C_0^{\bullet,\bullet} \subset \wedge_J^{\bullet,\bullet} M$ induces the cohomology isomorphism*

$$H_{\partial_0}^{\bullet,\bullet}(C_0^{\bullet,\bullet}) \cong H_{\partial}^{\bullet,\bullet}(M);$$

¹⁵Nakamura, 1975, “Complex parallelisable manifolds and their small deformations”.

¹⁶Sawai and Yamada, 2005, “Lattices on Benson-Gordon type solvable Lie groups”.

1. Deformations and cohomology

(A₄) there exists a smooth family $\{g_t\}_{t \in B}$ of J_t -Hermitian metrics such that $\bar{\kappa}_{g_t}(C_t^{\bullet,\bullet}) \subseteq C_t^{n-\bullet, n-\bullet}$, where we denote by $\bar{\kappa}_{g_t}$ the anti- \mathbb{C} -linear Hodge- $*$ -operator of g_t , and by $2n$ the real dimension of M .

Then, for sufficiently small t , the inclusion $C_t^{\bullet,\bullet} \subset \wedge_{J_t}^{\bullet,\bullet}(M)$ induces the cohomology isomorphism

$$H_{\bar{\partial}_t}^{\bullet,\bullet}(C_t^{\bullet,\bullet}) \cong H_{\bar{\partial}_t}^{\bullet,\bullet}(M).$$

Proof. Consider the operators $\bar{\partial}_t^* = -\bar{\kappa}_{g_t} \bar{\partial}_t \bar{\kappa}_{g_t}$ and $\Delta_{\bar{\partial}_t} = \bar{\partial}_t \bar{\partial}_t^* + \bar{\partial}_t^* \bar{\partial}_t$. Then by the assumptions (A₁) and (A₄) on the preceding page and on the current page, the operator $\Delta_{\bar{\partial}_t}$ can be defined on $C_t^{\bullet,\bullet}$. By a result by Kodaira and Spencer¹⁷, for each $t \in B$, we have a basis $\{e_1(t), \dots, e_i(t), \dots\}$ of $\wedge_{J_t}^{\bullet,\bullet} M$ and continuous functions $a_1(t) \leq \dots \leq a_i(t) \leq \dots$ on B such that $\Delta_{\bar{\partial}_t} e_i(t) = a_i(t) e_i(t)$ for any i . Since $\Delta_{\bar{\partial}_t}$ is defined on $C_t^{\bullet,\bullet}$, we can take a subset $\{e_{i_1}(t), \dots, e_{i_k}(t)\}$ of $\{e_i(t)\}_i$ that is a basis of $C_t^{\bullet,\bullet}$. Take $\{e_j(t), \dots, e_{j+k}(t)\} = \{e_i(t) \mid a_i(0) = 0\}$. Then $\{e_j(0), \dots, e_{j+k}(0)\}$ is a basis of $\ker \Delta_{\bar{\partial}_0}$. By the assumption (A₃) on the preceding page, we have $\ker \Delta_{\bar{\partial}_0} \subseteq C_0^{\bullet,\bullet}$. Hence we have $\{e_j(t), \dots, e_{j+k}(t)\} \subseteq C_t^{\bullet,\bullet}$ for any $t \in B$. Since each a_i is continuous, we have, for sufficiently small $t \in B$, that $a_{j-1}(t) < 0$ and $0 < a_{j+k+1}(t)$. Hence we have $\ker \Delta_{\bar{\partial}_t} \subseteq \{e_j(t), \dots, e_{j+k}(t)\} \subseteq C_t^{\bullet,\bullet}$. Hence the theorem follows. \square

Analogously, as regards the Bott-Chern cohomology, by considering the operators¹⁸,

$$\begin{aligned} \bar{\partial}_t^* &= -\bar{\kappa}_{g_t} \bar{\partial}_t \bar{\kappa}_{g_t}, \\ \partial_t^* &= -\bar{\kappa}_{g_t} \partial_t \bar{\kappa}_{g_t}, \text{ and} \\ \tilde{\Delta}_{BC_t} &= \partial_t \bar{\partial}_t \bar{\partial}_t^* \partial_t^* + \bar{\partial}_t^* \partial_t^* \partial_t \bar{\partial}_t + \bar{\partial}_t^* \partial_t \partial_t^* \bar{\partial}_t + \partial_t^* \bar{\partial}_t \bar{\partial}_t^* \partial_t + \bar{\partial}_t^* \partial_t + \partial_t^* \partial_t \end{aligned}$$

a similar argument yields the following result.

Theorem 2 – Let (M, J) be a compact complex manifold, and consider deformations $\{J_t\}_{t \in B}$ such that $J_0 = J$. We suppose that we have a family $\{C_t^{\bullet,\bullet} = \mathbb{C}\langle \phi_i^{\bullet,\bullet}(t) \rangle_i\}_{t \in B}$ of sub-vector spaces of $(\wedge_{J_t}^{\bullet,\bullet} M, \partial_t, \bar{\partial}_t)$ parametrized by $t \in B$ and spanned by linearly-independent vectors $\phi_i^{\bullet,\bullet}(t)$ so that:

1. for each $t \in B$, it holds that $(C_t^{\bullet,\bullet}, \partial_t, \bar{\partial}_t)$ is a sub-double-complex of $(\wedge_{J_t}^{\bullet,\bullet} M, \partial_t, \bar{\partial}_t)$;

¹⁷Kodaira and Spencer, 1960, “On deformations of complex analytic structures. III. Stability theorems for complex structures”, Theorem 11;

see also Kodaira, 2005, *Complex manifolds and deformation of complex structures*, Theorem 7.1.

¹⁸Kodaira and Spencer, 1960, “On deformations of complex analytic structures. III. Stability theorems for complex structures”, Proposition 5;

Schweitzer, 2007, “Autour de la cohomologie de Bott-Chern”, § 2.b.

2. $\phi_i^{\bullet,\bullet}(t)$ is smooth on $M \times B$, for any i ;
3. the inclusion $C_0^{\bullet,\bullet} \subset \wedge_{J_t}^{\bullet,\bullet} M$ induces the Bott-Chern cohomology isomorphism

$$H_{BC}^{\bullet,\bullet}(C_0^{\bullet,\bullet}) \cong H_{BC}^{\bullet,\bullet}(M);$$

4. there exists a smooth family $\{g_t\}_{t \in B}$ of J_t -Hermitian metrics such that $\bar{x}_{g_t}(C_t^{\bullet,\bullet}) \subseteq C_t^{n-\bullet, n-\bullet}$, where we denote by \bar{x}_{g_t} the anti- \mathbb{C} -linear Hodge- $*$ -operator of g_t , and by $2n$ the real dimension of M .

Then, for sufficiently small t , the inclusion $C_t^{\bullet,\bullet} \subset \wedge_{J_t}^{\bullet,\bullet} M$ induces the Bott-Chern cohomology isomorphism

$$H_{BC}^{\bullet,\bullet}(C_t^{\bullet,\bullet}) \cong H_{BC}^{\bullet,\bullet}(M).$$

2 Applications: nilmanifolds

Consider nilmanifolds, that is, compact quotients of connected simply-connected nilpotent Lie groups by discrete co-compact subgroups, and take left-invariant complex structures. By considering the sub-double-complex $C^{\bullet,\bullet} = \wedge^{\bullet,\bullet}(\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C})^*$ of left-invariant differential forms, where \mathfrak{g} is the Lie algebra associated to the nilmanifold, one recovers the stability results in Angella (2013) and Console and Fino (2001) by Theorems 1 and 2 on p. 76 and on the previous page.

Corollary 1 (Console and Fino¹⁹, Angella²⁰) – *Let $X = \Gamma \backslash G$ be a nilmanifold, and denote the Lie algebra associated to G by \mathfrak{g} and its complexification by $\mathfrak{g}_{\mathbb{C}} := \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$. The set of G -left-invariant complex structures on X such that the inclusion $\wedge^{\bullet,\bullet} \mathfrak{g}_{\mathbb{C}}^* \subset \wedge^{\bullet,\bullet} X$ induces the isomorphism $H_{\bar{\partial}}^{\bullet,\bullet}(\wedge^{\bullet,\bullet} \mathfrak{g}^*) \cong H_{\bar{\partial}}^{\bullet,\bullet}(X)$, respectively $H_{BC}^{\bullet,\bullet}(\wedge^{\bullet,\bullet} \mathfrak{g}^*) \cong H_{BC}^{\bullet,\bullet}(X)$, is open in the set of G -left-invariant complex structures on X .*

We recall that, in view of Sakane (1976, Theorem 1), Cordero et al. (2000, main Theorem), Console and Fino (2001, Theorem 2, Remark 4), Rollenske (2009, Theorem 1.10, 2011a, Corollary 3.10), and Angella (2013, Theorem 3.8), the above set contains several classes of left-invariant complex structures, among which holomorphically parallelizable, Abelian, nilpotent, and rational.

¹⁹Console and Fino, 2001, “Dolbeault cohomology of compact nilmanifolds”, Theorem 1.

²⁰Angella, 2013, “The cohomologies of the Iwasawa manifold and of its small deformations”, Theorem 3.9.

3 Applications: solvmanifolds

In order to investigate explicit examples, we recall some results concerning the computations of Dolbeault cohomology for solvmanifolds of two special classes, namely, *solvmanifolds of splitting-type* (that is, satisfying Assumption 1)²¹ and *holomorphically parallelizable solvmanifolds* (that is, with holomorphically-trivial holomorphic tangent bundle)²².

We start by considering solvmanifolds of the following type²³. We call them *solvmanifolds of splitting-type*.

Assumption 1 – Consider a solvmanifold $X = \Gamma \backslash G$ endowed with a G -left-invariant complex structure J . Assume that G is the semi-direct product $\mathbb{C}^n \rtimes_{\phi} N$ so that:

- (C₁) N is a connected simply-connected $2m$ -dimensional nilpotent Lie group endowed with an N -left-invariant complex structure J_N ;
- (C₂) for any $t \in \mathbb{C}^n$, it holds that $\phi(t) \in \text{GL}(N)$ is a holomorphic automorphism of N with respect to J_N ;
- (C₃) ϕ induces a semi-simple action on the Lie algebra \mathfrak{n} associated to N ;
- (C₄) G has a lattice Γ ; (then Γ can be written as $\Gamma = \Gamma_{\mathbb{C}^n} \rtimes_{\phi} \Gamma_N$ such that $\Gamma_{\mathbb{C}^n}$ and Γ_N are lattices of \mathbb{C}^n and, respectively, N , and, for any $t \in \Gamma'$, it holds $\phi(t)(\Gamma_N) \subseteq \Gamma_N$);
- (C₅) the inclusion $\wedge^{\bullet, \bullet}(\mathfrak{n} \otimes_{\mathbb{R}} \mathbb{C})^* \hookrightarrow \wedge^{\bullet, \bullet}(\Gamma_N \backslash N)$ induces the isomorphism

$$H_{\partial}^{\bullet, \bullet}(\wedge^{\bullet, \bullet}(\mathfrak{n} \otimes_{\mathbb{R}} \mathbb{C})^*) \xrightarrow{\cong} H_{\partial}^{\bullet, \bullet}(\Gamma_N \backslash N).$$

Consider the standard basis $\{X_1, \dots, X_n\}$ of \mathbb{C}^n . Consider the decomposition $\mathfrak{n} \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{n}^{1,0} \oplus \mathfrak{n}^{0,1}$ induced by J_N . By the condition condition (C₂), this decomposition is a direct sum of \mathbb{C}^n -modules. By the condition condition (C₃), we have a basis $\{Y_1, \dots, Y_m\}$ of $\mathfrak{n}^{1,0}$ and characters $\alpha_1, \dots, \alpha_m \in \text{Hom}(\mathbb{C}^n; \mathbb{C}^*)$ such that the induced action ϕ on $\mathfrak{n}^{1,0}$ is represented by

$$\mathbb{C}^n \ni t \mapsto \phi(t) = \text{diag}(\alpha_1(t), \dots, \alpha_m(t)) \in \text{GL}(\mathfrak{n}^{1,0}).$$

For any $j \in \{1, \dots, m\}$, since Y_j is an N -left-invariant $(1, 0)$ -vector field on N , the $(1, 0)$ -vector field $\alpha_j Y_j$ on $\mathbb{C}^n \rtimes_{\phi} N$ is $(\mathbb{C}^n \rtimes_{\phi} N)$ -left-invariant. Consider the Lie algebra \mathfrak{g} of G and the decomposition $\mathfrak{g}_{\mathbb{C}} := \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{g}^{1,0} \oplus \mathfrak{g}^{0,1}$ induced by J . Hence we have

²¹Kasuya, 2013a, “Minimal models, formality, and hard Lefschetz properties of solvmanifolds with local systems”.

²²Wang, 1954, “Complex parallisable manifolds”.

²³See Kasuya, 2013a, “Minimal models, formality, and hard Lefschetz properties of solvmanifolds with local systems”.

a basis $\{X_1, \dots, X_n, \alpha_1 Y_1, \dots, \alpha_m Y_m\}$ of $\mathfrak{g}^{1,0}$, and let $\{x_1, \dots, x_n, \alpha_1^{-1} y_1, \dots, \alpha_m^{-1} y_m\}$ be its dual basis of $\wedge^{1,0} \mathfrak{g}_{\mathbb{C}}^*$. Then we have

$$\wedge^{p,q} \mathfrak{g}_{\mathbb{C}}^* = \wedge^p \langle x_1, \dots, x_n, \alpha_1^{-1} y_1, \dots, \alpha_m^{-1} y_m \rangle \otimes \wedge^q \langle \bar{x}_1, \dots, \bar{x}_n, \bar{\alpha}_1^{-1} \bar{y}_1, \dots, \bar{\alpha}_m^{-1} \bar{y}_m \rangle.$$

The following lemma holds.

Lemma 1 (Kasuya²⁴) – *Let $X = \Gamma \backslash G$ be a solvmanifold endowed with a G -left-invariant complex structure J as in Assumption 1 on the previous page. With the above notations, for any $j \in \{1, \dots, m\}$, there exist unique unitary characters $\beta_j \in \text{Hom}(\mathbb{C}^n; \mathbb{C}^*)$ and $\gamma_j \in \text{Hom}(\mathbb{C}^n; \mathbb{C}^*)$ on \mathbb{C}^n such that $\alpha_j \beta_j^{-1}$ and $\bar{\alpha}_j \gamma_j^{-1}$ are holomorphic.*

Hence, we define the differential bi-graded sub-algebra $B_{\Gamma}^{\bullet, \bullet} \subset \wedge^{\bullet, \bullet} \Gamma \backslash G$, for $(p, q) \in \mathbb{Z}^2$, as

$$\begin{aligned} B_{\Gamma}^{p,q} := & \mathbb{C} \langle x_I \wedge (\alpha_J^{-1} \beta_J) y_J \wedge \bar{x}_K \wedge (\bar{\alpha}_L^{-1} \gamma_L) \bar{y}_L \mid |I| + |J| = p \text{ and } |K| + |L| = q \\ & \text{such that } (\beta_J \gamma_L) \Big|_{\Gamma} = 1 \rangle \end{aligned} \quad (1)$$

(where we shorten, e.g., $\alpha_I := \alpha_{i_1} \dots \alpha_{i_k}$ and $x_I := x_{i_1} \wedge \dots \wedge x_{i_k}$ for a multi-index $I = (i_1, \dots, i_k)$ of length $|I| = k$).

We recall the following result by the second author.

Theorem 3 (Kasuya²⁵) – *Let $X = \Gamma \backslash G$ be a solvmanifold endowed with a G -left-invariant complex structure J as in Assumption 1 on the previous page. Consider the differential bi-graded sub-algebra $B_{\Gamma}^{\bullet, \bullet} \subset \wedge^{\bullet, \bullet} \Gamma \backslash G$ defined in Equation (1). Then the inclusion $B_{\Gamma}^{\bullet, \bullet} \subset \wedge^{\bullet, \bullet} \Gamma \backslash G$ induces the cohomology isomorphism*

$$H_{\bar{\partial}}^{\bullet, \bullet} (B_{\Gamma}^{\bullet, \bullet}) \xrightarrow{\cong} H_{\bar{\partial}}^{\bullet, \bullet} (\Gamma \backslash G).$$

As regards the Bott-Chern cohomology, define $\bar{B}_{\Gamma}^{\bullet, \bullet} := \{\bar{\omega} \in \wedge^{\bullet, \bullet} \Gamma \backslash G \mid \omega \in B_{\Gamma}^{\bullet, \bullet}\}$ and

$$C_{\Gamma}^{\bullet, \bullet} := B_{\Gamma}^{\bullet, \bullet} + \bar{B}_{\Gamma}^{\bullet, \bullet}. \quad (2)$$

The authors proved the following result.

Theorem 4 (Angella and Kasuya²⁶) – *Let $\Gamma \backslash G$ be a solvmanifold endowed with a G -left-invariant complex structure J as in Assumption 1 on the previous page. Consider $C_{\Gamma}^{\bullet, \bullet}$ as in Equation (2). Then the inclusion $C_{\Gamma}^{\bullet, \bullet} \subset \wedge^{\bullet, \bullet} \Gamma \backslash G$ induces the isomorphisms*

$$H_{\bar{\partial}}^{\bullet, \bullet} (C_{\Gamma}^{\bullet, \bullet}) \xrightarrow{\cong} H_{\bar{\partial}}^{\bullet, \bullet} (\Gamma \backslash G) \quad \text{and} \quad H_{BC}^{\bullet, \bullet} (C_{\Gamma}^{\bullet, \bullet}) \xrightarrow{\cong} H_{BC}^{\bullet, \bullet} (\Gamma \backslash G).$$

²⁴Kasuya, 2013a, “Minimal models, formality, and hard Lefschetz properties of solvmanifolds with local systems”, Lemma 2.2.

²⁵Ibid., Corollary 4.2.

3. Applications: solvmanifolds

Another class of “cohomologically-computable” solvmanifolds is given by *holomorphically parallelizable solvmanifolds*, namely, compact quotients of connected simply-connected complex solvable Lie groups by co-compact discrete subgroups²⁷.

Let G be a connected simply-connected complex solvable Lie group admitting a lattice Γ , and denote by $2n$ the real dimension of G . Denote the Lie algebra naturally associated to G by \mathfrak{g} .

Denote by \mathfrak{g}_+ (respectively, \mathfrak{g}_-) the Lie algebra of the G -left-invariant holomorphic (respectively, anti-holomorphic) vector fields on G . As a (real) Lie algebra, we have an isomorphism $\mathfrak{g}_+ \cong \mathfrak{g}_-$ by means of the complex conjugation.

Let N be the nilradical of G . We can take a connected simply-connected complex nilpotent subgroup $C \subseteq G$ such that $G = C \cdot N$ ²⁸. Since C is nilpotent, the map

$$C \ni c \mapsto (\text{Ad}_c)_s \in \text{Aut}(\mathfrak{g}_+)$$

is a homomorphism, where $(\text{Ad}_c)_s$ is the semi-simple part of the Jordan decomposition of Ad_c . We have a basis $\{X_1, \dots, X_n\}$ of \mathfrak{g}_+ such that, for $c \in C$,

$$(\text{Ad}_c)_s = \text{diag}(\alpha_1(c), \dots, \alpha_n(c))$$

for some characters $\alpha_1, \dots, \alpha_n$ of C . By $G = C \cdot N$, we have $G/N = C/C \cap N$ and regard $\alpha_1, \dots, \alpha_n$ as characters of G . Let $\{x_1, \dots, x_n\}$ be the basis of \mathfrak{g}_+^* which is dual to $\{X_1, \dots, X_n\}$.

Let B_Γ^\bullet be the sub-complex of $(\wedge^{0,\bullet} \Gamma \backslash G, \bar{\partial})$ defined as

$$B_\Gamma^\bullet = \left\langle \frac{\bar{\alpha}_I}{\alpha_I} \bar{x}_I \mid I \subseteq \{1, \dots, n\} \text{ such that } \left(\frac{\bar{\alpha}_I}{\alpha_I} \right)_{\Gamma} = 1 \right\rangle \quad (3)$$

(where we shorten, e.g., $\alpha_I := \alpha_{i_1} \cdots \alpha_{i_k}$ and $x_I := x_{i_1} \wedge \cdots \wedge x_{i_k}$ for a multi-index $I = (i_1, \dots, i_k)$ of length $|I| = k$).

The second author proved the following result.

Theorem 5 (Kasuya²⁹) – *Let G be a connected simply-connected complex solvable Lie group admitting a lattice Γ . Consider the finite-dimensional sub-complex $B_\Gamma^\bullet \subset (\wedge^{0,\bullet} \Gamma \backslash G, \bar{\partial})$ defined in Equation (3). Then the inclusion $B_\Gamma^\bullet \hookrightarrow \wedge^{0,\bullet} \Gamma \backslash G$ induces the cohomology isomorphism*

$$H^\bullet(B_\Gamma^\bullet, \bar{\partial}) \xrightarrow{\cong} H_\bar{\partial}^{0,\bullet}(\Gamma \backslash G).$$

²⁶Angella and Kasuya, 2017, “Bott-Chern cohomology of solvmanifolds”, Theorem 2.16.

²⁷Wang, 1954, “Complex parallelisable manifolds”;

see also Nakamura, 1975, “Complex parallelisable manifolds and their small deformations”.

²⁸See, e.g., Dekimpe, 2000, “Semi-simple splittings for solvable Lie groups and polynomial structures”, Proposition 3.3.

As regards Bott-Chern cohomology, define

$$\bar{B}_\Gamma^\bullet := \left\langle \frac{\alpha_I}{\bar{\alpha}_I} x_I \mid I \subseteq \{1, \dots, n\} \text{ such that } \left(\frac{\alpha_I}{\bar{\alpha}_I} \right) \Big|_\Gamma = 1 \right\rangle,$$

and

$$C_\Gamma^{\bullet, \bullet, 2} := \wedge^{\bullet, 1} \mathfrak{g}_+^* \otimes B_\Gamma^{\bullet, 2} + \bar{B}_\Gamma^{\bullet, 1} \otimes \wedge^{\bullet, 2} \mathfrak{g}_-^*. \quad (4)$$

The authors proved the following result.

Theorem 6 (Angella and Kasuya³⁰) – *Let G be a connected simply-connected complex solvable Lie group admitting a lattice Γ . Consider the finite-dimensional sub-double-complex $C_\Gamma^{\bullet, \bullet} \subset \wedge^{\bullet, \bullet} \Gamma \backslash G$ defined in Equation (4). Then the inclusion $C_\Gamma^{\bullet, \bullet} \hookrightarrow \wedge^{\bullet, \bullet} \Gamma \backslash G$ induces the cohomology isomorphism*

$$H_{BC}^{\bullet, \bullet}(C_\Gamma^{\bullet, \bullet}) \xrightarrow{\cong} H_{BC}^{\bullet, \bullet}(\Gamma \backslash G).$$

Therefore, by Theorems 1 and 2 on p. 76 and on p. 77, we get the following result, for which we provide explicit applications in the following.

Corollary 2 – *Let X be either a solvmanifold of splitting-type or a holomorphically parallelizable solvmanifold. Then the Dolbeault cohomology and the Bott-Chern cohomology both of X and of some suitable small deformations of X are computable by means of a finite-dimensional sub-double-complex of $(\wedge^{\bullet, \bullet} X, \partial, \bar{\partial})$.*

We note that small deformations of a holomorphically parallelizable solvmanifolds does not necessarily remain holomorphically parallelizable. This was firstly proved by Nakamura³¹, providing explicit examples on the Iwasawa manifold. Rollenske studied conditions for which a small deformation of a holomorphically parallelizable nilmanifold is still holomorphically parallelizable³², proving that non-tori holomorphically parallelizable nilmanifolds admit non-holomorphically parallelizable small deformations³³. We prove that the same holds true for holomorphically parallelizable solvmanifolds.

Theorem 7 – *Let $X = \Gamma \backslash G$ be a holomorphically parallelizable solvmanifold which is not a torus. Then there exists a non-holomorphically parallelizable small deformation of $\Gamma \backslash G$.*

²⁹Kasuya, 2014a, “de Rham and Dolbeault cohomology of solvmanifolds with local systems”, Corollary 6.2 and its proof.

³⁰Angella and Kasuya, 2017, “Bott-Chern cohomology of solvmanifolds”, Theorem 2.24.

³¹Nakamura, 1975, “Complex parallelisable manifolds and their small deformations”, pp. 86, 96.

³²Rollenske, 2011b, “The Kuranishi space of complex parallelisable nilmanifolds”, Theorem 5.1.

³³Ibid., Corollary 5.2.

3. Applications: solvmanifolds

Proof. By Rollenske (2011b, Corollary 5.2), we can assume that $\Gamma \backslash G$ is not a nilmanifold. Take a connected simply-connected complex nilpotent subgroup $C \subset G$ such that $G = C \cdot N$, where N is the nilradical of G . We can take a 1-dimensional complex Lie subgroup $A \cong \mathbb{C}$ with $A \subset C$ and a 1-codimensional complex Lie subgroup G' with $N \subset G'$ such that we have decomposition $G = A \ltimes G'$. Take a basis $\{x_1, \dots, x_n\}$ of \mathfrak{g}_+^* which diagonalizes the semi-simple part of the C -action (where \mathfrak{g}_+ denotes the Lie algebra of G -left-invariant holomorphic vector fields on G). With respect to the above decomposition, we can take $x_1 = dz$ for a coordinate z of the 1-dimensional complex Lie subgroup A , and $x_2 = e^{a_2 z} x'_2$ for a non-trivial character $e^{a_2 z}$ of A and a holomorphic form x'_j on G' , by trigonalizing the A -action. Then the Dolbeault cohomology of $\Gamma \backslash G$ is computed by means of

$$C_0^{\bullet, \bullet} := \wedge^{\bullet} \mathfrak{g}_+^* \otimes B_{\Gamma}^{\bullet}$$

where

$$B_{\Gamma}^{\bullet} := \left\langle \frac{\bar{\alpha}_I}{\alpha_I} \bar{x}_I \mid I \subseteq \{1, \dots, n\} \text{ such that } \left(\frac{\bar{\alpha}_I}{\alpha_I} \right) \Big|_{\Gamma} = 1 \right\rangle,$$

(and where we shorten, e.g., $\alpha_I := \alpha_{i_1} \cdots \alpha_{i_k}$ and $x_I := x_{i_1} \wedge \cdots \wedge x_{i_k}$ for a multi-index $I = (i_1, \dots, i_k)$ of length $|I| = k$).

We consider the family $\{J_t\}_t$ of deformations given by

$$t \frac{\partial}{\partial z} \otimes d\bar{z} \in H^{0,1}(X; T^{1,0}X).$$

Then, for any t , we consider the double-complex

$$D_t^{\bullet, \bullet} := \wedge^{\bullet} \mathfrak{g}_+^*(t) \otimes B_{\Gamma}^{\bullet}(t)$$

so that

$$\wedge^{\bullet} \mathfrak{g}_+^*(t) = \wedge^{\bullet} \langle dz - t d\bar{z}, x_2, \dots, x_n \rangle$$

and

$$B_{\Gamma}^{\bullet}(t) = \wedge \langle d\bar{z} - \bar{t} dz \rangle \otimes \left\langle \frac{\bar{\alpha}_I}{\alpha_I} \bar{x}'_I \mid I \subseteq \{2, \dots, n\} \text{ such that } \left(\frac{\bar{\alpha}_I}{\alpha_I} \right) \Big|_{\Gamma} = 1 \right\rangle,$$

and the J_t -Hermitian metrics

$$g_t := (dz - t d\bar{z}) \odot (d\bar{z} - \bar{t} dz) + \sum_{j=2}^n x_j \odot \bar{x}_j.$$

We can apply Theorem 1 on p. 76.

Now we have

$$\bar{\partial}_t(e^{a_2 z} x'_2) = \frac{a_2 t (d\bar{z} - \bar{t} dz)}{1 - |t|^2} e^{a_2 z} x'_2.$$

Hence we have, for $t \neq 0$,

$$H_{\bar{\partial}_t}^{1,0}(\Gamma \backslash G) = \ker \bar{\partial}_t|_{\wedge^1 \mathfrak{g}_+^*(t)} \neq \wedge^1 \mathfrak{g}_+^*(t).$$

By this, for $t \neq 0$, we have $\dim_{\mathbb{C}} H_{\bar{\partial}_t}^{1,0}(\Gamma \backslash G) < \dim_{\mathbb{C}} G$ and hence $(\Gamma \backslash G, J_t)$ is not holomorphically parallelizable. \square

4 Example: deformations of the Nakamura manifold

Consider the Lie group $G = \mathbb{C} \rtimes_{\phi} \mathbb{C}^2$ where

$$\phi(z) = \begin{pmatrix} e^z & 0 \\ 0 & e^{-z} \end{pmatrix}.$$

Then there exists a lattice $\Gamma = (a\mathbb{Z} + 2\pi\mathbb{Z}) \rtimes \Gamma''$ where Γ'' is a lattice in \mathbb{C}^2 . The solvmanifold $X := \Gamma \backslash G$ is called (*holomorphically parallelizable*) *Nakamura manifold*³⁴.

In order to compute the Dolbeault, respectively Bott-Chern cohomologies of the Nakamura manifold, consider the sub-double-complexes $B_{\Gamma}^{\bullet,\bullet}$ and $C_{\Gamma}^{\bullet,\bullet}$ given in Tables 1 and 2 on p. 90 and on p. 91³⁵. (For the sake of simplicity, we shorten, e.g., $dz_{2\bar{3}} := dz_2 \wedge d\bar{z}_3$, where z_1 is the holomorphic coordinate on \mathbb{C} and $\{z_2, z_3\}$ is the set of holomorphic coordinates on \mathbb{C}^2 .)

Then, by Kasuya (2014a, Corollary 6.2) and by Angella and Kasuya (2017, Theorem 2.24), the inclusions $B_{\Gamma}^{\bullet,\bullet} \subset \wedge^{\bullet,\bullet}(X)$ and $C_{\Gamma}^{\bullet,\bullet} \subset \wedge^{\bullet,\bullet}(X)$ induce isomorphisms

$$H_{\bar{\partial}}^{\bullet,\bullet}(B_{\Gamma}^{\bullet,\bullet}) \cong H_{\bar{\partial}}^{\bullet,\bullet}(X) \quad \text{and} \quad H_{BC}^{\bullet,\bullet}(C_{\Gamma}^{\bullet,\bullet}) \cong H_{BC}^{\bullet,\bullet}(X)$$

We consider deformations $\{J_t\}_{t \in B}$ over a ball $B \subset \mathbb{C}$ given by

$$(C_1) \quad t \frac{\partial}{\partial z_1} \otimes d\bar{z}_1 \in H^{0,1}(X; T^{1,0}X), \text{ or}$$

$$(C_2) \quad t \frac{\partial}{\partial z_1} \otimes e^{z_1} d\bar{z}_3 \in H^{0,1}(X; T^{1,0}X).$$

As for deformations in case (C_1) , we can compute the Dolbeault and Bott-Chern cohomologies by applying Theorems 1 and 2 on p. 76 and on p. 77 to the complexes $B_{\Gamma}^{\bullet,\bullet}(t)$ and $C_{\Gamma}^{\bullet,\bullet}(t)$ in Tables 5 and 6 on p. 94 and on p. 95, respectively, and by considering the J_t -Hermitian metrics $g_t := \phi_1^{1,0}(t) \odot \phi_1^{0,1}(t) + \phi_2^{1,0}(t) \odot \phi_2^{0,1}(t) + \phi_3^{1,0}(t) \odot \phi_3^{0,1}(t)$; the generators of the complexes are defined starting from the forms in

³⁴Nakamura, 1975, "Complex parallelisable manifolds and their small deformations".

³⁵See Angella and Kasuya, 2017, "Bott-Chern cohomology of solvmanifolds";

Kasuya, 2013a, "Minimal models, formality, and hard Lefschetz properties of solvmanifolds with local systems".

5. Example: Sawai and Yamada generalized manifolds

Table 3 on p. 92, and we summarize the results of the computations of the Dolbeault and Bott-Chern cohomologies in Tables 7 and 8 on p. 96 and on p. 97, respectively.

As for deformations in case (C_2) on the preceding page, we can compute the Dolbeault cohomology by applying Theorem 1 on p. 76 to the complex $B_{\Gamma}^{\bullet, \bullet}(t)$ in Table 5 on p. 94, and by considering the J_t -Hermitian metrics $g_t := \phi_1^{1,0}(t) \odot \phi_1^{0,1}(t) + \phi_2^{1,0}(t) \odot \phi_2^{0,1}(t) + \phi_3^{1,0}(t) \odot \phi_3^{0,1}(t)$; the generators of the complex are defined starting from the forms in Table 4 on p. 93, and we summarize the results of the computation of the Dolbeault cohomology in Table 9 on p. 98. (As regards the Bott-Chern cohomology for deformations in case (C_2) on the preceding page, the vector space $C_{\Gamma}^{\bullet, \bullet}(t)$ does not provide a sub-double-complex for $t \neq 0$, and, by modifying it in order to be closed for both ∂_t and $\bar{\partial}_t$, and $\bar{\ast}_{g_t}$, as required in Theorem 2 on p. 77, it seems that the finite-dimensionality is no more guaranteed.)

Remark 1 – Hasegawa (2010, Theorem 4) showed that deformations in case (C_2) on the preceding page are not left-invariant. Hence our method is effective for computing the Dolbeault cohomology of non-left-invariant complex structures.

(As a matter of notations, we shorten, e.g., $\phi_1^{1,0}(t) \wedge \phi_{12}^{0,1}(t) := \phi_1^{1,0}(t) \wedge \phi_1^{0,1}(t) \wedge \phi_2^{0,1}(t)$.)

Straightforwardly (e.g., from Table 10 on p. 99 and by Angella and Tomassini (2013, Theorem B)), we get the following result.³⁶

Proposition 1 – Consider the holomorphically parallelizable Nakamura manifold (X, J_0) , and its small deformations $\{J_t\}_{t \in B}$ as in case (C_1) or case (C_2) on the preceding page. Then

1. the deformations (X, J_t) as in case (C_1) satisfy the $\partial\bar{\partial}$ -Lemma.
2. the deformations (X, J_t) as in case (C_2) satisfy the E_1 -degeneration of the Hodge and Frölicher spectral sequences, but do not satisfy the $\partial\bar{\partial}$ -Lemma.

5 Example: Sawai and Yamada generalized manifolds

In this section, we study the cohomology of the generalized examples introduced and studied by Sawai and Yamada³⁷ in order to generalize Benson and Gordon manifold³⁸.

Following Sawai and Yamada (2005), let \mathfrak{n} be a complex nilpotent Lie algebra. We assume that

$$\mathfrak{n} = \mathbb{C}\langle Y_1, \dots, Y_\ell, Y_{\ell+1}, \dots, Y_m \rangle$$

³⁶See Kasuya, 2014b, “Hodge symmetry and decomposition on non-Kähler solvmanifolds”, for other examples of non-Kähler solvmanifolds satisfying the $\partial\bar{\partial}$ -Lemma.

³⁷Sawai and Yamada, 2005, “Lattices on Benson-Gordon type solvable Lie groups”.

³⁸Benson and Gordon, 1990, “Kähler structures on compact solvmanifolds”.

so that $[\mathfrak{n}, \mathfrak{n}] = \mathbb{C}\langle Y_{\ell+1}, \dots, Y_m \rangle$ and $[Y_i, Y_j] = C_{ij}^k Y_k$ for some $C_{ij}^k \in \mathbb{Z}$, varying $i, j, k \in \{1, \dots, m\}$. Define

$$\tilde{\mathfrak{n}} := \mathbb{C}\langle Y_{1,1}, \dots, Y_{1,\ell}, Y_{1,\ell+1}, Y_{1,m} \rangle \oplus \mathbb{C}\langle Y_{2,1}, \dots, Y_{2,\ell}, Y_{2,\ell+1}, Y_{2,m} \rangle$$

where $\mathbb{C}\langle Y_{1,1}, \dots, Y_{1,\ell}, Y_{1,\ell+1}, \dots, Y_{1,m} \rangle \cong \mathbb{C}\langle Y_{2,1}, \dots, Y_{2,\ell}, Y_{2,\ell+1}, \dots, Y_{2,m} \rangle \cong \mathfrak{n}$. Consider the semi-direct product $\mathfrak{g} := \mathbb{C}\langle X \rangle \ltimes \tilde{\mathfrak{n}}$ given by

$$[X, Y_{1,j}] := k_j Y_{1,j}, \quad [X, Y_{2,j}] := -k_j Y_{2,j}$$

where $\{k_j\}_j \subset \mathbb{N} \setminus \{0\}$ is such that the Jacobi identity holds.

Let $G = \mathbb{C} \ltimes \tilde{N}$ be the connected simply-connected complex Lie group corresponding to \mathfrak{g} . Then we have

$$G = \left\{ \left(z, \begin{pmatrix} w_{1,1} \\ w_{2,1} \end{pmatrix}, \dots, \begin{pmatrix} w_{1,m} \\ w_{2,m} \end{pmatrix} \right) : z, w_{1,j}, w_{2,j} \in \mathbb{C} \right\}$$

with the product

$$\begin{aligned} & \left(z, \begin{pmatrix} w_{1,1} \\ w_{2,1} \end{pmatrix}, \dots, \begin{pmatrix} w_{1,m} \\ w_{2,m} \end{pmatrix} \right) \cdot \left(z', \begin{pmatrix} w'_{1,1} \\ w'_{2,1} \end{pmatrix}, \dots, \begin{pmatrix} w'_{1,m} \\ w'_{2,m} \end{pmatrix} \right) \\ &= \left(z + z', \begin{pmatrix} f_{1,1}(z, w_{1,1}, \dots, w_{1,m}, w'_{1,1}, \dots, w'_{1,m}) \\ f_{2,1}(z, w_{2,1}, \dots, w_{2,m}, w'_{2,1}, \dots, w'_{2,m}) \end{pmatrix}, \dots, \right. \\ & \quad \left. \begin{pmatrix} f_{1,m}(z, w_{1,1}, \dots, w_{1,m}, w'_{1,1}, \dots, w'_{1,m}) \\ f_{2,m}(z, w_{2,1}, \dots, w_{2,m}, w'_{2,1}, \dots, w'_{2,m}) \end{pmatrix} \right), \end{aligned}$$

for certain functions $f_{1,1}, \dots, f_{1,m}, f_{2,1}, \dots, f_{2,m}$ ³⁹.

Take a unimodular matrix $B \in \mathrm{SL}(2, \mathbb{Z})$ with distinct positive eigenvalues λ and λ^{-1} , and set $a := \ln \lambda$. Consider

$$\begin{aligned} \Gamma := & \left\{ \left(a s + 2\pi\sqrt{-1}t, \begin{pmatrix} w_{1,1} + \lambda w_{2,1} \\ w_{1,1} + \lambda^{-1} w_{2,1} \end{pmatrix}, \dots, \begin{pmatrix} w_{1,m} + \lambda w_{2,m} \\ w_{1,m} + \lambda^{-1} w_{2,m} \end{pmatrix} \right) \right. \\ & \left. : s, t \in \mathbb{Z}, w_{1,j}, w_{2,j} \in \mathbb{Z} + \sqrt{-1}\mathbb{Z} \right\}. \end{aligned}$$

Then, as proved by Sawai and Yamada (2005, Theorem 2.1), Γ is a lattice in G . Hence we have $\Gamma = (a\mathbb{Z} + 2\pi\sqrt{-1}\mathbb{Z}) \ltimes \Gamma''$ such that Γ'' is a lattice in \tilde{N} .

Let $\{y_{1,1}, \dots, y_{1,\ell}, y_{1,\ell+1}, \dots, y_{1,m}, y_{2,1}, \dots, y_{2,\ell}, y_{2,\ell+1}, \dots, y_{2,m}\}$ be the dual basis of the space $(\tilde{\mathfrak{n}}^{1,0})^*$ of the left-invariant $(1,0)$ -forms on \tilde{N} . Then, by the assumption, we

³⁹See Sawai and Yamada, 2005, "Lattices on Benson-Gordon type solvable Lie groups", Section 2.

5. Example: Sawai and Yamada generalized manifolds

have $dy_{1,j} = dy_{2,j} = 0$ for $1 \leq j \leq \ell$. The space $(\mathfrak{g}^{1,0})^*$ of the left-invariant $(1,0)$ -forms on G is given by

$$(\mathfrak{g}^{1,0})^* = \mathbb{C} \langle dz, e^{-k_1 z} y_{1,1}, \dots, e^{-k_m z} y_{1,m}, e^{k_1 z} y_{2,1}, \dots, e^{k_m z} y_{2,m} \rangle.$$

Consider

$$B_{\Gamma}^{\bullet,\bullet} := \wedge^{\bullet,\bullet} \mathbb{C} \langle dz, e^{-k_1 z} y_{1,1}, \dots, e^{-k_m z} y_{1,m}, e^{k_1 z} y_{2,1}, \dots, e^{k_m z} y_{2,m} \rangle \\ \otimes \mathbb{C} \langle d\bar{z}, e^{-k_1 z} \bar{y}_{1,1}, \dots, e^{-k_m z} \bar{y}_{1,m}, e^{k_1 z} \bar{y}_{2,1}, \dots, e^{k_m z} \bar{y}_{2,m} \rangle.$$

Then we have

$$H_{\bar{\partial}}^{\bullet,\bullet}(B_{\Gamma}^{\bullet,\bullet}) \cong H_{\bar{\partial}}^{\bullet,\bullet}(\Gamma \backslash G).$$

We consider deformations $\{J_t\}_{t \in B}$ over a ball $B \subset \mathbb{C}$ given by:

$$t \frac{\partial}{\partial z} \otimes e^{k_1 z} \bar{y}_{2,1} \in H^{0,1}(\Gamma \backslash G; T^{1,0} \Gamma \backslash G).$$

To compute the Dolbeault cohomology of $(\Gamma \backslash G, J_t)$, consider the forms defined in Table 11 on p. 100.

More precisely, by applying Theorem 1 on p. 76 to the double-complex

$$B_{\Gamma}^{\bullet,\bullet}(t) = \wedge^{\bullet,\bullet} \mathbb{C} \langle \phi_0^{1,0}(t), \phi_{1,1}^{1,0}(t), \dots, \phi_{1,m}^{1,0}(t), \phi_{2,1}^{1,0}(t), \dots, \phi_{2,m}^{1,0}(t) \rangle \\ \otimes \mathbb{C} \langle \phi_0^{0,1}(t), \phi_{1,1}^{0,1}(t), \dots, \phi_{1,m}^{0,1}(t), \phi_{2,1}^{0,1}(t), \dots, \phi_{2,m}^{0,1}(t) \rangle \quad (5)$$

and to the J_t -Hermitian metric

$$g_t := \phi_0^{1,0}(t) \otimes \phi_0^{0,1}(t) + \sum_{j=1}^m \phi_{1,j}^{1,0}(t) \otimes \phi_{1,j}^{0,1}(t) + \sum_{j=1}^m \phi_{2,j}^{1,0}(t) \otimes \phi_{2,j}^{0,1}(t),$$

since $(B_{\Gamma}^{\bullet,\bullet}(t), \bar{\partial}_t)$ is a sub-complex of $(\wedge^{\bullet,\bullet}(\Gamma \backslash G), \bar{\partial}_t)$ and $\bar{*}_t(B_{\Gamma}^{\bullet,\bullet}(t)) \subseteq B_{\Gamma}^{2m+1-\bullet, 2m+1-\bullet}(t)$, then we have

$$H_{\bar{\partial}_t}^{\bullet,\bullet}(B_{\Gamma}^{\bullet,\bullet}(t)) \cong H_{\bar{\partial}_t}^{\bullet,\bullet}(\Gamma \backslash G).$$

By simple computations we have the following result.

Proposition 2 – Consider the Sawai and Yamada generalized manifold $X = \Gamma \backslash G$ of complex dimension $2m + 1$, and its small deformations $\{J_t\}_{t \in B \subset \mathbb{C}}$ induced by $t \frac{\partial}{\partial z} \otimes e^{k_1 z} \bar{y}_{2,1} \in H^{0,1}(X; T^{1,0} X)$. Then

$$\dim H_{\bar{\partial}_t}^{1,0}(X) = 0 \quad \text{and} \quad \dim H_{\bar{\partial}_t}^{2m+1,0}(X) = 0.$$

Remark 2 – In Console, Fino, and Kasuya (2016) and Kasuya (2013a, 2014a), structures of holomorphic fibre bundles over complex tori with nilmanifold-fibres play a very important role for computing the Dolbeault cohomology of certain solvmanifolds. But, by Proposition 2 on the previous page, such deformed complex solvmanifolds are not holomorphic fibre bundles over complex tori. Hence they provide new examples of “Dolbeault-cohomologically-computable” complex solvmanifolds.

6 Closedness and openness under holomorphic deformation

We recall that a property \mathcal{P} concerning complex manifolds is called *open* under holomorphic deformations if, whenever it holds for a compact complex manifold X , it holds also for any small deformations of X . It is called (*Zariski*-)closed (simply, *closed*) if, for any family $\{X_t\}_{t \in \Delta}$ of compact complex manifolds such that \mathcal{P} holds for any $t \in \Delta \setminus \{0\}$ in the punctured-disk, then \mathcal{P} holds also for X_0 .

It is known that the $\partial\bar{\partial}$ -Lemma is open under holomorphic deformations⁴⁰. Angella and Kasuya (2017, Theorem 2.20) proved that the $\partial\bar{\partial}$ -Lemma is not *strongly-closed* under holomorphic deformations, namely, there exists a family $\{X_t\}_{t \in \Delta}$ of compact complex manifolds and a sequence $\{t_k\}_{k \in \mathbb{N}} \subset \Delta$ converging to $0 \in \Delta$ in the Euclidean topology of Δ such that X_{t_k} satisfies the $\partial\bar{\partial}$ -Lemma and X_0 does not; more precisely, in Angella and Kasuya (2017, Example 2.17), X_0 is the completely-solvable Nakamura manifold.

We prove now that the $\partial\bar{\partial}$ -Lemma is also non-(Zariski-)closed. Indeed, consider the holomorphically parallelizable Nakamura manifold $\Gamma \backslash G$ and its small deformations as in Section 4 on p. 84. While $\Gamma \backslash G$ does not satisfy the E_1 -degeneration of the Hodge and Frölicher spectral sequences, deformations as in cases (C_1) and (C_2) on p. 84 do. While $\Gamma \backslash G$ does not satisfy the $\partial\bar{\partial}$ -Lemma, deformations as in case (C_1) on p. 84 do. Hence we get the following result.

Corollary 3 – *The properties of E_1 -degeneration of the Hodge and Frölicher spectral sequences and the $\partial\bar{\partial}$ -Lemma are not closed under holomorphic deformations.*

The non-closedness of the property of E_1 -degeneration of the Hodge and Frölicher spectral sequences was firstly proven by Eastwood and Singer⁴¹, by considering twistor spaces.

⁴⁰See, e.g. Voisin, 2002, *Hodge Theory and Complex Algebraic Geometry I*, Proposition 9.21;

or Wu, 2006, “On the geometry of superstrings with torsion”, Theorem 5.12;

or Tomasiello, 2008, “Reformulating supersymmetry with a generalized Dolbeault operator”, § B;

or Angella and Tomassini, 2013, “On the $\partial\bar{\partial}$ -lemma and Bott-Chern cohomology”, Corollary 2.7.

⁴¹Eastwood and Singer, 1993, “The Frölicher spectral sequence on a twistor space”, Theorem 5.4.

6. Closedness and openness under holomorphic deformation

Remark 3 – Note that the small deformations $(\Gamma \backslash G, J_t)$ as in case (C_1) on p. 84 of the holomorphically parallelizable Nakamura manifold $\Gamma \backslash G$ provide examples of compact complex manifolds that are not in *Fujiki class* \mathcal{C} ⁴² but satisfy the $\partial\bar{\partial}$ -Lemma. This follows from Chiose (2014, Theorem 2.3). See also Arapura (2004, Theorem 9), or Angella and Kasuya (2014, Theorem 3.3).

This is in accord with the conjectures that the property of being Moishezon is closed under holomorphic deformations⁴³, and that the Fujiki class \mathcal{C} is closed under holomorphic deformations⁴⁴.

⁴²Fujiki, 1978, “On automorphism groups of compact Kähler manifolds”.

⁴³See Popovici, 2010, “Limits of Moishezon Manifolds under Holomorphic Deformations”.

⁴⁴Popovici, 2014, “Deformation openness and closedness of various classes of compact complex manifolds; examples”, standard Conjecture 1.17;

compare also Popovici, 2013, “Deformation limits of projective manifolds: Hodge numbers and strongly Gauduchon metrics”, Question 1.5.

	$B_{\Gamma}^{\bullet, \bullet}$
(0, 0)	$\mathbb{C}\langle 1 \rangle$
(1, 0)	$\mathbb{C}\langle dz_1, e^{-z_1} dz_2, e^{z_1} dz_3 \rangle$
(0, 1)	$\mathbb{C}\langle dz_{\bar{1}}, e^{-z_1} dz_{\bar{2}}, e^{z_1} dz_{\bar{3}} \rangle$
(2, 0)	$\mathbb{C}\langle e^{-z_1} dz_{12}, e^{z_1} dz_{13}, dz_{23} \rangle$
(1, 1)	$\mathbb{C}\langle dz_{1\bar{1}}, e^{-z_1} dz_{1\bar{2}}, e^{z_1} dz_{1\bar{3}}, e^{-z_1} dz_{2\bar{1}}, e^{-2z_1} dz_{2\bar{2}}, dz_{2\bar{3}}, e^{z_1} dz_{3\bar{1}}, dz_{3\bar{2}}, e^{2z_1} dz_{3\bar{3}} \rangle$
(0, 2)	$\mathbb{C}\langle e^{-z_1} dz_{\bar{1}\bar{2}}, e^{z_1} dz_{\bar{1}\bar{3}}, dz_{\bar{2}\bar{3}} \rangle$
(3, 0)	$\mathbb{C}\langle dz_{123} \rangle$
(2, 1)	$\mathbb{C}\langle e^{-z_1} dz_{12\bar{1}}, e^{-2z_1} dz_{12\bar{2}}, dz_{12\bar{3}}, e^{z_1} dz_{13\bar{1}}, dz_{13\bar{2}}, e^{2z_1} dz_{13\bar{3}}, dz_{23\bar{1}}, e^{-z_1} dz_{23\bar{2}}, e^{z_1} dz_{23\bar{3}} \rangle$
(1, 2)	$\mathbb{C}\langle dz_{3\bar{1}\bar{2}}, dz_{2\bar{1}\bar{3}}, dz_{1\bar{2}\bar{3}}, e^{-z_1} dz_{1\bar{1}\bar{2}}, e^{z_1} dz_{1\bar{1}\bar{3}}, e^{-2z_1} dz_{2\bar{1}\bar{2}}, e^{-z_1} dz_{2\bar{2}\bar{3}}, e^{2z_1} dz_{3\bar{1}\bar{3}}, e^{z_1} dz_{3\bar{2}\bar{3}} \rangle$
(0, 3)	$\mathbb{C}\langle dz_{\bar{1}\bar{2}\bar{3}} \rangle$
(3, 1)	$\mathbb{C}\langle dz_{123\bar{1}}, e^{-z_1} dz_{123\bar{2}}, e^{z_1} dz_{123\bar{3}} \rangle$
(2, 2)	$\mathbb{C}\langle e^{-2z_1} dz_{12\bar{1}\bar{2}}, dz_{12\bar{1}\bar{3}}, e^{-z_1} dz_{12\bar{2}\bar{3}}, dz_{13\bar{1}\bar{2}}, e^{2z_1} dz_{13\bar{1}\bar{3}}, e^{z_1} dz_{13\bar{2}\bar{3}}, e^{-z_1} dz_{23\bar{1}\bar{2}}, e^{z_1} dz_{23\bar{1}\bar{3}}, dz_{23\bar{2}\bar{3}} \rangle$
(1, 3)	$\mathbb{C}\langle dz_{1\bar{1}\bar{2}\bar{3}}, e^{-z_1} dz_{2\bar{1}\bar{2}\bar{3}}, e^{z_1} dz_{3\bar{1}\bar{2}\bar{3}} \rangle$
(3, 2)	$\mathbb{C}\langle e^{-z_1} dz_{123\bar{1}\bar{2}}, e^{z_1} dz_{123\bar{1}\bar{3}}, dz_{123\bar{2}\bar{3}} \rangle$
(2, 3)	$\mathbb{C}\langle e^{-z_1} dz_{12\bar{1}\bar{2}\bar{3}}, e^{z_1} dz_{13\bar{1}\bar{2}\bar{3}}, dz_{23\bar{1}\bar{2}\bar{3}} \rangle$
(3, 3)	$\mathbb{C}\langle dz_{123\bar{1}\bar{2}\bar{3}} \rangle$

Table 1 – The double-complex $B_{\Gamma}^{\bullet, \bullet}$ for computing the Dolbeault cohomology of the holomorphically parallelizable Nakamura manifold $\Gamma \backslash G$.

6. Closedness and openness under holomorphic deformation

	$C_{\Gamma}^{\bullet, \bullet}$
(0, 0)	$\mathbb{C}\langle 1 \rangle$
(1, 0)	$\mathbb{C}\langle dz_1, e^{-z_1} dz_2, e^{z_1} dz_3, e^{-\bar{z}_1} dz_2, e^{\bar{z}_1} dz_3 \rangle$
(0, 1)	$\mathbb{C}\langle dz_{\bar{1}}, e^{-z_1} dz_2, e^{z_1} dz_3, e^{-\bar{z}_1} dz_2, e^{\bar{z}_1} dz_3 \rangle$
(2, 0)	$\mathbb{C}\langle e^{-z_1} dz_{12}, e^{z_1} dz_{13}, dz_{23}, e^{-\bar{z}_1} dz_{12}, e^{\bar{z}_1} dz_{13} \rangle$
(1, 1)	$\mathbb{C}\langle dz_{1\bar{1}}, e^{-z_1} dz_{1\bar{2}}, e^{z_1} dz_{1\bar{3}}, e^{-z_1} dz_{2\bar{1}}, e^{-2z_1} dz_{2\bar{2}}, dz_{2\bar{3}}, e^{z_1} dz_{3\bar{1}}, dz_{3\bar{2}}, e^{2z_1} dz_{3\bar{3}}, e^{-\bar{z}_1} dz_{2\bar{1}}, e^{-\bar{z}_1} dz_{1\bar{2}}, e^{\bar{z}_1} dz_{1\bar{3}}, e^{\bar{z}_1} dz_{3\bar{1}}, e^{-2\bar{z}_1} dz_{2\bar{2}}, e^{2\bar{z}_1} dz_{3\bar{3}} \rangle$
(0, 2)	$\mathbb{C}\langle e^{-z_1} dz_{\bar{1}\bar{2}}, e^{z_1} dz_{\bar{1}\bar{3}}, dz_{2\bar{3}}, e^{-\bar{z}_1} dz_{\bar{1}\bar{2}}, e^{\bar{z}_1} dz_{\bar{1}\bar{3}} \rangle$
(3, 0)	$\mathbb{C}\langle dz_{123} \rangle$
(2, 1)	$\mathbb{C}\langle e^{-z_1} dz_{12\bar{1}}, e^{-2z_1} dz_{12\bar{2}}, dz_{12\bar{3}}, e^{z_1} dz_{13\bar{1}}, dz_{13\bar{2}}, e^{2z_1} dz_{13\bar{3}}, dz_{23\bar{1}}, e^{-z_1} dz_{23\bar{2}}, e^{z_1} dz_{23\bar{3}}, e^{-\bar{z}_1} dz_{12\bar{1}}, e^{\bar{z}_1} dz_{13\bar{1}}, e^{-2\bar{z}_1} dz_{12\bar{2}}, e^{-\bar{z}_1} dz_{23\bar{2}}, e^{2\bar{z}_1} dz_{13\bar{3}}, e^{\bar{z}_1} dz_{23\bar{3}} \rangle$
(1, 2)	$\mathbb{C}\langle e^{-\bar{z}_1} dz_{1\bar{1}\bar{2}}, e^{-2\bar{z}_1} dz_{2\bar{1}\bar{2}}, dz_{3\bar{1}\bar{2}}, e^{\bar{z}_1} dz_{1\bar{1}\bar{3}}, dz_{2\bar{1}\bar{3}}, e^{2\bar{z}_1} dz_{3\bar{1}\bar{3}}, dz_{1\bar{2}\bar{3}}, e^{-\bar{z}_1} dz_{2\bar{2}\bar{3}}, e^{\bar{z}_1} dz_{3\bar{2}\bar{3}}, e^{-z_1} dz_{1\bar{1}\bar{2}}, e^{z_1} dz_{1\bar{1}\bar{3}}, e^{-2z_1} dz_{2\bar{1}\bar{2}}, e^{-z_1} dz_{2\bar{2}\bar{3}}, e^{2z_1} dz_{3\bar{1}\bar{3}}, e^{z_1} dz_{3\bar{2}\bar{3}} \rangle$
(0, 3)	$\mathbb{C}\langle dz_{\bar{1}\bar{2}\bar{3}} \rangle$
(3, 1)	$\mathbb{C}\langle dz_{123\bar{1}}, e^{-z_1} dz_{123\bar{2}}, e^{z_1} dz_{123\bar{3}}, e^{-\bar{z}_1} dz_{123\bar{2}}, e^{\bar{z}_1} dz_{123\bar{3}} \rangle$
(2, 2)	$\mathbb{C}\langle e^{-2z_1} dz_{12\bar{1}\bar{2}}, dz_{12\bar{1}\bar{3}}, e^{-z_1} dz_{12\bar{2}\bar{3}}, dz_{13\bar{1}\bar{2}}, e^{2z_1} dz_{13\bar{1}\bar{3}}, e^{z_1} dz_{13\bar{2}\bar{3}}, e^{-z_1} dz_{23\bar{1}\bar{2}}, e^{z_1} dz_{23\bar{1}\bar{3}}, dz_{23\bar{2}\bar{3}}, e^{-2\bar{z}_1} dz_{12\bar{1}\bar{2}}, e^{-\bar{z}_1} dz_{23\bar{1}\bar{2}}, e^{-\bar{z}_1} dz_{12\bar{2}\bar{3}}, e^{\bar{z}_1} dz_{13\bar{2}\bar{3}}, e^{2\bar{z}_1} dz_{13\bar{1}\bar{3}}, e^{\bar{z}_1} dz_{23\bar{1}\bar{3}} \rangle$
(1, 3)	$\mathbb{C}\langle dz_{1\bar{1}\bar{2}\bar{3}}, e^{-\bar{z}_1} dz_{2\bar{1}\bar{2}\bar{3}}, e^{\bar{z}_1} dz_{3\bar{1}\bar{2}\bar{3}}, e^{-z_1} dz_{2\bar{1}\bar{2}\bar{3}}, e^{z_1} dz_{3\bar{1}\bar{2}\bar{3}} \rangle$
(3, 2)	$\mathbb{C}\langle e^{-z_1} dz_{123\bar{1}\bar{2}}, e^{z_1} dz_{123\bar{1}\bar{3}}, dz_{123\bar{2}\bar{3}}, e^{-\bar{z}_1} dz_{123\bar{1}\bar{2}}, e^{\bar{z}_1} dz_{123\bar{1}\bar{3}} \rangle$
(2, 3)	$\mathbb{C}\langle e^{-z_1} dz_{12\bar{1}\bar{2}\bar{3}}, e^{z_1} dz_{13\bar{1}\bar{2}\bar{3}}, dz_{23\bar{1}\bar{2}\bar{3}}, e^{-\bar{z}_1} dz_{12\bar{1}\bar{2}\bar{3}}, e^{\bar{z}_1} dz_{13\bar{1}\bar{2}\bar{3}} \rangle$
(3, 3)	$\mathbb{C}\langle dz_{\bar{1}\bar{2}\bar{3}\bar{1}} \rangle$

Table 2 – The double-complex $C_{\Gamma}^{\bullet, \bullet}$ for computing the Bott-Chern cohomology of the holomorphically parallelizable Nakamura manifold $\Gamma \backslash G$.

case (C ₁)	$d\psi$
ψ	$d\psi$
$\phi_1^{1,0}(t) := dz_1 - t d\bar{z}_1$	$d\phi_1^{1,0}(t) = 0$
$\phi_2^{1,0}(t) := e^{-z_1} dz_2$	$d\phi_2^{1,0}(t) = -\frac{1}{1- t ^2} \phi_1^{1,0}(t) \wedge \phi_2^{1,0}(t) + \frac{t}{1- t ^2} \phi_2^{1,0}(t) \wedge \phi_1^{0,1}(t)$
$\phi_3^{1,0}(t) := e^{z_1} dz_3$	$d\phi_3^{1,0}(t) = \frac{1}{1- t ^2} \phi_1^{1,0}(t) \wedge \phi_3^{1,0}(t) - \frac{t}{1- t ^2} \phi_3^{1,0}(t) \wedge \phi_1^{0,1}(t)$
$\varphi_2^{1,0}(t) := e^{-\bar{z}_1} dz_2$	$d\varphi_2^{1,0}(t) = -\frac{\bar{t}}{1- t ^2} \phi_1^{1,0}(t) \wedge \varphi_2^{1,0}(t) + \frac{1}{1- t ^2} \varphi_2^{1,0}(t) \wedge \phi_1^{0,1}(t)$
$\varphi_3^{1,0}(t) := e^{\bar{z}_1} dz_3$	$d\varphi_3^{1,0}(t) = \frac{\bar{t}}{1- t ^2} \phi_1^{1,0}(t) \wedge \varphi_3^{1,0}(t) - \frac{1}{1- t ^2} \varphi_3^{1,0}(t) \wedge \phi_1^{0,1}(t)$
$\phi_1^{0,1}(t) := d\bar{z}_1 - \bar{t} dz_1$	$d\phi_1^{0,1}(t) = 0$
$\phi_2^{0,1}(t) := e^{-z_1} d\bar{z}_2$	$d\phi_2^{0,1}(t) = -\frac{1}{1- t ^2} \phi_1^{1,0}(t) \wedge \phi_2^{0,1}(t) - \frac{t}{1- t ^2} \phi_1^{0,1}(t) \wedge \phi_2^{0,1}(t)$
$\phi_3^{0,1}(t) := e^{z_1} d\bar{z}_3$	$d\phi_3^{0,1}(t) = \frac{1}{1- t ^2} \phi_1^{1,0}(t) \wedge \phi_3^{0,1}(t) + \frac{t}{1- t ^2} \phi_1^{0,1}(t) \wedge \phi_3^{0,1}(t)$
$\varphi_2^{0,1}(t) := e^{-\bar{z}_1} d\bar{z}_2$	$d\varphi_2^{0,1}(t) = -\frac{\bar{t}}{1- t ^2} \phi_1^{1,0}(t) \wedge \varphi_2^{0,1}(t) - \frac{1}{1- t ^2} \phi_1^{0,1}(t) \wedge \varphi_2^{0,1}(t)$
$\varphi_3^{0,1}(t) := e^{\bar{z}_1} d\bar{z}_3$	$d\varphi_3^{0,1}(t) = \frac{\bar{t}}{1- t ^2} \phi_1^{1,0}(t) \wedge \varphi_3^{0,1}(t) + \frac{1}{1- t ^2} \phi_1^{0,1}(t) \wedge \varphi_3^{0,1}(t)$

Table 3 – Definitions for setting the generators of the complexes $B_\Gamma^{\bullet,\bullet}(t)$, see Table 5 on p. 94, and $C_\Gamma^{\bullet,\bullet}(t)$, see Table 6 on p. 95, for the deformations in case (C₁) on p. 84, which are given by $t \frac{\partial}{\partial z_1} \otimes d\bar{z}_1$, of the holomorphically parallelizable Nakamura manifold $\Gamma \backslash G$.

6. Closedness and openness under holomorphic deformation

case (C ₂) ψ	$d\psi$
$\phi_1^{1,0}(t) := dz_1 - te^{z_1} d\bar{z}_3$	$d\phi_1^{1,0}(t) = -t\phi_1^{1,0}(t) \wedge \phi_3^{0,1}(t)$
$\phi_2^{1,0}(t) := e^{-z_1} dz_2$	$d\phi_2^{1,0}(t) = -\phi_1^{1,0}(t) \wedge \phi_2^{1,0}(t) + t\phi_2^{1,0}(t) \wedge \phi_3^{0,1}(t)$
$\phi_3^{1,0}(t) := e^{\bar{z}_1} dz_3$	$d\phi_3^{1,0}(t) = \phi_1^{1,0}(t) \wedge \phi_3^{1,0}(t) - t\phi_3^{1,0}(t) \wedge \phi_3^{0,1}(t)$
$\varphi_2^{1,0}(t) := e^{-\bar{z}_1} dz_2$	$d\varphi_2^{1,0}(t) = \bar{t}\varphi_2^{1,0}(t) \wedge \varphi_3^{1,0}(t) + \varphi_2^{1,0}(t) \wedge \phi_1^{0,1}(t)$
$\varphi_3^{1,0}(t) := e^{\bar{z}_1} dz_3$	$d\varphi_3^{1,0}(t) = -\varphi_3^{1,0}(t) \wedge \phi_1^{0,1}(t)$
$\phi_1^{0,1}(t) := d\bar{z}_1 - \bar{t}e^{\bar{z}_1} dz_3$	$d\phi_1^{0,1}(t) = \bar{t}\varphi_3^{1,0}(t) \wedge \phi_1^{0,1}(t)$
$\phi_2^{0,1}(t) := e^{-z_1} d\bar{z}_2$	$d\phi_2^{0,1}(t) = -\phi_1^{1,0}(t) \wedge \phi_2^{0,1}(t) + t\phi_2^{0,1}(t) \wedge \phi_3^{0,1}(t)$
$\phi_3^{0,1}(t) := e^{z_1} d\bar{z}_3$	$d\phi_3^{0,1}(t) = \phi_1^{1,0}(t) \wedge \phi_3^{0,1}(t)$
$\varphi_2^{0,1}(t) := e^{-\bar{z}_1} d\bar{z}_2$	$d\varphi_2^{0,1}(t) = -\bar{t}\varphi_3^{1,0}(t) \wedge \varphi_2^{0,1}(t) - \phi_1^{0,1}(t) \wedge \varphi_2^{0,1}(t)$
$\varphi_3^{0,1}(t) := e^{\bar{z}_1} d\bar{z}_3$	$d\varphi_3^{0,1}(t) = \bar{t}\varphi_3^{1,0}(t) \wedge \varphi_3^{0,1}(t) + \phi_1^{0,1}(t) \wedge \varphi_3^{0,1}(t)$

Table 4 – Definitions for setting the generators of the complex $B_\Gamma^{\bullet,\bullet}(t)$, see Table 5 on the next page, for the deformations in case (C₂) on p. 84, which are given by $t \frac{\partial}{\partial z_1} \otimes e^{z_1} d\bar{z}_3$, of the holomorphically parallelizable Nakamura manifold $\Gamma \backslash G$.

	$B_{\Gamma}^{\bullet, \bullet}(t)$
(0, 0)	$\mathbb{C}\langle 1 \rangle$
(1, 0)	$\mathbb{C}\langle \phi_1^{1,0}(t), \phi_2^{1,0}(t), \phi_3^{1,0}(t) \rangle$
(0, 1)	$\mathbb{C}\langle \phi_1^{0,1}(t), \phi_2^{0,1}(t), \phi_3^{0,1}(t) \rangle$
(2, 0)	$\mathbb{C}\langle \phi_{12}^{1,0}(t), \phi_{13}^{1,0}(t), \phi_{23}^{1,0}(t) \rangle$
(1, 1)	$\mathbb{C}\langle \phi_1^{1,0}(t) \wedge \phi_1^{0,1}(t), \phi_1^{1,0}(t) \wedge \phi_2^{0,1}(t), \phi_1^{1,0}(t) \wedge \phi_3^{0,1}(t), \phi_2^{1,0}(t) \wedge \phi_1^{0,1}(t), \phi_2^{1,0}(t) \wedge \phi_2^{0,1}(t), \phi_2^{1,0}(t) \wedge \phi_3^{0,1}(t), \phi_3^{1,0}(t) \wedge \phi_1^{0,1}(t), \phi_3^{1,0}(t) \wedge \phi_2^{0,1}(t), \phi_3^{1,0}(t) \wedge \phi_3^{0,1}(t) \rangle$
(0, 2)	$\mathbb{C}\langle \phi_{12}^{0,1}(t), \phi_{13}^{0,1}(t), \phi_{23}^{0,1}(t) \rangle$
(3, 0)	$\mathbb{C}\langle \phi_{123}^{1,0}(t) \rangle$
(2, 1)	$\mathbb{C}\langle \phi_{12}^{1,0}(t) \wedge \phi_1^{0,1}(t), \phi_{12}^{1,0}(t) \wedge \phi_2^{0,1}(t), \phi_{12}^{1,0}(t) \wedge \phi_3^{0,1}(t), \phi_{13}^{1,0}(t) \wedge \phi_1^{0,1}(t), \phi_{13}^{1,0}(t) \wedge \phi_2^{0,1}(t), \phi_{13}^{1,0}(t) \wedge \phi_3^{0,1}(t) \rangle$
(1, 2)	$\mathbb{C}\langle \phi_3^{1,0}(t) \wedge \phi_{12}^{0,1}(t), \phi_2^{1,0}(t) \wedge \phi_{13}^{0,1}(t), \phi_1^{1,0}(t) \wedge \phi_{23}^{0,1}(t), \phi_1^{1,0}(t) \wedge \phi_{12}^{0,1}(t), \phi_1^{1,0}(t) \wedge \phi_{13}^{0,1}(t), \phi_2^{1,0}(t) \wedge \phi_{12}^{0,1}(t), \phi_2^{1,0}(t) \wedge \phi_{23}^{0,1}(t), \phi_3^{1,0}(t) \wedge \phi_{13}^{0,1}(t), \phi_3^{1,0}(t) \wedge \phi_{23}^{0,1}(t) \rangle$
(0, 3)	$\mathbb{C}\langle \phi_{123}^{0,1}(t) \rangle$
(3, 1)	$\mathbb{C}\langle \phi_{123}^{1,0}(t) \wedge \phi_1^{0,1}(t), \phi_{123}^{1,0}(t) \wedge \phi_2^{0,1}(t), \phi_{123}^{1,0}(t) \wedge \phi_3^{0,1}(t) \rangle$
(2, 2)	$\mathbb{C}\langle \phi_{12}^{1,0}(t) \wedge \phi_{12}^{0,1}(t), \phi_{12}^{1,0}(t) \wedge \phi_{13}^{0,1}(t), \phi_{12}^{1,0}(t) \wedge \phi_{23}^{0,1}(t), \phi_{13}^{1,0}(t) \wedge \phi_{12}^{0,1}(t), \phi_{13}^{1,0}(t) \wedge \phi_{13}^{0,1}(t), \phi_{13}^{1,0}(t) \wedge \phi_{23}^{0,1}(t), \phi_{23}^{1,0}(t) \wedge \phi_{12}^{0,1}(t), \phi_{23}^{1,0}(t) \wedge \phi_{13}^{0,1}(t), \phi_{23}^{1,0}(t) \wedge \phi_{23}^{0,1}(t) \rangle$
(1, 3)	$\mathbb{C}\langle \phi_1^{1,0}(t) \wedge \phi_{123}^{0,1}(t), \phi_2^{1,0}(t) \wedge \phi_{123}^{0,1}(t), \phi_3^{1,0}(t) \wedge \phi_{123}^{0,1}(t) \rangle$
(3, 2)	$\mathbb{C}\langle \phi_{123}^{1,0}(t) \wedge \phi_{12}^{0,1}(t), \phi_{123}^{1,0}(t) \wedge \phi_{13}^{0,1}(t), \phi_{123}^{1,0}(t) \wedge \phi_{23}^{0,1}(t) \rangle$
(2, 3)	$\mathbb{C}\langle \phi_{12}^{1,0}(t) \wedge \phi_{123}^{0,1}(t), \phi_{13}^{1,0}(t) \wedge \phi_{123}^{0,1}(t), \phi_{23}^{1,0}(t) \wedge \phi_{123}^{0,1}(t) \rangle$
(3, 3)	$\mathbb{C}\langle \phi_{123}^{1,0}(t) \wedge \phi_{123}^{0,1}(t) \rangle$

Table 5 – The double-complex $B_{\Gamma}^{\bullet, \bullet}(t)$ for computing the Dolbeault cohomology of the small deformations in cases (C_1) and (C_2) on p. 84 of the holomorphically parallelizable Nakamura manifold $\Gamma \backslash G$.

6. Closedness and openness under holomorphic deformation

	$\mathbb{C}_{\Gamma}^{\bullet,\bullet}(t)$
(0,0)	$\mathbb{C}\langle 1 \rangle$
(1,0)	$\mathbb{C}\langle \phi_1^{1,0}(t), \phi_2^{1,0}(t), \phi_3^{1,0}(t), \varphi_2^{1,0}(t), \varphi_3^{1,0}(t) \rangle$
(0,1)	$\mathbb{C}\langle \phi_1^{0,1}(t), \phi_2^{0,1}(t), \phi_3^{0,1}(t), \varphi_2^{0,1}(t), \varphi_3^{0,1}(t) \rangle$
(2,0)	$\mathbb{C}\langle \phi_{12}^{1,0}(t), \phi_{13}^{1,0}(t), \phi_{23}^{1,0}(t), \phi_1^{1,0}(t) \wedge \varphi_2^{1,0}(t), \phi_1^{1,0}(t) \wedge \varphi_3^{1,0}(t) \rangle$
(1,1)	$\mathbb{C}\langle \phi_1^{1,0}(t) \wedge \phi_1^{0,1}(t), \phi_1^{1,0}(t) \wedge \phi_2^{0,1}(t), \phi_1^{1,0}(t) \wedge \phi_3^{0,1}(t), \phi_2^{1,0}(t) \wedge \phi_1^{0,1}(t), \phi_2^{1,0}(t) \wedge \phi_2^{0,1}(t), \phi_2^{1,0}(t) \wedge \phi_3^{0,1}(t), \phi_3^{1,0}(t) \wedge \phi_1^{0,1}(t), \phi_3^{1,0}(t) \wedge \phi_2^{0,1}(t), \phi_3^{1,0}(t) \wedge \phi_3^{0,1}(t), \phi_1^{1,0}(t) \wedge \varphi_2^{0,1}(t), \phi_1^{1,0}(t) \wedge \varphi_3^{0,1}(t), \phi_2^{1,0}(t) \wedge \varphi_2^{0,1}(t), \phi_2^{1,0}(t) \wedge \varphi_3^{0,1}(t), \phi_3^{1,0}(t) \wedge \varphi_2^{0,1}(t), \phi_3^{1,0}(t) \wedge \varphi_3^{0,1}(t) \rangle$
(0,2)	$\mathbb{C}\langle \phi_{12}^{0,1}(t), \phi_{13}^{0,1}(t), \phi_{23}^{0,1}(t), \phi_1^{0,1}(t) \wedge \varphi_2^{0,1}(t), \phi_1^{0,1}(t) \wedge \varphi_3^{0,1}(t) \rangle$
(3,0)	$\mathbb{C}\langle \phi_{123}^{1,0}(t) \rangle$
(2,1)	$\mathbb{C}\langle \phi_{12}^{1,0}(t) \wedge \phi_1^{0,1}(t), \phi_{12}^{1,0}(t) \wedge \phi_2^{0,1}(t), \phi_{12}^{1,0}(t) \wedge \phi_3^{0,1}(t), \phi_{13}^{1,0}(t) \wedge \phi_1^{0,1}(t), \phi_{13}^{1,0}(t) \wedge \phi_2^{0,1}(t), \phi_{13}^{1,0}(t) \wedge \phi_3^{0,1}(t), \phi_{23}^{1,0}(t) \wedge \phi_1^{0,1}(t), \phi_{23}^{1,0}(t) \wedge \phi_2^{0,1}(t), \phi_{23}^{1,0}(t) \wedge \phi_3^{0,1}(t), \phi_1^{1,0}(t) \wedge \varphi_2^{0,1}(t), \phi_1^{1,0}(t) \wedge \varphi_3^{0,1}(t), \phi_2^{1,0}(t) \wedge \varphi_2^{0,1}(t), \phi_2^{1,0}(t) \wedge \varphi_3^{0,1}(t), \phi_3^{1,0}(t) \wedge \varphi_2^{0,1}(t), \phi_3^{1,0}(t) \wedge \varphi_3^{0,1}(t) \rangle$
(1,2)	$\mathbb{C}\langle \phi_1^{1,0}(t) \wedge \phi_1^{0,1}(t) \wedge \varphi_2^{0,1}(t), \phi_2^{1,0}(t) \wedge \phi_1^{0,1}(t) \wedge \varphi_2^{0,1}(t), \phi_3^{1,0}(t) \wedge \phi_1^{0,1}(t) \wedge \varphi_2^{0,1}(t), \phi_1^{1,0}(t) \wedge \phi_1^{0,1}(t) \wedge \varphi_3^{0,1}(t), \phi_2^{1,0}(t) \wedge \phi_1^{0,1}(t) \wedge \varphi_3^{0,1}(t), \phi_3^{1,0}(t) \wedge \phi_1^{0,1}(t) \wedge \varphi_3^{0,1}(t), \phi_1^{1,0}(t) \wedge \phi_2^{0,1}(t) \wedge \varphi_2^{0,1}(t), \phi_2^{1,0}(t) \wedge \phi_2^{0,1}(t) \wedge \varphi_2^{0,1}(t), \phi_3^{1,0}(t) \wedge \phi_2^{0,1}(t) \wedge \varphi_2^{0,1}(t), \phi_1^{1,0}(t) \wedge \phi_2^{0,1}(t) \wedge \varphi_3^{0,1}(t), \phi_2^{1,0}(t) \wedge \phi_2^{0,1}(t) \wedge \varphi_3^{0,1}(t), \phi_3^{1,0}(t) \wedge \phi_2^{0,1}(t) \wedge \varphi_3^{0,1}(t), \phi_1^{1,0}(t) \wedge \phi_3^{0,1}(t) \wedge \varphi_2^{0,1}(t), \phi_2^{1,0}(t) \wedge \phi_3^{0,1}(t) \wedge \varphi_2^{0,1}(t), \phi_3^{1,0}(t) \wedge \phi_3^{0,1}(t) \wedge \varphi_2^{0,1}(t), \phi_1^{1,0}(t) \wedge \phi_3^{0,1}(t) \wedge \varphi_3^{0,1}(t), \phi_2^{1,0}(t) \wedge \phi_3^{0,1}(t) \wedge \varphi_3^{0,1}(t), \phi_3^{1,0}(t) \wedge \phi_3^{0,1}(t) \wedge \varphi_3^{0,1}(t) \rangle$
(0,3)	$\mathbb{C}\langle \phi_{123}^{0,1}(t) \rangle$
(3,1)	$\mathbb{C}\langle \phi_{123}^{1,0}(t) \wedge \phi_1^{0,1}(t), \phi_{123}^{1,0}(t) \wedge \phi_2^{0,1}(t), \phi_{123}^{1,0}(t) \wedge \phi_3^{0,1}(t), \phi_{123}^{1,0}(t) \wedge \varphi_2^{0,1}(t), \phi_{123}^{1,0}(t) \wedge \varphi_3^{0,1}(t) \rangle$
(2,2)	$\mathbb{C}\langle \phi_{12}^{1,0}(t) \wedge \phi_{12}^{0,1}(t), \phi_{12}^{1,0}(t) \wedge \phi_{13}^{0,1}(t), \phi_{12}^{1,0}(t) \wedge \phi_{23}^{0,1}(t), \phi_{13}^{1,0}(t) \wedge \phi_{12}^{0,1}(t), \phi_{13}^{1,0}(t) \wedge \phi_{13}^{0,1}(t), \phi_{13}^{1,0}(t) \wedge \phi_{23}^{0,1}(t), \phi_1^{1,0}(t) \wedge \phi_{23}^{0,1}(t), \phi_1^{1,0}(t) \wedge \varphi_2^{0,1}(t) \wedge \phi_1^{0,1}(t) \wedge \varphi_2^{0,1}(t), \phi_1^{1,0}(t) \wedge \varphi_3^{0,1}(t) \wedge \phi_1^{0,1}(t) \wedge \varphi_3^{0,1}(t), \phi_2^{1,0}(t) \wedge \varphi_2^{0,1}(t) \wedge \phi_2^{0,1}(t) \wedge \varphi_2^{0,1}(t), \phi_3^{1,0}(t) \wedge \varphi_2^{0,1}(t) \wedge \phi_3^{0,1}(t) \wedge \varphi_2^{0,1}(t), \phi_1^{1,0}(t) \wedge \varphi_3^{0,1}(t) \wedge \phi_2^{0,1}(t) \wedge \varphi_3^{0,1}(t), \phi_2^{1,0}(t) \wedge \varphi_3^{0,1}(t) \wedge \phi_2^{0,1}(t) \wedge \varphi_3^{0,1}(t), \phi_3^{1,0}(t) \wedge \varphi_3^{0,1}(t) \wedge \phi_3^{0,1}(t) \wedge \varphi_3^{0,1}(t) \rangle$
(1,3)	$\mathbb{C}\langle \phi_1^{1,0}(t) \wedge \phi_{123}^{0,1}(t), \phi_2^{1,0}(t) \wedge \phi_{123}^{0,1}(t), \phi_3^{1,0}(t) \wedge \phi_{123}^{0,1}(t), \varphi_2^{1,0}(t) \wedge \phi_{123}^{0,1}(t), \varphi_3^{1,0}(t) \wedge \phi_{123}^{0,1}(t) \rangle$
(3,2)	$\mathbb{C}\langle \phi_{123}^{1,0}(t) \wedge \phi_{12}^{0,1}(t), \phi_{123}^{1,0}(t) \wedge \phi_{13}^{0,1}(t), \phi_{123}^{1,0}(t) \wedge \phi_{23}^{0,1}(t), \phi_{123}^{1,0}(t) \wedge \phi_1^{0,1}(t) \wedge \varphi_2^{0,1}(t), \phi_{123}^{1,0}(t) \wedge \phi_1^{0,1}(t) \wedge \varphi_3^{0,1}(t) \rangle$
(2,3)	$\mathbb{C}\langle \phi_{12}^{1,0}(t) \wedge \phi_{123}^{0,1}(t), \phi_{13}^{1,0}(t) \wedge \phi_{123}^{0,1}(t), \phi_{23}^{1,0}(t) \wedge \phi_{123}^{0,1}(t), \phi_1^{1,0}(t) \wedge \varphi_2^{1,0}(t) \wedge \phi_{123}^{0,1}(t), \phi_1^{1,0}(t) \wedge \varphi_3^{1,0}(t) \wedge \phi_{123}^{0,1}(t) \rangle$
(3,3)	$\mathbb{C}\langle \phi_{123}^{1,0}(t) \wedge \phi_{123}^{0,1}(t) \rangle$

Table 6 – The double-complex $\mathbb{C}_{\Gamma}^{\bullet,\bullet}(t)$ for computing the Bott-Chern cohomology of the small deformations in case (C_1) on p. 84 of the holomorphically parallelizable Nakamura manifold $\Gamma \setminus G$.

case (C ₁)	$H_{\partial_t}^{\bullet,\bullet}(X)$
(0,0)	$\mathbb{C}\langle 1 \rangle$
(1,0)	$\mathbb{C}\langle \phi_1^{1,0}(t) \rangle$
(0,1)	$\mathbb{C}\langle \phi_1^{0,1}(t) \rangle$
(2,0)	$\mathbb{C}\langle \phi_{23}^{1,0}(t) \rangle$
(1,1)	$\mathbb{C}\langle \phi_1^{1,0}(t) \wedge \phi_1^{0,1}(t), \phi_2^{1,0}(t) \wedge \phi_3^{0,1}(t), \phi_3^{1,0}(t) \wedge \phi_2^{0,1}(t) \rangle$
(0,2)	$\mathbb{C}\langle \phi_{23}^{0,1}(t) \rangle$
(3,0)	$\mathbb{C}\langle \phi_{123}^{1,0}(t) \rangle$
(2,1)	$\mathbb{C}\langle \phi_{12}^{1,0}(t) \wedge \phi_3^{0,1}(t), \phi_{13}^{1,0}(t) \wedge \phi_2^{0,1}(t), \phi_{23}^{1,0}(t) \wedge \phi_1^{0,1}(t) \rangle$
(1,2)	$\mathbb{C}\langle \phi_3^{1,0}(t) \wedge \phi_{12}^{0,1}(t), \phi_2^{1,0}(t) \wedge \phi_{13}^{0,1}(t), \phi_1^{1,0}(t) \wedge \phi_{23}^{0,1}(t) \rangle$
(0,3)	$\mathbb{C}\langle \phi_{123}^{0,1}(t) \rangle$
(3,1)	$\mathbb{C}\langle \phi_{123}^{1,0}(t) \wedge \phi_1^{0,1}(t) \rangle$
(2,2)	$\mathbb{C}\langle \phi_{12}^{1,0}(t) \wedge \phi_{13}^{0,1}(t), \phi_{13}^{1,0}(t) \wedge \phi_{12}^{0,1}(t), \phi_{23}^{1,0}(t) \wedge \phi_{23}^{0,1}(t) \rangle$
(1,3)	$\mathbb{C}\langle \phi_1^{1,0}(t) \wedge \phi_{123}^{0,1}(t) \rangle$
(3,2)	$\mathbb{C}\langle \phi_{123}^{1,0}(t) \wedge \phi_{23}^{0,1}(t) \rangle$
(2,3)	$\mathbb{C}\langle \phi_{23}^{1,0}(t) \wedge \phi_{123}^{0,1}(t) \rangle$
(3,3)	$\mathbb{C}\langle \phi_{123}^{1,0}(t) \wedge \phi_{123}^{0,1}(t) \rangle$

Table 7 – The harmonic representatives of the Dolbeault cohomology of the small deformations in case (C₁) on p. 84, which are given by $t \frac{\partial}{\partial z_1} \otimes d\bar{z}_1$, of the holomorphically parallelizable Nakamura manifold, with respect to the Hermitian metric $g_t := \phi_1^{1,0}(t) \odot \phi_1^{0,1}(t) + \phi_2^{1,0}(t) \odot \phi_2^{0,1}(t) + \phi_3^{1,0}(t) \odot \phi_3^{0,1}(t)$.

6. Closedness and openness under holomorphic deformation

case (C ₁)	$H_{BC,t}^{\bullet,\bullet}(X)$
(0,0)	$\mathbb{C}\langle 1 \rangle$
(1,0)	$\mathbb{C}\langle \phi_1^{1,0}(t) \rangle$
(0,1)	$\mathbb{C}\langle \phi_1^{0,1}(t) \rangle$
(2,0)	$\mathbb{C}\langle \phi_{23}^{1,0}(t) \rangle$
(1,1)	$\mathbb{C}\langle \phi_1^{1,0}(t) \wedge \phi_1^{0,1}(t), \phi_2^{1,0}(t) \wedge \phi_3^{0,1}(t), \phi_3^{1,0}(t) \wedge \phi_2^{0,1}(t) \rangle$
(0,2)	$\mathbb{C}\langle \phi_{23}^{0,1}(t) \rangle$
(3,0)	$\mathbb{C}\langle \phi_{123}^{1,0}(t) \rangle$
(2,1)	$\mathbb{C}\langle \phi_{12}^{1,0}(t) \wedge \phi_3^{0,1}(t), \phi_{13}^{1,0}(t) \wedge \phi_2^{0,1}(t), \phi_{23}^{1,0}(t) \wedge \phi_1^{0,1}(t) \rangle$
(1,2)	$\mathbb{C}\langle \phi_3^{1,0}(t) \wedge \phi_{12}^{0,1}(t), \phi_2^{1,0}(t) \wedge \phi_{13}^{0,1}(t), \phi_1^{1,0}(t) \wedge \phi_{23}^{0,1}(t) \rangle$
(0,3)	$\mathbb{C}\langle \phi_{123}^{0,1}(t) \rangle$
(3,1)	$\mathbb{C}\langle \phi_{123}^{1,0}(t) \wedge \phi_1^{0,1}(t) \rangle$
(2,2)	$\mathbb{C}\langle \phi_{12}^{1,0}(t) \wedge \phi_{13}^{0,1}(t), \phi_{13}^{1,0}(t) \wedge \phi_{12}^{0,1}(t), \phi_{23}^{1,0}(t) \wedge \phi_{23}^{0,1}(t) \rangle$
(1,3)	$\mathbb{C}\langle \phi_1^{1,0}(t) \wedge \phi_{123}^{0,1}(t) \rangle$
(3,2)	$\mathbb{C}\langle \phi_{123}^{1,0}(t) \wedge \phi_{23}^{0,1}(t) \rangle$
(2,3)	$\mathbb{C}\langle \phi_{23}^{1,0}(t) \wedge \phi_{123}^{0,1}(t) \rangle$
(3,3)	$\mathbb{C}\langle \phi_{123}^{1,0}(t) \wedge \phi_{123}^{0,1}(t) \rangle$

Table 8 – The harmonic representatives of the Bott-Chern cohomology of the small deformations in case (C₁) on p. 84, which are given by $t \frac{\partial}{\partial z_1} \otimes d\bar{z}_1$, of the holomorphically parallelizable Nakamura manifold, with respect to the Hermitian metric $g_t := \phi_1^{1,0}(t) \odot \phi_1^{0,1}(t) + \phi_2^{1,0}(t) \odot \phi_2^{0,1}(t) + \phi_3^{1,0}(t) \odot \phi_3^{0,1}(t)$.

case (C ₂)	$H_{\bar{\partial}_t}^{\bullet,\bullet}(X)$
(0,0)	$\mathbb{C}\langle 1 \rangle$
(1,0)	0
(0,1)	$\mathbb{C}\langle \phi_1^{0,1}(t), \phi_3^{0,1}(t) \rangle$
(2,0)	$\mathbb{C}\langle \phi_{12}^{1,0}(t), \phi_{23}^{1,0}(t) \rangle$
(1,1)	$\mathbb{C}\langle \phi_1^{1,0}(t) \wedge \phi_2^{0,1}(t), \phi_3^{1,0}(t) \wedge \phi_2^{0,1}(t) \rangle$
(0,2)	$\mathbb{C}\langle \phi_{13}^{0,1}(t) \rangle$
(3,0)	0
(2,1)	$\mathbb{C}\langle \phi_{12}^{1,0}(t) \wedge \phi_1^{0,1}(t), \phi_{12}^{1,0}(t) \wedge \phi_3^{0,1}(t), \phi_{23}^{1,0}(t) \wedge \phi_1^{0,1}(t), \phi_{23}^{1,0}(t) \wedge \phi_3^{0,1}(t) \rangle$
(1,2)	$\mathbb{C}\langle \phi_3^{1,0}(t) \wedge \phi_{12}^{0,1}(t), \phi_1^{1,0}(t) \wedge \phi_{23}^{0,1}(t), \phi_1^{1,0}(t) \wedge \phi_{12}^{0,1}(t), \phi_3^{1,0}(t) \wedge \phi_{23}^{0,1}(t) \rangle$
(0,3)	0
(3,1)	$\mathbb{C}\langle \phi_{123}^{1,0}(t) \wedge \phi_2^{0,1} \rangle$
(2,2)	$\mathbb{C}\langle \phi_{12}^{1,0}(t) \wedge \phi_{13}^{0,1}(t), \phi_{23}^{1,0}(t) \wedge \phi_{13}^{0,1}(t) \rangle$
(1,3)	$\mathbb{C}\langle \phi_1^{1,0}(t) \wedge \phi_{123}^{0,1}(t), \phi_3^{1,0}(t) \wedge \phi_{123}^{0,1}(t) \rangle$
(3,2)	$\mathbb{C}\langle \phi_{123}^{1,0}(t) \wedge \phi_{12}^{0,1}(t), \phi_{123}^{1,0}(t) \wedge \phi_{23}^{0,1}(t) \rangle$
(2,3)	0
(3,3)	$\mathbb{C}\langle \phi_{123}^{1,0}(t) \wedge \phi_{123}^{0,1}(t) \rangle$

Table 9 – The harmonic representatives of the Dolbeault cohomology of the small deformations in case (C₂) on p. 84, which are given by $t \frac{\partial}{\partial z_1} \otimes e^{\bar{z}_1} d\bar{z}_3$, of the holomorphically parallelizable Nakamura manifold, with respect to the Hermitian metric $g_t := \phi_1^{1,0}(t) \odot \phi_1^{0,1}(t) + \phi_2^{1,0}(t) \odot \phi_2^{0,1}(t) + \phi_3^{1,0}(t) \odot \phi_3^{0,1}(t)$.

6. Closedness and openness under holomorphic deformation

$\dim_{\mathbb{C}} H_{\sharp}^{\bullet, \bullet}$	Nakamura			case (C ₁)			case (C ₂)	
	dR	$\bar{\partial}$	BC	dR	$\bar{\partial}$	BC	dR	$\bar{\partial}$
(0,0)	1	1	1	1	1	1	1	1
(1,0)	2	3	1	2	1	1	2	0
(0,1)		3	1		1	1		
(2,0)	5	3	3	5	1	1	5	2
(1,1)		9	7		3	3		2
(0,2)		3	3		1	1		1
(3,0)	8	1	1	8	1	1	8	0
(2,1)		9	9		3	3		4
(1,2)		9	9		3	3		4
(0,3)		1	1		1	1		0
(3,1)	5	3	3	5	1	1	5	1
(2,2)		9	11		3	3		2
(1,3)		3	3		1	1		2
(3,2)	2	3	5	2	1	1	2	2
(2,3)		3	5		1	1		0
(3,3)	1	1	1	1	1	1	1	1

Table 10 – Summary of the dimensions of the cohomologies of the holomorphically parallelizable Nakamura manifold X (Angella and Kasuya 2017, Example 2.25) and of its small deformations in cases (C₁) and (C₂) on p. 84, given, respectively, by $t \frac{\partial}{\partial z_1} \otimes d\bar{z}_1$ and by $t \frac{\partial}{\partial z_1} \otimes e^{z_1} d\bar{z}_3$.

ψ	$d\psi$
$\phi_0^{1,0}(t) := dz - t e^{k_1 z} \bar{y}_{2,1}$	$d\phi_0^{1,0}(t) = -t k_1 \phi_0^{1,0}(t) \wedge \phi_{2,1}^{0,1}(t)$
$\phi_{1,j}^{1,0}(t) := e^{-k_j z} y_{1,j}$	$d\phi_{1,j}^{1,0}(t) = -k_j \phi_0^{1,0}(t) \wedge \phi_{1,j}^{1,0}(t) + t k_j \phi_{1,j}^{1,0}(t) \wedge \phi_{2,1}^{0,1}(t) + e^{-k_j z} dy_{1,j}$
$\phi_{2,j}^{1,0}(t) := e^{k_j z} y_{2,j}$	$d\phi_{2,j}^{1,0}(t) = k_j \phi_0^{1,0}(t) \wedge \phi_{2,j}^{1,0}(t) - t k_j \phi_{2,j}^{1,0}(t) \wedge \phi_{2,1}^{0,1}(t) + e^{k_j z} dy_{2,j}$
$\varphi_{1,j}^{1,0}(t) := e^{-k_j \bar{z}} y_{1,j}$	$d\varphi_{1,j}^{1,0}(t) = -k_j \phi_0^{0,1}(t) \wedge \varphi_{1,j}^{1,0}(t) + \bar{t} k_j \varphi_{1,j}^{1,0}(t) \wedge \varphi_{2,1}^{1,0}(t) + e^{-k_j \bar{z}} dy_{1,j}$
$\varphi_{2,j}^{1,0}(t) := e^{k_j \bar{z}} y_{2,j}$	$d\varphi_{2,j}^{1,0}(t) = k_j \phi_0^{0,1}(t) \wedge \varphi_{2,j}^{1,0}(t) - \bar{t} k_j \varphi_{2,j}^{1,0}(t) \wedge \varphi_{2,1}^{1,0}(t) + e^{k_j \bar{z}} dy_{2,j}$
$\phi_0^{0,1}(t) := d\bar{z} - \bar{t} e^{k_1 \bar{z}} y_{2,1}$	$d\phi_0^{0,1}(t) = -t k_1 \phi_0^{0,1}(t) \wedge \varphi_{2,1}^{1,0}(t)$
$\phi_{1,j}^{0,1}(t) := e^{-k_j \bar{z}} \bar{y}_{1,j}$	$d\phi_{1,j}^{0,1}(t) = -k_j \phi_0^{0,1}(t) \wedge \phi_{1,j}^{0,1}(t) + t k_j \phi_{1,j}^{0,1}(t) \wedge \varphi_{2,1}^{1,0}(t) + e^{-k_j \bar{z}} d\bar{y}_{1,j}$
$\phi_{2,j}^{0,1}(t) := e^{k_j \bar{z}} \bar{y}_{2,j}$	$d\phi_{2,j}^{0,1}(t) = k_j \phi_0^{0,1}(t) \wedge \phi_{2,j}^{0,1}(t) - t k_j \phi_{2,j}^{0,1}(t) \wedge \varphi_{2,1}^{1,0}(t) + e^{k_j \bar{z}} d\bar{y}_{2,j}$
$\varphi_{1,j}^{0,1}(t) := e^{-k_j \bar{z}} \bar{y}_{1,j}$	$d\varphi_{1,j}^{0,1}(t) = -k_j \phi_0^{0,1}(t) \wedge \varphi_{1,j}^{0,1}(t) + \bar{t} k_j \varphi_{1,j}^{0,1}(t) \wedge \varphi_{2,1}^{1,0}(t) + e^{-k_j \bar{z}} d\bar{y}_{1,j}$
$\varphi_{2,j}^{0,1}(t) := e^{k_j \bar{z}} \bar{y}_{2,j}$	$d\varphi_{2,j}^{0,1}(t) = k_j \phi_0^{0,1}(t) \wedge \varphi_{2,j}^{0,1}(t) - \bar{t} k_j \varphi_{2,j}^{0,1}(t) \wedge \varphi_{2,1}^{1,0}(t) + e^{k_j \bar{z}} d\bar{y}_{2,j}$

Table 11 – Definitions for setting the generators of the complex $B_\Gamma^{\bullet,\bullet}(t)$, see Equation (5) on p. 87, for the deformations induced by $t \frac{\partial}{\partial z} \otimes e^{k_1 z} \bar{y}_{2,1}$, of the Sawai and Yamada generalized manifold $\Gamma \backslash G$.

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Contents

Introduction	73
1 Deformations and cohomology	76
2 Applications: nilmanifolds	78
3 Applications: solvmanifolds	79
4 Example: deformations of the Nakamura manifold	84
5 Example: Sawai and Yamada generalized manifolds	85
6 Closedness and openness under holomorphic deformation	88
Acknowledgments	101
References	101
Contents	i