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Limited operators and differentiability

Mohammed Bachir[1]

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Abstract

We characterize the limited operators by differentiability of convex continuous functions. Given Banach spaces Y and X and a linear continuous operator $T : Y \to X$, we prove that T is a limited operator if and only if, for every convex continuous function $f : X \to \mathbb{R}$ and every point $y \in Y$, $f \circ T$ is Fréchet differentiable at $y \in Y$ whenever f is Gâteaux differentiable at $T(y) \in X$.

Keywords: Limited operators, Gâteaux differentiability, Fréchet differentiability, convex functions.

MSC: 47B07, 49J50, 58C20.

1 Introduction

A subset A of a Banach space X is called limited, if every weak* null sequence $(p_n)_n$ in X^* converges uniformly on A, that is,

$$\lim_{n\to+\infty} \sup_{x\in A} |\langle p_n, x\rangle| = 0.$$

We know that every relatively compact subset of X is limited, but the converse is false in general. A bounded linear operator $T : Y \to X$ between Banach spaces Y and X is called limited, if T takes the closed unit ball B_Y of Y to a limited subset of X. It is easy to see that $T : Y \to X$ is limited if and only if, the adjoint operator $T^* : X^* \to Y^*$ takes weak* null sequence to norm null sequence. For useful properties of limited sets and limited operators we refer to the papers Andrews (1979), Bourgain and Diestel (1984), Carrión, H. Galindo, and Lourenco (2006), and Schlumprecht (1987).

The goal of this paper, is to prove the result mentioned in the abstract (Theorem 1 on the next page), which gives a characterization of limited operators in terms of differentiability of convex continuous functions.

There exists a class of Banach spaces $(E, \|.\|_E)$ such that the canonical embedding $i : E \to E^{**}$ is a limited operator. This class contains in particular the space c_0 and

[1]Laboratoire SAMM 4543, Université Paris 1 Panthéon-Sorbonne, Centre P.M.F. 90 rue Tolbiac 75634 Paris cedex 13

any closed subspace F of c_0 (this class is also stable by product and quotient[2]). As consequence of Theorem 1, we prove in Corollary 1 on p. 65 that when i is a limited operator, then for each convex lower semicontinuous function $g : E^{**} \to \mathbb{R} \cup \{+\infty\}$, if g is Gâteaux differentiable at some point $a \in E$ which is in the interior of its domain, then the restriction of g to E is Fréchet differentiable at a. If moreover we assume that g is convex and weak* lower semicontinuous function, then using a result of Godefroy[3], we also get that the Gâteaux and Fréchet differentiability of g coincides at each point of $E \cap \int(\mathrm{dom}(g))$, where $\int(\mathrm{dom}(g))$ denotes the norm interior of $\mathrm{dom}(g)$.

This note is organized as follows. In Section 2, we give the proof of the main result Theorem 1 and some consequences. In Section 3 on p. 67 we give a canonical construction in infinite dimensional of convex Lipschitz continuous functions f for which there exists a point at which f is Gâteaux differentiable but not Fréchet differentiable.

2 Limited operators and differentiability

Recall that the domain of a function $f : X \to \mathbb{R} \cup \{+\infty\}$, is the set

$$\mathrm{dom}(f) := \{x \in X / f(x) < +\infty\}.$$

For a function f with $\mathrm{dom}(f) \neq \emptyset$, the Fenchel transform of f is defined on the dual space for all $p \in X^*$ by

$$f^*(p) := \sup_{x \in X} \{\langle p, x \rangle - f(x)\}.$$

The second transform $(f^*)^*$ is defined on the bidual X^{**} by the same formula. We denote by f^{**}, the restriction of $(f^*)^*$ to X, where X is identified to a subspace of X^{**} by the canonical embedding. Recall that the Fenchel theorem state that $f = f^{**}$ if and only if f is convex lower semicontinuous on X.

Definition 1 – We say that a function g on X^* has a norm-strong minimum (resp. weak*-strong minimum) at $p \in X^*$ if $g(p) = \inf_{q \in X^*} g(q)$ and $(p_n)_n \subset X^*$ norm converges (resp. weak* converges) to p whenever $g(p_n) \to g(p)$.

A norm-strong minimum and weak*-strong minimum are in particular unique.

Theorem 1 – *Let Y and X be two Banach spaces and $T : Y \to X$ be a linear continuous operator. Then, the following assertions are equivalent:*

[2] For more information, see Carrión, H. Galindo, and Lourenco, 2006, "Banach spaces whose bounded sets are bounding in the bidual".

[3] Godefroy, 1981, "Prolongement de fonctions convexes définies sur un espace de Banach E au bidual E^{**}".

(A_1) *The operator T is limited.*

(A_2) *For every convex lower semicontinuous function $f : X \to \mathbb{R} \cup \{+\infty\}$ and every $a \in Y$ such that $T(a)$ belongs to the interior of* $\mathrm{dom}(f)$*, we have that $f \circ T$ is Fréchet differentiable at $a \in Y$ with Fréchet-derivative $T^*(Q) \in Y^*$, whenever f is Gâteaux differentiable at $T(a) \in X$ with Gâteaux-derivative $Q \in X^*$.*

(A_3) *for every convex Lipschitz continuous function $f : X \to \mathbb{R}$, we have that $f \circ T$ is Fréchet differentiable at* 0 *whenever f is Gâteaux differentiable at* 0.

Proof.

(A_1) $\Longrightarrow$ (A_2). We can assume that $T \neq 0$. Since f is convex lower semicontinuous and $T(a)$ is in the interior of $\mathrm{dom}(f)$, there exists $r_a > 0$ such that f is Lipschitz continuous on the closed ball $B_X(T(a), r_a)$. It is well known that there exists a convex Lipschitz continuous function $\tilde{f}_a$ on X such that $\tilde{f}_a = f$ on $B_X(T(a), r_a)$[4]. It follows that $\tilde{f}_a \circ T = f \circ T$ on $B_Y(a, \frac{r_a}{\|T\|})$, since $T(B_X(a, \frac{r_a}{\|T\|}))$ is a subset of $B_X(T(a), r_a)$. Replacing f by $\frac{1}{L_a}\tilde{f}_a$ (where L_a denotes the Lipschitz constant of $\tilde{f}_a$), we can assume without loss of generality that f is convex 1-Lipschitz continuous on X. It follows that $\mathrm{dom}(f^*) \subset B_{X^*}$ (the closed unit ball of X^*).

Claim 1 – *Suppose that f is Gâteaux differentiable at $T(a) \in X$ with Gâteaux-derivative $Q \in X^*$, then the function $q \mapsto f^*(q) - \langle q, T(a)\rangle$ has a weak*-strong minimum on B_{X^*} at Q.*

Proof (of the claim). See Asplund and Rockafellar (1969, Corollary 1). □

Proof (return to Theorem 1 on the preceding page).

(A_1) $\Longrightarrow$ (A_2) (continue). Now, suppose by contradiction that $T^*(Q)$ is not the Fréchet derivative of $f \circ T$ at a. There exist $\varepsilon > 0$, $t_n \to 0^+$ and $h_n \in Y$, $\|h_n\|_Y = 1$ such that for all $n \in \mathbb{N}^*$,

$$f \circ T(a + t_n h_n) - f \circ T(a) - \langle T^*(Q), t_n h_n\rangle > \varepsilon t_n. \tag{1}$$

Let $r_n = t_n/n$ for all $n \in \mathbb{N}^*$ and choose $p_n \in B_{X^*}$ such that

$$f^*(p_n) - \langle p_n, T(a + t_n h_n)\rangle < \inf_{p \in B_{X^*}} \{f^*(p) - \langle p, T(a + t_n h_n)\rangle\} + r_n. \tag{2}$$

From (2) we get

$$f^*(p_n) - \langle p_n, T(a)\rangle < \inf_{p \in B_{X^*}} \{f^*(p) - \langle p, T(a)\rangle\} + 2t_n\|T\| + r_n.$$

[4]See e.g. Phelps, 1993, *Convex Functions, Monotone Operators and Differentiability*, Lemma 2.31.

This implies that $(p_n)_n$ is a minimizing sequence for the function $q \mapsto f^*(q) - \langle q, T(a)\rangle$ on B_{X^*}. Using the claim, we get that the sequence $(p_n)_n$ weak* converges to Q. Now, since T is a limited operator, we have

$$\|T^*(p_n - Q)\|_{Y^*} \to 0. \tag{3}$$

On the other hand, since

$$f(T(a+t_nh_n)) = f^{**}(T(a+t_nh_n)) := -\inf_{p\in B_{X^*}}\{f^*(p) - \langle p, T(a+t_nh_n)\rangle\},$$

using (2) we obtain that, for all $y \in Y$

$$\begin{aligned} f\circ T(a+t_nh_n) - \langle p_n, T(a+t_nh_n)\rangle &< -f^*(p_n) + r_n \\ &\le f\circ T(y) - \langle p_n, T(y)\rangle + r_n. \end{aligned}$$

Replacing y by a in the above inequality we obtain

$$f\circ T(a+t_nh_n) - \langle p_n, T(t_nh_n)\rangle \le f\circ T(a) + r_n. \tag{4}$$

Combining (1) and (4) we get, for all $n \in \mathbb{N}^*$,

$$\begin{aligned} \varepsilon &< \langle p_n, T(h_n)\rangle - \langle T^*(Q), h_n\rangle + \frac{r_n}{t_n} \\ &= \langle T^*(p_n), h_n\rangle - \langle T^*(Q), h_n\rangle + \frac{1}{n} \\ &\le \|T^*(p_n - Q)\|_{Y^*} + \frac{1}{n} \end{aligned}$$

which is a contradiction with (3). Hence, $f\circ T$ is Fréchet differentiable at a with Fréchet derivative $T^*(Q)$.

$(\mathsf{A}_2) \Longrightarrow (\mathsf{A}_3)$ is trivial.

$(\mathsf{A}_3) \Longrightarrow (\mathsf{A}_1)$. Let $(p_n)_n$ be a weak* null sequence in X^*. We want to prove that $\|T^*(p_n)\|_{Y^*} \to 0$. Let $f : X \to \mathbb{R}$ be the function defined for all $x \in X$ by

$$f(x) = \sup_{n\in\mathbb{N}^*}\left\{p_n(x) - \frac{1}{n}, 0\right\}.$$

Since $(p_n)_n$ is weak* null sequence in X^*, the convex function f is Lipschitz continuous and Gâteaux differentiable at 0 with Gâteaux derivative $\nabla f(0) = 0$[5]. By assumption $f\circ T$ is Fréchet differentiable at 0 with Fréchet derivative equal to 0. It follows from Asplund and Rockafellar (1969, Corollary 2) that $(f\circ T)^*$ has a norm-strong minimum at 0. Now, we prove that $(T^*(p_n))_n$ is a minimizing sequence for $(f \circ T)^*$, which will implies that $\|T^*(p_n)\|_{Y^*} \to 0$. Indeed, since $f(0) = 0$, we have

$$\begin{aligned} 0 = -f(0) &\leq \sup_{y\in Y}\{-f \circ T(y)\} := (f \circ T)^*(0) \\ &\leq \sup_{x\in X}\{-f(x)\} \\ &\leq 0. \end{aligned}$$

It follows that $(f \circ T)^*(0) = 0$. Since $(f \circ T)^*$ has a minimum at 0, we obtain

$$\begin{aligned} 0 = (f \circ T)^*(0) \leq (f \circ T)^*(T^*(p_n)) &:= \sup_{y\in Y}\{\langle T^*(p_n), y\rangle - f \circ T(y)\} \\ &= \sup_{y\in Y}\{\langle p_n, T(y)\rangle - f(T(y))\} \\ &\leq \sup_{x\in X}\{\langle p_n, x\rangle - f(x)\} \\ &= f^*(p_n) \\ &\leq \frac{1}{n}. \end{aligned}$$

It follows that $(f \circ T)^*(T^*(p_n)) \to 0 = (f \circ T)^*(0) = \min_{Y^*}(f \circ T)^*$. In other words, $(T^*(p_n))_n$ is a minimizing sequence for $(f \circ T)^*$. Since $(f \circ T)^*$ has a norm-strong minimum at 0, we obtain that $\|T^*(p_n)\|_{Y^*} \to 0$, which implies that T is a limited operator. □

Corollary 1 – *Suppose that the canonical embedding $i : E \to E^{**}$ is a limited operator (we use the identification $i(x) = x$). Let $g : E^{**} \to \mathbb{R} \cup \{+\infty\}$ be a convex lower semicontinuous function. Then, the restriction $g_{|E}$ is Fréchet differentiable on E at any point of $E \cap \int(\mathrm{dom}(g))$ at which g is Gâteaux differentiable. If moreover, we assume that g is convex weak* lower semicontinuous function. Then, Gâteaux and Fréchet differentiability coincides for g at each point of $E \cap \int(\mathrm{dom}(g))$, where $\int(\mathrm{dom}(g))$ denotes the norm interior of* $\mathrm{dom}(g)$.

Proof. Suppose that g is Gâteaux differentiable at a point $x \in E \cap \int(\mathrm{dom}(g))$. Since i is a limited operator, it follows from Theorem 1 on p. 62, that the restriction $g_{|E}$ is Fréchet differentiable on E at x. If moreover, we assume that g is convex weak* lower semicontinuous function, then by using Godefroy (1981, Proposition 5), we get that the Fréchet differentiability of $g_{|E}$ is preserved at points in $E \subset E^{**}$ by any weak* lower semicontinuous extension of $g_{|E}$ to E^{**}, in particular g is Fréchet differentiable at x. □

[5]See Borwein, Montesinos, and Vanderwerff, 2006, "Boundedness, Differentiability and Extensions of Convex Functions", Proposition 2.1.

We will call a Banach space X a Gelfand-Phillips space, if all limited sets in X are relatively norm-compact[6]. In this case, for every Banach space Y, if $T : Y \to X$ is a limited operator then it is a compact operator. We give the following characterization of compact operators by differentiability of locally Lipschitz function.

Corollary 2 – *Let Y be a Banach space and X be a Gelfand-Phillips space. Let $T : Y \to X$ be a linear continuous operator. Then, T is a compact operator if and only if, for every locally Lipschitz function $f : X \to \mathbb{R}$ and every point $y \in Y$, $f \circ T$ is Fréchet differentiable at $y \in Y$ whenever f is Gâteaux differentiable at $T(y) \in X$.*

Proof. Suppose that, for every locally Lipschitz function $f : X \to \mathbb{R}$ and every point $y \in Y$, $f \circ T$ is Fréchet differentiable at $y \in Y$ whenever f is Gâteaux differentiable at $T(y) \in X$. It follows from Theorem 1 on p. 62 that T is a limited operator. Since X is a Gelfand-Phillips space, then T is a compact operator. The converse follows from Bachir and Lancien (2003, Lemma 3.1). □

We obtain the Corollary 3 below, by combining Proposition 1 and a delicate result due to Zajicek[7], which says that in a separable Banach space, the set of the points where a convex continuous function is not Gâteaux differentiable, can be covered by countably many *d.c* (that is, delta-convex) *hypersurface*. Recall that in a separable Banach space Y, each set A which can be covered by countably many d.c hypersurfaces is σ-lower porous, also σ-directionally porous; in particular it is both Aronszajn (equivalent to Gauss) null and Γ-null. For details about this notions of small sets we refer to Zajicek (2005) and references therein. Note that a limited set in a separable Banach space is relatively compact[8].

Proposition 1 – *Let Y and X be Banach spaces and $T : Y \to X$ be a limited operator with a dense range. Let $f : X \to \mathbb{R}$ be a convex continuous function. Then $f \circ T$ is Gâteaux differentiable at $a \in Y$ if and only if, $f \circ T$ is Fréchet differentiable at $a \in Y$.*

Proof. Suppose that $f \circ T$ is Gâteaux differentiable at $a \in Y$. It follows that f is Gâteaux differentiable at $T(a)$ with respect to the direction $T(Y)$ which is dense in X. It follows (from a classical fact on locally Lipschitz continuous functions) that f is Gâteaux differentiable at $T(a)$ on X. So by Theorem 1 on p. 62, $f \circ T$ is Fréchet differentiable at $a \in Y$. The converse is always true. □

Corollary 3 – *Let Y be a separable Banach space, X be a Banach space and $T : Y \to X$ be a compact operator with a dense range. Let $f : X \to \mathbb{R}$, be a convex and continuous function. Then, the set of all points at which $f \circ T$ is not Fréchet differentiable can be covered by countably many d.c hypersurfaces.*

Proof. The proof is a consequence of Zajicek (1979, Theorem 2) and Proposition 1.□

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[6] See Schlumprecht, 1987, *Limited sets in Banach spaces*.

[7] See Zajicek, 1979, "On the differentiation of convex functions in finite and infinite dimensional spaces", Theorem 2.

[8] See Bourgain and Diestel, 1984, "Limited operators and strict cosingularity".

3 Canonical construction of PGNF-function

A real valued function f on a Banach space will be called a PGNF-function[9] if there exists a point at which f is Gâteaux but not Fréchet differentiable. A JN-sequence[10] is a sequence $(p_n)_n$ in a dual space Y^* that is weak* null and $\inf_n \|p_n\| > 0$. There exist different way to build a PGNF-function in infinite dimentional Banach spaces. We can find examples of such constructions in Borwein and Fabian (1993). We present below a different method for constructing a PGNF-function on a Banach space X canonically from a JN-sequence.

We need the following probably known lemma. Since we do not know a specific reference to this lemma, we give its proof, for completeness. If B is a subset of a dual Banach space X^*, we denote by $\overline{\text{co}}^{w^*}(B)$ the weak* closed convex hull of B.

Lemma 1 – *Let X be a Banach space and K be a subset of X^*.*

1. *Suppose that K is norm separable, then there exists a sequence $(x_n)_n$ in the unit sphere S_X of X which separate the points of K i.e. for all $p, p' \in K$, if $\langle p, x_n \rangle = \langle p', x_n \rangle$ for all $n \in \mathbb{N}$, then $p = p'$. Consequently, if K is a weak* compact and norm separable set of X^*, then the weak* topology of X^* restricted to K is metrizable.*

2. *Let $(p_n)_n$ be a weak* null sequence in X^*. Then, the set $\overline{\text{co}}^{w^*}\{p_n : n \in \mathbb{N}\}$ is convex weak* compact and norm separable.*

Proof.

1. Since K is norm separable, then $K - K := \{a - b/(a,b) \in K \times K\}$ is also norm separable and so there exists a sequence $(q_n)_n$ of $K - K$ which is dense in $K - K$. According to the Bishop-Phelps theorem[11], the set

$$D = \{r \in X^* \mid r \text{ attains its supremum on the sphere } S_X\}$$

is norm-dense in the dual X^*. Thus, for each $n \in \mathbb{N}^*$, there exists $r_n \in D$ such that $\|q_n - r_n\| < \frac{1}{n}$. For each $n \in \mathbb{N}^*$, let $x_n \in S_X$ be such that $\|r_n\| = \langle r_n, x_n \rangle$. We claim that the sequence $(x_n)_n$ separate the points of K. Indeed, let $q \in K - K$ and suppose that $\langle q, x_n \rangle = 0$, for all $n \in \mathbb{N}^*$. There exists a subsequence $(q_{n_k})_k \subset K - K$ such that $\|q_{n_k} - q\| < \frac{1}{k}$ for all $k \in \mathbb{N}^*$ and so we have $\|r_{n_k} - q\| < \frac{1}{n_k} + \frac{1}{k}$. It follows that

(Cont. next page)

$$\|r_{n_k}\| = \langle r_{n_k}, x_{n_k} \rangle$$

[9]See Borwein and Fabian, 1993, “On convex functions having points of Gâteaux differentiability which are not points of Féchet-differentiability”.

[10]Due to Josefson-Nissenzweig theorem, see Diestel, 1984, *Sequences and series in Banach spaces*, Chapter XII.

$$= \langle r_{n_k}, x_{n_k}\rangle - \langle q, x_{n_k}\rangle$$
$$\leq \|r_{n_k} - q\|$$
$$< \frac{1}{n_k} + \frac{1}{k}.$$

Hence, for all $k \in \mathbb{N}^*$, $\|q\| \leq \|q - r_{n_k}\| + \|r_{n_k}\| < 2(\frac{1}{n_k} + \frac{1}{k})$, which implies that $q = 0$, and so that $(x_n)_n$ separate the points of K. Now, if we assume that K is separable and weak* compact subset of X^*, it is then classical to see that the weak* topology on K is induced from the metric

$$d(p,p') := \sum_{n=0}^{+\infty} 2^{-n} \frac{|\langle p - p', x_n\rangle|}{1 + |\langle p - p', x_n\rangle|}.$$

Hence the weak* topology on K is metrizable.

2. Let $(p_n)_n$ be a weak* null sequence in X^* and set $K = \overline{\text{co}}^{w^*}\{p_n : n \in \mathbb{N}\}$. Clearly K is a convex and weak* compact subset of X^*. According to Haydon's theorem[12] the weak* compact convex set K is the norm closed convex hull of its extreme points whenever $\text{ex}(K)$ (the set of extreme points of K) is norm separable. By the Milman theorem[13] $\text{ex}(K) \subset \overline{\{p_n : n \in \mathbb{N}\}}^{w^*} = \{p_n : n \in \mathbb{N}\} \cup \{0\}$ so that $\text{ex}(K)$ is norm separable and, hence, by Haydon's theorem, K itself is weak* compact, convex, and norm separable. □

We also need the following proposition.

Proposition 2 – *Let X be a Banach space and K be a weak* compact subset of X^* containing 0. Suppose that there exists a sequence $(x_n)_n \subset S_X$ that separates the points of K (in this case K is weak* metrizable). Then, the function $h : X^* \to \mathbb{R}$ defined by:*

$$h(x^*) = \left(\sum_{k\geq 0} 2^{-k}(\langle x^*, x_k\rangle)^2\right)^{\frac{1}{2}}, \ \forall x^* \in X^*,$$

has the following properties:

1. *h is a continuous seminorm on X^**

2. *h is weak* lower semicontinuous and sequentially weak* continuous,*

[11]See Bishop and Phelps, 1961, "A proof that every Banach space is subreflexive".

[12]See Haydon, 1976, "An extreme point criterion for separability of a dual Banach space, and a new proof of a theorem of Corson", Theorem 3.3.

[13]See Phelps, 1966, *Lectures on Chaquet's theorem*, p. 9.

3. *the restriction* $h_{|K}$ *of* h *to* K *has a weak***-strong minimum at* 0, *with* $\min_K h_{|K} = h(0) = 0$.

Proof. It is clear that h is a seminorm, and since $h(x^*) \leq \|x^*\|$ for all $x^* \in X^*$, it is also continuous. Since h is the supremum of a sequence of weak* continuous functions, it is weak* lower semicontinuous. On the other hand, the series $\sum_{k\geq 0} 2^{-k}(\langle x^*, x_k\rangle)^2$ uniformly converges on bounded sets of X^* and the maps $\hat{x}_k : x^* \mapsto \langle x^*, x_k\rangle$ are weak* continuous for all $k \in \mathbb{N}$, it follows that h is sequentially weak* continuous. If $p \in K$ and $h(p) = 0$, then $\langle p, x_k\rangle = 0$ for all $k \in \mathbb{N}$ which implies that $p = 0$, since the sequence $(x_k)_k$ separate the points of K. Hence, the restriction of h to K has a unique minimum at 0. This minimum is necessarily a weak*-strong minimum since K is weak* metrizable, this follows from a general fact which say that for every lower semicontinuous function on a compact metric space (K, d), a unique minimum is necessarily a strong minimum for the metric d in question. □

Canonical construction of PGNF-function. Let X be an infinite dimensional Banach space. Given a JN-sequence $(p_n)_n \subset X^*$, we set $K = \overline{\text{co}}^{w^*}\{p_n : n \in \mathbb{N}\}$. Using Lemma 1 on p. 67, there exists a sequence $(x_n)_n \in S_X$ which separates the points of K, and by Proposition 2 on the preceding page, the function $h : X^* \to \mathbb{R}$ defined by:

$$h(x^*) = \left(\sum_{k\geq 0} 2^{-k}(\langle x^*, x_k\rangle)^2\right)^{\frac{1}{2}}, \ \forall x^* \in X^*,$$

is weak* lower semicontinuous and weak* sequentially continuous such that $h_{|K}$ has a weak*-strong minimum at 0.

Since $(p_n)_n$ weak* converges to 0 and h is weak* sequentially continuous, then $(p_n)_n$ is a minimizing sequence for $h_{|K}$. Since $(p_n)_n$ is a JN-sequence, it follows that 0 is not a norm-strong minimum for $h_{|K}$. Define the function f by

$$f(x) = (h + \delta_K)^*(\hat{x}), \ \forall x \in X,$$

where δ_K denotes the indicator function, which is equal to 0 on K and equal to $+\infty$ otherwise and where for each $x \in X$, we denote by $\hat{x} \in X^{**}$ the linear and weak* continuous map $x^* \mapsto \langle x^*, x\rangle$ for all $x^* \in X^*$. Then f is convex Lipschitz continuous, Gâteaux differentiable at 0 (since $h + \delta_K$ has a weak*-strong minimum[14]) but is not Fréchet differentiable at 0 (because 0 is not a norm-strong minimum for $h + \delta_K$[15].

Remark 1 – Let Y be a Banach space. Then the following assertions are equivalent.

[14]See Asplund and Rockafellar, 1969, "Gradients of convex functions", Corollary 1.
[15]See ibid., Corollary 2.

(A_1) Y is infinite dimensional.

(A_2) There exists a convex weak* compact and norm separable subset K of Y^* contaning 0 and a continuous seminorm h on Y^* which is weak* lower semicontinuous and weak* sequentially continuous, such that the restriction $h_{|K}$ has a weak*-strong minimum but not norm-strong minimum at 0.

(A_3) There exists a Banach space X and a linear continuous non-limited operator $T : Y \to X$.

(A_4) There exists on Y a convex Lipschitz continuous PGNF-function.

Proof.

(A_1) $\Longrightarrow$ (A_2). We know from the Josefson-Nissenzweig theorem[16] that there exists a weak* null sequence $(p_n)_n$ in Y^* such that $\inf_n \|p_n\| > 0$. Set $K = \overline{\text{co}}^{w^*}\{p_n : n \in \mathbb{N}\}$. By Lemma 1 on p. 67, the set K is convex weak* compact and norm separable. On the other hand, from Proposition 2 on p. 68, there exists a continuous seminorm h which is weak* lower semicontinuous and weak* sequentially continuous on Y^* such that the restriction of h to K has a weak*-strong minimum at 0. It remains to show that 0 is not a norm-strong minimum for $h_{|K}$. Indeed, since $(p_n)_n$ is weak* null and h is weak* sequentially continuous, then $\lim_n h(p_n) = h(0) = \min_K h$. So $(p_n)_n$ is a minimizing sequence for $h_{|K}$ which not norm converges to 0 since $\inf_n \|p_n\| > 0$. Hence, 0 is not a norm-strong minimum for $h_{|K}$.

(A_2) $\Longrightarrow$ (A_3). Since 0 is not a norm-strong minimum for the restriction $h_{|K}$, there exists a sequence $(p_n)_n$ that minimize h on K but $\|p_n\| \nrightarrow 0$. Since $h_{|K}$ has a weak*-strong minimum at 0, it follows that $(p_n)_n$ weak* converges to 0. Hence, $(p_n)_n$ weak* converges to 0 but $\|p_n\| \nrightarrow 0$. It follows that the identity operator on Y is not limited, so we can take $X = Y$.

(A_3) $\Longrightarrow$ (A_4). If there exists a Banach space X and a non-limited operator $T : Y \to X$, by using Theorem 1 on p. 62, there exists a convex Lipschitz continuous function $f : X \to \mathbb{R}$ and a point $y \in Y$ such that f is Gâteaux differentiable at $T(y) \in X$ but $f \circ T$ is not Fréchet differentiable at y. So $f \circ T$ is Gâteaux but not Fréchet differentiable at y. Hence, $f \circ T$ is a convex Lipschitz continuous PGNF-function on Y.

(A_4) $\Longrightarrow$ (A_1) is well known. □

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[16] See Diestel, 1984, *Sequences and series in Banach spaces*, Chapter XII.

References

Andrews, K. T. (1979). "Dunford-Pettis sets in the space of Bochner integrable functions". *Math. Ann.*, pp. 35–41 (cit. on p. 61).

Asplund, E. and R. T. Rockafellar (1969). "Gradients of convex functions". *Trans. Amer. Math. Soc.*, pp. 443–467 (cit. on pp. 63, 64, 69).

Bachir, M. and G. Lancien (2003). "On the composition of differentiable functions". *Canadian Bull. of Math.*, pp. 481–494 (cit. on p. 66).

Bishop, E. and R. R. Phelps (1961). "A proof that every Banach space is subreflexive". *Bull. Amer. Math. Soc.*, pp. 97–98 (cit. on p. 68).

Borwein, J. M. and M. Fabian (1993). "On convex functions having points of Gâteaux differentiability which are not points of Féchet-differentiability". *Can. J. Math.*, pp. 1121–1134 (cit. on p. 67).

Borwein, J. M., V. Montesinos, and J. D. Vanderwerff (2006). "Boundedness, Differentiability and Extensions of Convex Functions". *J. Convex Analysis*, pp. 587–602 (cit. on p. 65).

Bourgain, J. and J. Diestel (1984). "Limited operators and strict cosingularity". *Math. Nachr.*, pp. 55–58 (cit. on pp. 61, 66).

Carrión, P. H. Galindo, and M. L. Lourenco (2006). "Banach spaces whose bounded sets are bounding in the bidual". *Annales Academiae Scientiarum Fennicae Mathematica*, pp. 61–70 (cit. on pp. 61, 62).

Diestel, J. (1984). *Sequences and series in Banach spaces*. Graduate texts in Mathematics, Springer Verlag, N.Y., Berlin, Tokyo (cit. on pp. 67, 70).

Godefroy, G. (1981). "Prolongement de fonctions convexes définies sur un espace de Banach E au bidual E^{**}". *C. R. Acad. Sci. Paris*, pp. 371–374 (cit. on pp. 62, 65).

Haydon, R. (1976). "An extreme point criterion for separability of a dual Banach space, and a new proof of a theorem of Corson". *Quart. J. Math. Oxford Ser.*, pp. 379–385 (cit. on p. 68).

Phelps, R. R. (1966). *Lectures on Chaquet's theorem*. Van Nostrand, Princeton, N. J. (cit. on p. 68).

Phelps, R. R. (1993). *Convex Functions, Monotone Operators and Differentiability*. Lecture Notes in Mathematics 1364, Springer-Verlag, Berlin (cit. on p. 63).

Schlumprecht, T. (1987). *Limited sets in Banach spaces*. Dissertation, München (cit. on pp. 61, 66).

Zajicek, L. (1979). "On the differentiation of convex functions in finite and infinite dimensional spaces". *Czechoslovak Math. J.*, pp. 340–348 (cit. on p. 66).

Zajicek, L. (2005). "On sigma-porous sets in abstract spaces". *Abstract and Applied Analysis*, pp. 509–534 (cit. on p. 66).

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