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Semi-groups and the mean reverting SABR stochastic volatility model

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Abstract

We continue our study of solutions to linear parabolic partial differential equations (PDEs) by means of an asymptotic method that is based on approximate Green functions. A substantial part of this method is devoted to constructing the approximate Green function. In this paper, we approximate the Green function (or heat kernel) by asymptotically developing it in a small parameter other than time. While the method is general, in order to better illustrate it, we concentrate on the λ-SABR partial differential equation (PDE for short), which we study in detail. The λSABR PDE is a particular evolution PDE that arises in applications to stochastic volatility models (Hagan, Kumar, Lesniewski, and Woodward, Wilmott Magazine, 2002). Concretely, we study the generation and approximation of several semi-groups associated to the SABR PDE, some of which are non-standard because their generators are *not* uniformly elliptic and have *unbounded coefficients.* These type of generators appear also in the study of quasi-linear evolution equations. For some of the resulting semi-groups, we obtain *explicit* formulas by using a general technique based on solvable Lie groups that we develop in this paper. We thus obtain a simple, explicit approximation for the solution of the λ-SABR PDE and we prove explicit error bounds. In view of the potential applications, we have tried to make our paper as self-contained as reasonably possible.

Keywords: Degenerate parabolic equations, solvable Lie algebra, semi-groups, fundamental solution, option pricing, SABR model, mean reversion.

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1 Introduction

We continue our study begun in Constantinescu et al. (2010) of solutions to linear parabolic partial differential equations (PDE for short) by means of an asymptotic method that is based on approximate Green's functions. We will refer to this method, informally, as the "Green's function method." Among the first papers to use this method, we mention Hagan, Kumar, et al. (2002) and Henry-Labordère (n.d.), their motivation stemming from financial applications. In this paper, we develop a variant of this method developed in Constantinescu et al. (2010). In that paper, we developed a systematic method to derive asymptotic expansions of the Green function (or heat kernel) using time as a small parameter exploiting parabolic rescaling, thus obtaining an approximation of the Green function valid in principle at any order. The approximation of the solution of the parabolic equation is obtained by integrating the approximate Green function with respect to the initial data. In this work, the asymptotic expansion is in a small parameter other than time. From a theoretical point of view, this set up presents new challenges. To keep our treatment self contained, we illustrate the method for the λSABR PDE, (1).

1.1 Formulation of the problem and main results

We use the Green method to study the λSABR PDE, that is, the parabolic partial differential equation:

$$\partial_t u - \kappa(\theta - \sigma)\partial_\sigma u - \frac{\sigma^2}{2}\Big[(\partial_x^2 u - \partial_x u) + 2\nu\rho\partial_x\partial_\sigma u + \nu^2\partial_\sigma^2 u\Big] = 0. \quad (1)$$

Here $u = u(t, \sigma, x)$, where $t \geq 0$, $x \in \mathbb{R}$, and $\sigma > 0$, while $|\rho| < 1$, κ, $\nu > 0$ are a given constant parameter. The equation is complemented with initial and boundary conditions to be determined. We shall refer to this PDE as the *λSABR PDE*; it has been first introduced in financial applications[4]. Originally, it was formulated in terms of a variable S that had the meaning of the (forward) stock price. Our formulation is obtained after the substitution $S = e^x$. However, no closed-form solution formula is known. Hence, our goal is to obtain an asymptotic expansion of the solution operators in powers of ν considered as the small parameter.

Our theoretical results, exemplified by the case of the SABR PDE, are obtained using the theory of semi-groups of operators. The main difficulty here is that the principal term in the ν-expansion is a degenerate operator with unbounded coefficient. Moreover, some of our semi-groups combine parabolic and hyperbolic features. We reconcile these difficulties in our paper by considering solvable Lie algebras.

[4] Hagan, Kumar, et al., 2002, "Managing smile risk";
Hagan, Lesniewski, and Woodward, 2015, "Probability distribution in the SABR model of stochastic volatility".

One of the main results in this paper, explained below, is an explicit approximation formula with a rigorous error bound for the solution of λSABR PDE. This result is relevant because the λSABR PDE is interesting in itself and has been studied in many other papers, including Cheng, Mazzucato, and Nistor (n.d.), Grishchenko, Han, and Nistor (n.d.), Hagan, Kumar, et al. (2002), Henry-Labordère (n.d.), and Lorig, Pagliarani, and Pascucci (2013). We stress, however, that the general framework of the paper and several of the results we obtain apply to a more general setting.

We shall denote the generator of Equation (1) by

$$L := \frac{\sigma^2}{2}\Big[(\partial_x^2 - \partial_x) + 2\nu\rho\partial_x\partial_\sigma + \nu^2\partial_\sigma^2\Big] + \kappa(\theta-\sigma)\partial_\sigma\,, \tag{2}$$

a notation that would remain in place throughout the paper. We let

$$L_0 := \frac{\sigma^2}{2}(\partial_x^2 - \partial_x) + \kappa(\theta-\sigma)\partial_\sigma\,,\; L_1 := \rho\sigma^2\partial_x\partial_\sigma\,, \quad \text{and} \quad L_2 := \frac{1}{2}\sigma^2\partial_\sigma^2\,, \tag{3}$$

so that $L = L_0 + \nu L_1 + \nu^2 L_2$. This decomposition is the basis of our asymptotic expansion in powers of ν for the solution operator e^{tL} to equation (1). In practical applications, ν is approximately equal to 1, which is not immediately thought of as a "small number." However, the numerical tests performed by one of the authors[5], the approximation established in this work performs well even for values of the parameter ν close to 1.

The point of departure for our approach is a careful study of the operator L_0. To this end, we further decompose L_0 as $L_0 := A + \frac{\sigma^2}{2}B$, where:

$$A := \kappa(\theta-\sigma)\partial_\sigma\,, \quad B := \partial_x^2 - \partial_x. \tag{4}$$

We study separately the semi-groups generated by A, B, and L_0. We shall use the standard notation for semi-groups. That is, throughout the paper, if P is a linear operator whose closure generates a strongly continuous (or c_0) semi-group, we shall denote the semi-group it generates by e^{tP}, $t \geq 0$, as usual. We recall that then the (mild) solution to the abstract Initial Value Problem (IVP) $\partial_t u - Pu = 0$, $u(0) = h$, is given by $u = e^{tP}h$. (See the Appendix for the various types of "solutions" considered in this paper).

Our *first main result* is to show that L_0 generates c_0 semi-groups e^{tL_0} on various weighted Sobolev spaces provided that we restrict the variable σ to a *bounded interval*,

$$\sigma \in I := (\alpha, \beta)\,, \quad \text{where } 0 < \alpha < \theta < \beta < \infty\,. \tag{5}$$

[5]Grishchenko, Han, and Nistor, n.d., "A Volatility-of-Volatility Expansion of the Option Prices in the SABR Stochastic Volatility".

Moreover, we obtain an explicit formula for e^{tL_0}, which justifies trying to approximate the solution $u(t)$ with $e^{tL_0}u(0)$. We stress that the semi-group associated to $L_0 := A + \frac{\sigma^2}{2}B$ is neither parabolic nor hyperbolic, so it does not fit into the classical frameworks that prove the generation of this type of semi-groups. In fact, we provide a *general* method to study semi-groups similar to those generated by L_0, which may have applications to quasi-linear evolution equations. While the semi-group generated by L_0 can also be studied by an elementary (but complicated) change of variables, our more general approach provides some subtle mapping properties for e^{tL_0} that seem difficult to obtain by a change of variables.

Unlike the time evolution associated to L_0, the time evolution equations associated to the operators A, B, and L have a definite type (the ones associated to L and B are parabolic, while the one associated to A is hyperbolic). Let $I := (\alpha, \beta)$, $0 < \alpha < \beta < \infty$ as above and restrict to $\Omega := I \times \mathbb{R}$ (from $(0,\infty)\times\mathbb{R}$). Although Ω is unbounded, classical arguments can be adapted to give that the operators L, A, and B generate c_0 semi-groups on suitable spaces of functions on Ω and that, in fact, the semi-groups generated by L and B are analytic[6]. We stress here that for L it is essential that we restrict to I, as the behavior on the full domain seems to be quite different (and to require significant additional insight). However, these issues disappear if $\kappa = 0$ (that is, if there is no mean-reverting term). The fact that L generates a c_0 semi-group follows from the results of Mazzucato and Nistor if $\kappa = 0$.

Since the existence of the semi-group generated by L_0 does not follows from standard arguments alone, we employ a new, general strategy, which allows us to establish that L_0 (and many other similar operators) generates a c_0 semi-group e^{tL_0} with an *explicit* kernel. The explicit formula for the distribution kernel of e^{tL_0} is obtained from those of the semi-groups e^{tB} and e^{tA}. A key observation for us is that *the operators A and $\frac{\sigma^2}{2}B$ generate a solvable, finite-dimensional Lie algebra.*

We prove several mapping properties of the semi-groups generated by L and L_0. This allow us to estimate the difference $e^{tL}h - e^{tL_0}h$, which is our *second main result.* More specifically, we derive an error estimate of the form:

$$\|e^{tL}h - e^{tL_0}h\|_{L^2_\lambda} \le C\nu\Big(\|h\|_{L^2_\lambda} + \nu\|\partial_\sigma h\|_{L^2_\lambda}\Big), \tag{6}$$

for $\nu \in (0,1]$ and with a constant C, possibly dependent on L and κ, but not on h and ν (see Theorem 6 on p. 144 for a complete statement). Above, L^2_λ denotes an exponentially weighted Lebesgue space (see Section 2.1 on p. 124). The method of proof of this estimate is to combine the usual perturbative argument and our mapping properties with the commutator method developed by two of the authors[7].

[6] Amann, 1995, *Linear and quasilinear parabolic problems. Vol. I*;
Lunardi, 1995, *Analytic semigroups and optimal regularity in parabolic problems*;
Pazy, 1983, *Semigroups of linear operators and applications to partial differential equations*.

[7] Cheng, Costanzino, et al., 2011, "Closed-form asymptotics and numerical approximations of 1D parabolic equations with applications to option pricing";

We notice that in many applications h is independent of σ, so our estimate is better in those cases.

To summarize, we interested in this paper both in closed form approximations of the solution *u and* in theoretical properties of the λSABR PDE that will rigorously justify numerical methods for the λSABR PDE. Thus we are interested in proving that L and suitable operators associated to L *generate strongly continuous* (or c_0) *semi-groups* of operators and in the properties and approximations of these semi-groups. The main difficulties that we encounter are that L (as well as other auxiliary operators) is not uniformly elliptic, it has unbounded coefficients, acts on an unbounded (spatial) domain, and the initial condition has exponential growth.

1.2 Background and previous results

The primary application of the λSABR PDE is in pricing of options in financial mathematics. Indeed, the model takes its name from "Stochastic Alpha Beta Rho"[8], where α, β, and ρ refer to some parameters in the model, whereas the λ means that it includes mean reversion[9]. Mean reversion makes the model more accurate, but also more difficult to solve explicitly. The initial condition for this equation has then typically *exponential* growth in x, which naturally lead us to consider exponentially weighted Sobolev spaces.

Quite recently, H. Amann has initiated a program of studying evolution equations on non-compact domains of the form used in this paper (manifolds with boundary and bounded geometry), see Amann (2016, 2017) and the references therein. This is relevant because the fact that our PDEs are defined on unbounded domains is one of the main difficulties in studying them. Earlier related results were obtained, for example, in Browder (1960/1961). See Ammann, Große, and Nistor (n.d.) for a review of manifolds with boundary and bounded geometry. Two of the auxiliary semi-groups used in this paper (e^{tB} and e^{tA}) fall into the scope of the results of those papers, and we thus take advantage of that to slightly simplify the presentation in certain places. However, the λSABR PDE (1) does not fall into a standard framework, and this probably explains why so few rigorous, theoretical results exist for this PDE. The related PDE $\partial_t u - L_0 u$, used to approximate the solution of the λSABR PDE (1) also does not fall into the scope of the results mentioned above. In fact, L_0 is a degenerate operator, in the sense that the diffusion matrix associated to L_0 is not of full rank. Moreover, the operator $\partial_t - L_0$ is not hypoelliptic, in particular the distributional kernel of the fundamental solution of $\partial_t - L_0$ does not agree with a smooth function for $t > 0$. See also Ammann, Große, and Nistor (n.d.), Christian, Omar, and Michal (n.d.), Disconzi, Shao, and Simonett

Cheng, Mazzucato, and Nistor, n.d., "Approximate solutions to second order parabolic equations II: time-dependent coefficients".

[8] Hagan, Kumar, et al., 2002, "Managing smile risk".

[9] Henry-Labordère, 2009, *Analysis, geometry, and modeling in finance*.

(n.d.), Große and Schneider (2013), Kordyukov (1988), and Mazzucato and Nistor (2006) for more related results on PDEs on manifolds with bounded geometry.

In applications, the solution u of the λSABR PDE has the meaning of the option price (after a suitable change of coordinates) and the initial condition $u(0) = u(0,x,\sigma)$ is the pay-off of the option. For instance, for European Calls, one has $u(0,x,\sigma) = |e^x - K|_+ := \max\{e^x - K, 0\}$, where K is a parameter associated to the given option contract (the "strike"). We thus notice that $u(0)$ has an exponential growth in x, which is an additional difficulty. However, $\partial_\sigma u(0) = 0$, which makes our approximation result of Equation (6) especially convenient (for this particular class of initial conditions). In practice, one is interested in *very fast* approximations of u. Of great importance are, thus, the so called "closed form" approximations, where u is approximated by an explicit formula, without using iterative methods. For $\kappa = 0$, the λSABR PDE reduces to the classical Black-Scholes-Merton PDE, for which an explicit solution (involving the normal distribution function $\sqrt{2\pi}N(x) := \int_{-\infty}^{x} e^{-t^2/2} dt$) exists. The approach used for the Black-Scholes-Merton PDE does not extend to the λSABR PDE, however, since the distributional kernel of the solution operator for $\partial_t - L$ is not known in explicit form.

Because of the lack of explicit solutions, new methods were devised. We would like to first mention here the pioneering work of Lesniewski and his collaborators[10] and the ground breaking work on Henry-Labordère on heat kernel asymptotics[11]. Our method extends these results. A similar method was developed by Pagliarani and Pascucci (2012) and the references therein. Heat kernel asymptotics were employed in this context also by Gatheral, Hsu, et al. (2012) and Gatheral and Wang (2012). See also Choulli, Kayser, and Ouhabaz (2015), Feehan and Pop (2015), Hilber et al. (2013), Jacquier and Lorig (2015), Lejay, Lenôtre, and Pichot (2015), and Nakagawa et al. (2014). See Hilber et al. (2013) for an introduction to the Finite Element Method in Computational Finance.

Similar ideas are used in analysis on polyhedral domains (see Ammann, Ionescu, and Nistor (2006), Bacuta, Nistor, and Zikatanov (2005), Costabel, Dauge, and Nicaise (2012), and Costabel, Dauge, and Schwab (2005) for some relevant results in this direction). In fact, the λSABR PDE is formally similar to the *edge* differential operators, as encountered, for example, in Apel and Nicaise (1998), Costabel and Dauge (1993), Dauge (1988), and Li (2009) whereas the Black-Scholes-Merton PDE is similar to the *cone* differential operators[12].

[10] Hagan, Kumar, et al., 2002, "Managing smile risk";
Hagan, Lesniewski, and Woodward, 2015, "Probability distribution in the SABR model of stochastic volatility".

[11] Henry-Labordère, 2009, *Analysis, geometry, and modeling in finance*.

[12] Dauge, 1988, *Elliptic boundary value problems on corner domains*;
Kondrat′ev, 1967, "Boundary value problems for elliptic equations in domains with conical or angular points";
Kozlov, Maz′ya, and Rossmann, 2001, *Spectral problems associated with corner singularities of solutions to elliptic equations*.

1.3 Contents of the paper

The paper is organized as follows. Section 2 on the next page contains the result that the operators L and B generate c_0 semi-groups of operators when restricted to suitable sets of the form $\Omega := I \times \mathbb{R}$, $\overline{I} \subset (0, \infty)$. Our approach is based on the Lumer-Phillips theorem, which works well in this situation since the operators L and B are both strongly elliptic with bounded coefficients once restricted to Ω, so we can use some standard techniques on evolution equation[13] to study them. The main difficulty encountered here in using the general theory is that we employ exponentially weighted function spaces and that we work on certain non-compact domains (with boundary). We thus treat in detail the mapping properties of the semi-groups generated by L and B. These results are needed to prove in Section 3 on p. 129 that L_0 also generates a c_0 semi-group in spite of the fact that it is not elliptic. From the explicit formula of e^{tA}, we obtain also an explicit formula for e^{tL_0}, by exploiting the commutator identities that A and $f(\sigma)B$ satisfy. This method applies, in fact, to a much larger class of operators, and we prove several results in this direction in Section 3.2 on p. 135. The main technique here is to use the observation that the operators A and $\sigma^2 B$ generate a *solvable Lie* algebra of operators. (This is true also for A and $f(\sigma)B$, for suitable functions f, as well as for other, more general operators. The techniques developed in this paper will thus be useful for the study of other PDEs as well.) The hyperbolic nature of the operator A and the local in σ nature of the semi-group generated by operators of the form $f(\sigma)B$ lead to well-posedness results in many other types of function spaces. In particular, we obtain the existence of classical solutions of $\partial_t u - L_0 u$ on $\mathbb{R}_+ \times \mathbb{R}$, for some rather general initial data. The last section, Section 4 on p. 140, contains the derivation of mapping properties and norm estimates for the semi-groups generated by L_0 and by the other operators. These mapping properties are then used to prove the error estimate (6). Lastly, in the Appendix, we review a few needed facts on evolution equations and semi-groups of operators and we introduce the exponentially weighted spaces used in this work.

[13] Amann, 1995, *Linear and quasilinear parabolic problems. Vol. I*;
Henry, 1981, *Geometric theory of semilinear parabolic equations*;
Lunardi, 1995, *Analytic semigroups and optimal regularity in parabolic problems*;
Martin, 1987, *Nonlinear operators and differential equations in Banach spaces*;
Pazy, 1983, *Semigroups of linear operators and applications to partial differential equations.*

2 The semi-group generated by L and B

In this section, we give a short proof that the operators L and B, defined in (2) and (3) respectively, generate analytic semi-group on exponentially weighted Sobolev spaces, using the Lumer-Phillips theorem. The methods we use are standard in classical Sobolev spaces, although the theory is somewhat less developed in exponentially weighted spaces (see, however, Ammann, Ionescu, and Nistor 2006; Große and Schneider 2013). The fact that L and B generate analytic semi-groups on the given weighted Sobolev spaces can also be obtained from the result in Amann (2016, 2017) or Browder (1960/1961). Our approach is, however, more direct and more elementary. We also provide additional mapping properties for the operators L and B. For the reader's sake, we include full details for the operator L.

2.1 Notation: function spaces and semi-groups

We introduce here the function spaces that we need in this paper and recall some of their main properties. We also fix the notation for semi-groups. Let $\Omega \subset \mathbb{R}^d$ be an open subset, as in the previous subsection, and let $w \in L^1_{loc}(\Omega)$ satisfy $w \geq 0$. If X is any Banach space of functions on Ω with norm $\|\cdot\|_X$, we define

$$wX := \{w\xi,\ \xi \in X\}, \tag{7}$$

with the norm $\|w\xi\|_{wX} := \|\xi\|_X$. Thus, if $p < \infty$, if $X = L^p(\Omega, d\mu)$, and if $w > 0$ almost everywhere with respect to μ, $\mu \geq 0$, then $wX = L^p(\Omega, w^{-1/p} d\mu)$. Of course, for any linear operator T we have that $T\colon wX \to wX$ is bounded if, and only if $w^{-1}Tw\colon X \to X$ is bounded. In fact, these two operators are unitarily equivalent. For example, let $\langle x \rangle := \sqrt{1+x^2}$ be the usual *Japanese bracket*. Let $\lambda \in \mathbb{R}$ and $m \in \mathbb{Z}_+$ and

$$\begin{aligned} H^m_\lambda(\mathbb{R}) := e^{\lambda\langle x\rangle} H^m(\mathbb{R}) &= \Big\{f : \mathbb{R} \to \mathbb{C},\ e^{-\lambda\langle x\rangle} f \in H^m(\mathbb{R})\Big\} \\ &= \Big\{f : \mathbb{R} \to \mathbb{C},\ e^{-\lambda\langle x\rangle} \partial^i f \in L^2(\mathbb{R}),\ i \leq m\Big\}, \end{aligned} \tag{8}$$

where the last equality is valid due to the fact that the weight $w(x) := e^{\lambda\langle x\rangle}$ has the property that $w^{-1}\partial^i w$ forms a bounded family as operators on $H^m(\mathbb{R})$ (by writing $f = wg$, with $g \in H^m(\mathbb{R})$). We also let $L^2_\lambda = H^0_\lambda$.

Let I be a closed interval in $\mathbb{R}$. We consider, similarly, the spaces

$$\begin{aligned} H^{i,j}_\lambda(I \times \mathbb{R})) := wH^i(I; H^j(\mathbb{R})) &= \Big\{u,\ \partial^\alpha_\sigma \partial^\beta_x u \in L^2_\lambda(I \times \mathbb{R}),\ \alpha \leq i, \beta \leq j\Big\} \\ &= \Big\{u,\ \partial^\alpha_\sigma \partial^\beta_x (e^{-\lambda\langle x\rangle} u) \in L^2(I \times \mathbb{R}),\ \alpha \leq i, \beta \leq j\Big\} \\ &= H^i(I; H^j_\lambda(\mathbb{R})), \end{aligned} \tag{9}$$

where $w := e^{\lambda\langle x\rangle}$ is viewed as a function of x and σ, constant in σ.

We study semi-groups as particular cases of the abstract problem

$$\partial_t u - Pu = F, \quad u(0) = h \in X, \tag{10}$$

where P is a (usually unbounded) operator on a Banach space X with domain $D(P)$. For the λSABR PDE, Equation (1), one takes $P = L$ acting on $L^2_\lambda(\Omega) := e^{\lambda\langle x\rangle} L^2(\Omega)$, $\Omega = I \times \mathbb{R}$, $F = 0$, and $h(\sigma, x) := |e^x - K|_+$, for $I = [\alpha, \beta] \subset (0, \infty)$.

2.2 Operators with totally bounded coefficients

Let $\Omega = \mathbb{R}$ or $\Omega = I \times \mathbb{R}$, with $I \subset \mathbb{R}$ an interval. Some of the results of this subsection can be derived from those of Amann (2016, 2017), Ammann, Große, and Nistor (n.d.), and Browder (1960/1961), with some slightly different arguments, so we omit a couple of standard proofs.

Definition 1 – A function $f : \Omega \to \mathbb{C}$ is *totally bounded* if it is smooth and bounded and all its derivatives are also bounded.

We have the following simple lemmas by a direct calculation.

Lemma 1 – *Let P be an order m differential operator on Ω with totally bounded coefficients. Then P defines continuous a map $H^s_\lambda(\Omega) \to H^{s-m}_\lambda(\Omega)$, for every $s \geq m$.*

Lemma 2 – *Let $P := \sum_{|\alpha| \leq m} a_\alpha \partial^\alpha$ be an order m differential operator on Ω with totally bounded coefficients. If $w(\sigma, x) = e^{\lambda\langle x\rangle}$, as before, then $w^{-1}Pw$ also has totally bounded coefficients and the same terms of order m as P.*

We formulate the following result in slightly greater generality than needed for the proof of the existence of the semi-group generated by L, for further possible applications. We recall the definition of a second order uniformly strongly elliptic differential operator on $\Omega = \mathbb{R}$ or $\Omega = I \times \mathbb{R}$, with real coefficients, in the form that we use in this paper.

Definition 2 – Let $P = a_{xx}(\sigma, x)\partial_x^2 + 2a_{\sigma x}(\sigma, x)\partial_\sigma \partial_x + a_{\sigma\sigma}(\sigma, x)\partial_\sigma^2 + b(\sigma, x)\partial_x + c(\sigma, x)\partial_\sigma + d(\sigma, x)$ be a differential operator with real coefficients on $I \times \mathbb{R}$. We say that P is *uniformly strongly elliptic* if it has bounded coefficients and if there exists $\epsilon > 0$ such that $a_{xx} \geq \epsilon$ and $a_{xx}a_{\sigma\sigma} - a_{\sigma x}^2 \geq \epsilon$.

If $\Omega = \mathbb{R}$, we simply set $a_{\sigma\sigma} = a_{\sigma x} = c = 0$. We have the following standard regularity results, where we continue to assume that $\Omega = I \times \mathbb{R}$ or $\Omega = \mathbb{R}$.

Theorem 1 – *Let P be second order, uniformly strongly elliptic differential operator with totally bounded coefficients on Ω. Assume $u \in H^1_\lambda(\Omega)$ is such that $Pu \in H^{m-1}_\lambda(\Omega)$. If $\Omega = I \times \mathbb{R}$, we also assume that u vanishes at the endpoints of I. Then $u \in H^{m+1}_\lambda(\Omega)$. Moreover, there exists $C > 0$, independent of u, such that $\|u\|_{H^{m+1}_\lambda(\Omega)} \leq C\big(\|Pu\|_{H^{m-1}_\lambda(\Omega)} + \|u\|_{H^1_\lambda(\Omega)}\big)$.*

This result was proved in the greater generality of Lie manifolds in Ammann, Ionescu, and Nistor (2006, Theorem 0.1). A direct proof of can also be obtained by first reducing to the case $\lambda = 0$, that is, $w = 1$, using Lemma 2 on the previous page and then either by using a dyadic partition of unity (see also Ariche, Coster, and Nicaise n.d. and the references therein).

Corollary 1 – *Let P be second order, uniformly strongly elliptic differential operator with totally bounded coefficients on $\mathbb{R}$. Then $\|u\|_{L^2_\lambda} + \|P^k u\|_{L^2_\lambda}$ defines an equivalent norm on $H^{2k}_\lambda(\mathbb{R})$.*

Definition 3 – Let $\mathcal{P}$ denote the set of second-order differential operators

$$T := a_{xx}(\sigma,x)\partial_x^2 + 2a_{\sigma x}(\sigma,x)\partial_\sigma\partial_x + a_{\sigma\sigma}(\sigma,x)\partial_\sigma^2 + b(\sigma,x)\partial_x + c(\sigma,x)\partial_\sigma + d(\sigma,x)$$

with real, totally bounded coefficients on $I \times \mathbb{R}$, satisfying

$$a_{xx},\ a_{\sigma\sigma},\ a_{xx}a_{\sigma\sigma} - a_{\sigma x}^2 \geq 0.$$

To an operator $T \in \mathcal{P}$, we associate the matrix of highest-order coefficients

$$M_T := \begin{bmatrix} a_{xx} & a_{\sigma x} \\ a_{\sigma x} & a_{\sigma x} \end{bmatrix}. \tag{11}$$

Then, $\xi^t M_T \xi$, $\xi \in \mathbb{R}^2$, is the principal symbol of T. Let $\mathcal{K}_0 := H^2_\lambda(I \times \mathbb{R}) \cap \{u = 0 \text{ on } \partial I \times \mathbb{R}\}$.

Proposition 1 – *If $w(\sigma,x) = e^{\lambda\langle x\rangle}$ and $T \in \mathcal{P}$, then $w^{-1}Tw \in \mathcal{P}$. Let M_T be as in Equation (11), then there exists $C > 0$ such that*

$$(Tu,u)_{L^2_\lambda(I\times\mathbb{R})} \leq -\int_{I\times\mathbb{R}} (M_T\nabla u, \nabla u)e^{-2\lambda\langle x\rangle}\, d\sigma dx + C\|u\|^2_{L^2_\lambda(I\times\mathbb{R})}, \quad u \in \mathcal{K}_0,$$

and hence, T with domain $\mathcal{K}_0 := H^2_\lambda(I \times \mathbb{R}) \cap \{u = 0 \text{ on } \partial I \times \mathbb{R}\}$ is quasi dissipative on $L^2_\lambda(I \times \mathbb{R})$.

Under the hypotheses of Proposition 1, it follows immediately that there exists a constant $C > 0$ such that

$$|(Tu,u)| \leq C\|u\|_{H^1_\lambda(I\times\mathbb{R})}. \tag{12}$$

This result follows by using again Lemma 2 on the previous page and the total boundedness of the coefficients.

Under the condition of strong, uniform ellipticity on T, we have the standard Garding's inequality (stated for negative-definite operators). For the next three results, $T \in \mathcal{P}$, as in the statement of Proposition 1.

Lemma 3 – *Let $T \in \mathcal{P}$. Assume that there exists $\epsilon > 0$ such that $a_{xx}a_{\sigma\sigma} - a_{\sigma x}^2 \geq \epsilon$. Then there exist $C_1 > 0$ and C_2 such that*

$$\mathfrak{Re}(Tu, u) \leq -C_1 \|u\|^2_{H^1_\lambda(I\times\mathbb{R})} + C_2 \|u\|^2_{L^2_\lambda(I\times\mathbb{R})}.$$

Also, if $u \in H^1_\lambda(I \times \mathbb{R}) \cap \{u|_{\partial I\times\mathbb{R}} = 0\}$ satisfies $Tu \in L^2_\lambda(I \times \mathbb{R})$, then $u \in H^2_\lambda(I \times \mathbb{R})$. Consequently, $T - \mu_0 \colon \mathcal{K}_0 \to L^2_\lambda(I \times \mathbb{R})$ is invertible for $\mu_0 > C_2$.

We obtain as a consequence the following theorem.

Theorem 2 – *Let $T \in \mathcal{P}$. The operator T generates an analytic semi-group e^{tL} on $L^2_\lambda(I \times \mathbb{R})$. In particular, if $I = (\alpha, \beta)$ is a bounded interval with $0 < \alpha \leq \beta < \infty$, then L as given in* (1) *satisfies the hypothesis of Lemma 3, and hence it generates an analytic semi-group on $L^2_\lambda(I \times \mathbb{R})$.*

Corollary 2 – *Let $T \in \mathcal{P}$. Then, $u(t) := e^{tT}h$ is the unique strong solution of $\partial_t u - Tu = 0$, $u(0) = h$. It is also a classical solution on $(0, \tau]$, for all $\tau > 0$. Moreover, $u(t)$ does not depend on λ.*

Proof. We have that T generates an analytic semi-group $S(t) = e^{tT}$. Moreover, elliptic regularity gives $D(T^k) \subset H^{2k}(I \times \mathbb{R})$, for all $k \in \mathbb{Z}_+$. The Sobolev embedding theorem and standard results (see, for instance, Lunardi 1995, Section 4.3, Chapter 5) prove the first part of the result. The details are contained in Lemma 21 on p. 149 and Proposition 5 on p. 150 of the Appendix.

The independence of u on λ follows from the fact that the map $L^2_{\lambda'}(I \times \mathbb{R}) \to L^2_{\lambda''}(I \times \mathbb{R})$ is injective and continuous for all $\lambda' < \lambda''$ and from the uniqueness of strong solutions. □

Remark 1 – The assumption that I be a bounded interval in the second half of Theorem 2, is essential for our method to apply. Our method does not apply, for instance, if $I = (0, \infty)$. The problem lies in the fact that, at $\sigma = 0$, we lose uniform ellipticity and, at $\sigma = \infty$, the coefficient $\theta - \sigma$ becomes unbounded. However, if $\kappa = 0$, we do obtain that L generates an analytic semi-group using the results in Amann (2017) or Mazzucato and Nistor (2006). The degeneracy at $\sigma = 0$ and $\sigma = \infty$ could be addressed by introducing appropriate weights in σ. For the applications of interest in this work, it is enough to consider σ in a bounded interval, bounded away from zero.

We now consider the operator $B := \partial_x^2 - \partial_x$ (recall Equation (3) on p. 119). The analysis of B is a special case of the analysis of the operator L. We collect and state the result on the semi-group generated by B for clarity. It is also a consequence of Amann (2016, 2017), Browder (1960/1961), and Mazzucato and Nistor (2006).

Theorem 3 – *Let $T = a\partial_x^2 + b\partial_x + c$ be a uniformly strongly elliptic operator with totally bounded coefficients. Then, T generates an analytic semi-group on $L^2_\lambda(\mathbb{R})$. In particular, B generates an analytic semi-group on $L^2_\lambda(\mathbb{R})$.*

Since $D(B^k) = H^{2k}_\lambda(\mathbb{R})$, we also have the following corollary.

Corollary 3 – *The operator B generates an analytic semi-group on $H^j_\lambda(\mathbb{R})$, for all j.*

Remark 2 – As for L, $h \in L^2_\lambda(\mathbb{R})$, $u(t) := e^{tT}h$ is a strong solution of $\partial_t u - Tu = 0$, $u(0) = h$, a classical solution on any interval $(0,\tau]$, $\tau > 0$, and $u(t)$ does not depend on λ. In view of the independence of λ, the semi-group e^{tB} on L^2_λ is given by the usual explicit formula

$$e^{tB}h(x) = \frac{1}{\sqrt{4\pi t}} \int e^{-\frac{|x-y-t|^2}{4t}} h(y)\,dy. \tag{13}$$

2.3 Families

To treat the IVP associated to the λSABR PDE formulated in Equation (1), we will need to consider families of operators. In particular, we will show that the operator $P = \frac{\sigma^2}{2}B$ that appears in the λSABR PDE, also generates an analytic semi-group.

We begin by defining $P = \frac{\sigma^2}{2}B$ more precisely as a family of unbounded operators depending on a parameter, σ. If $p\colon I \to [0,\infty)$ is a bounded and continuous function, we shall write pB for the operator $(pBv)(\sigma) = p(\sigma)Bv(\sigma) \in L^2_\lambda(\mathbb{R})$, with domain $D(pB) := \{f \in L^2(I; L^2_\lambda(\mathbb{R})| \; pBf \in L^2(I; L^2_\lambda(\mathbb{R})\}$. Then e^{pB} acting on $L^2(I; L^2_\lambda(\mathbb{R})$ is given, in a standard way, by:

$$(e^{pB}v)(\sigma) := e^{p(\sigma)B}v(\sigma) \in L^2_\lambda(\mathbb{R}). \tag{14}$$

We can thus regard both pB and e^{pB} as a family of operators parameterized by $\sigma \in I$ and acting on $L^2_\lambda(\mathbb{R})$-valued measurable functions defined on I.

We will need the following standard results.

Lemma 4 – *Let $\xi \in \mathcal{C}([0,1]; X)$ and $[0,1] \ni t \to V(t) \in \mathcal{L}(X)$ be strongly continuous. Then the map $[0,1] \ni t \to V(t)\xi(t) \in X$ is continuous*

We can then prove the following.

Proposition 2 – *Let T be a differential operator as in Theorem 3 on the previous page. Let $I \subset \mathbb{R}$ be an interval and $p\colon I \to [0,\infty)$ be a* bounded, *continuous function. Then e^{tpT}, defined by $(e^{tpT}h)(\sigma) := e^{tp(\sigma)T}h(\sigma) \in L^2_\lambda(\mathbb{R})$, $\sigma \in I$, defines a c_0 semi-group on $L^2_\lambda(I \times \mathbb{R})$ with generator pT.*

Proof. Since T generates a c_0 semi-group, $e^{tp(\sigma)T}h(s)$ depends continuously on $\sigma \in I$ whenever $h \in L^2_\lambda(I \times \mathbb{R})$ is continuous in σ. Since $\|e^{tT}\|$ is uniformly bounded for t in a bounded interval, we obtain that the family of operators $e^{t\,p(\sigma)T}$ thus defines a bounded operator on $L^2_\lambda(I \times \mathbb{R})$. □

To deal with higher regularity, we need the following extension of Lemma 4 on the preceding page.

Lemma 5 – *Let $J := (0,1)$ and assume that $\xi \in \mathcal{C}^1(J;X)$, that T is the generator of c_0 semi-group $V(t)$ on X, and that one of the following two conditions is satisfied:*

(i) *$\xi(t) \in D(T)$ and the map $J \ni t \to T\xi(t) \in X$ is continuous;*

(ii) *the semi-group $V(t)$ generated by T is an analytic semi-group.*

Then $V(t)\xi(t) \in \mathcal{C}^1(J;X)$ with differential $TV(t)\xi(t) + V(t)\xi'(t)$.

We shall need the following well known fact[14]

Remark 3 – We recall that, if T is the generator of an analytic semi-group e^{tT} on a Banach space X, then $T^n e^{tT}$ extends to a bounded operator on X and there exists $C > 0$ such that

$$\|T^n e^{tT}\| \leq Ct^{-n}, \quad \text{for all } t \in (0,1]. \tag{15}$$

Corollary 4 – *Let $f : [\alpha,\beta] = \overline{I} \to [\epsilon,\infty)$, $\epsilon > 0$. Assume that f, f', and f'' are (defined and) continuous. Let $\mathcal{K}_1 := H^2_\lambda(I \times \mathbb{R})$. Then e^{fB} maps $\mathcal{K}_1$ to itself. Moreover, e^{tfB} defines a c_0 semi-group on $\mathcal{K}_1$, generated by fB as an operator with domain*

$$\{\xi \in \mathcal{K}_1,\ B\xi \in \mathcal{K}_1\} \supset H^{2,4}_\lambda(I \times \mathbb{R}) := H^2(I; H^4_\lambda(\mathbb{R})).$$

Proof. The first part is an immediate consequence of Lemma 5(ii) and of Remark 3. The second part follows using also Corollary 3 on the preceding page. □

3 The semi-group generated by L_0

In this section, we discuss the derivation of an explicit formula for the distributional kernel of the operator e^{tL_0} using Lie algebra techniques. Besides being of independent interest, in this work we utilize the explicit formula for e^{tL_0} to approximate e^{tL}, for which no closed form are available. This is achieved by means of a perturbative expansion in the parameter ν, the so-called "volvol" or "volatility of the volatility." We recall that $L_0 = A + \frac{\sigma^2}{2} B$ and $L = L_0 + \nu L_1 + \nu^2 L_2$, with L_i independent of ν (see Equations (2) and (3)).

[14] Amann, 1995, *Linear and quasilinear parabolic problems. Vol. I*;
Henry, 1981, *Geometric theory of semilinear parabolic equations*;
Lunardi, 1995, *Analytic semigroups and optimal regularity in parabolic problems*;
Martin, 1987, *Nonlinear operators and differential equations in Banach spaces*;
Pazy, 1983, *Semigroups of linear operators and applications to partial differential equations*.

There is an added difficulty in our problem, namely, the fact that L_0 is not strongly elliptic, and $\partial_t - L_0$ is not hypoelliptic in the sense of Hörmander[15] (although L_0 is). As a matter of fact, this expansion is only valid under additional regularity assumptions on the initial data h, which will be discussed in Section 4 on p. 140.

The explicit formula for e^{tL_0} is derived from the corresponding formulas for e^{tA} and $e^{\frac{t\sigma^2}{2}B}$, where the later is defined using Proposition 2 on p. 128. The existence of the group e^{tA}, $t \in \mathbb{R}$, follows directly from the transport character of the operator $A = (\theta - \sigma)\partial_\sigma$, as recalled below.

We thus assume that $I = (\alpha, \beta)$ *satisfies* $0 < \alpha < \theta < \beta < \infty$, *as in Proposition 2 on p. 128. We will make the further assumption that* $\kappa > 0$.

This last assumption implies that the characteristics of the operator A are incoming at $\sigma = \alpha$ and $\sigma = \beta$, as long as $\alpha < \theta < \beta$ and $\kappa > 0$. Therefore, no boundary conditions need to be imposed at $\sigma = \alpha$ and $\sigma = \beta$ (cf. the seminal work of Feller[16] and the references therein). The case $\kappa < 0$ is similar provided one imposes suitable boundary conditions. However, this case will not be needed for our purposes.

We next briefly discuss e^{tA} and its properties. These will be used in deriving an explicit formula for e^{tL_0}.

3.1 The generation property for L_0

Let $I = (\alpha, \beta) \subset \mathbb{R}$ and $A := \kappa(\theta - \sigma)\partial_\sigma$, as before.

The transport equation generated by A

We consider first the transport equation

$$\partial_t v - Av = 0, \tag{16}$$

where v depends on σ and, possibly, on some parameters. As is well known, this equation is solved explicitly by the method of characteristics. The behavior of this equation is somewhat different according to the sign of κ. Thus, for $\kappa < 0$ we need to impose boundary conditions, whereas for $\kappa > 0$, we do not. The techniques to treat the two cases are very similar, but in this paper we will concentrate, for simplicity, on the case $\kappa > 0$, because this is the case of greatest interest in applications (it expresses the "mean reversion" of the volatility). For $t, \sigma \in \mathbb{R}$, let

$$\delta_t(\sigma) := \theta(1 - e^{-\kappa t}) + \sigma e^{-\kappa t}, \tag{17}$$

[15] Hörmander, 2007, *The analysis of linear partial differential operators. III*.

[16] Feller, 1952, "The parabolic differential equations and the associated semi-groups of transformations".

be the characteristic line starting at σ, that is, $\delta(t,\sigma) = \delta_t(\sigma)$, $t \in \mathbb{R}$, $\sigma \in \mathbb{R}$, is the flow map generated by A. Then, $\delta_t \circ \delta_s = \delta_{t+s}$. In addition, by the assumptions on I, $\delta_t(I) \subset I$ for $t \geq 0$.

By property of the flow, for any $h \in L^1_{loc}(I)$, there is a unique weak solution of (16), which is a classical solution if if $h \in \mathcal{C}^1(I)$, and given by the formula:

$$v(t,\sigma) := h(\delta_t(\sigma)). \tag{18}$$

Properties of the flow also immediately give that the family of operators $T(t)$, $t \in \mathbb{R}$, defined by $T(t)h = v(t)$ form a group on any $L^p(\mathbb{R})$, $1 \leq p \leq \infty$, and a semi-group if we restrict to I.

In what follows, we consider A as operator acting of functions of σ with values in a Hilbert space $\mathcal{H}$. For the application at hand, $\mathcal{H}$ will be an exponentially weighted Sobolev space. We record the generation of the semi-group in this case and present a brief proof for clarity.

Proposition 3 – *Let $\mathcal{H}$ be a Hilbert space and $h \in L^2(I;\mathcal{H})$. Then, $\|T(t)h\| \leq e^{\kappa t/2}\|h\|$, where $\|\cdot\|$ denotes the Hilbert space norm on $L^2(I;\mathcal{H})$. Moreover, $T(t) =: e^{tA}$ is a c_0 semi-group whose generator coincides with A on $\mathcal{C}^1(I;\mathcal{H})$.*

Proof. The relation $\|T(t)h\| \leq e^{\kappa t/2}\|h\|$ follows by a change of variables (note also that, for $I = \mathbb{R}$, we have equality). The identity $T(t_1)T(t_2)h = T(t_1+t_2)h$ again follows from the flow properties. If $h \in \mathcal{C}^1(I;\mathcal{H})$, then (18) gives $t^{-1}(T(t)h - h) \to Ah$. Since $\|T(t)\|$ is uniformly bounded for $t \leq 1$, this gives that $T(t)h \to h$ as $t \to 0$ for all h. □

In particular, v is a strong solution of Equation (16) for $h \in \mathcal{C}^1(I;L^2_\lambda(\mathbb{R}))$. If $h \in \mathcal{C}^1(I;H^1_\lambda(\mathbb{R}))$, it is also a classical solution. We recall that $\mathcal{K}_1 := H^2_\lambda(I \times \mathbb{R})$. The explicit formula (18) for $v(t)$ and the fact that $\mathcal{K}_1 \subset \mathcal{C}^1(I;L^2_\lambda(\mathbb{R}))$, by Sobolev embedding, implies also the following result needed later in the paper.

Corollary 5 – *For $h \in \mathcal{K}_1$, v is a strong solution of Equation* (16), *$e^{tA}(\mathcal{K}_1) \subset \mathcal{K}_1$, and e^{tA} defines a c_0 semi-group on $\mathcal{K}_1$.*

The generation property for L_0

We now turn to the study of the operator L_0. It seems difficult to apply directly the Lumer-Philips Theorem to a degenerate operator like L_0. We will therefore adopt a different strategy and directly prove that the solution operator of $\partial_t - L_0$, which we still denote by e^{tL_0}, is a semi-group generated by L_0, justifying the notation. This strategy is accomplished by an implicit operator splitting of L_0 into multiples of A and B, using that A and B almost commute.

Recall that $\delta_t(\sigma) = \theta(1-e^{-\kappa t}) + \sigma e^{-\kappa t}$.

Lemma 6 – *Let $g\colon I \to [0,\infty)$ be a continuous function. Assume that either g is bounded or that the parameter $\lambda = 0$ in the definition of the weight $w(x) = e^{\lambda\langle x\rangle}$. Then $e^{tA}e^{gB} = e^{(g\circ\delta_t)B}e^{tA}$.*

Proof. The result follows from $e^{(g\circ\delta_t)B}\xi \circ \delta_t = (e^{gB}\xi)\circ\delta_t = e^{tA}e^{gB}\xi$. □

The formula for the solution operator of $\partial_t - L_0$ will be conveniently expressed through the use of the following auxiliary function:

$$\mathfrak{D}(t) = \mathfrak{D}(t,\sigma) := \frac{(\theta-\sigma)^2}{4\kappa}(1-e^{-2\kappa t}) - \frac{\theta(\theta-\sigma)}{\kappa}(1-e^{-\kappa t}) + \frac{\theta^2 t}{2}. \tag{19}$$

Proposition 4 – *The function $\mathfrak{D}(t,\sigma)$ of* (19) *is analytic in $(\kappa,t,\sigma)\in\mathbb{R}^3$ and satisfies $\mathfrak{D}(0,\sigma)=0$ and $\mathfrak{D}(t,\sigma)>0$ for any $t>0$ and any $\sigma\in\mathbb{R}$.*

Proof. The function $\mathfrak{D}(t,\sigma)$ is analytic on $\mathbb{R}^3$ since the singularity at $\kappa=0$ is removable. We shall regard $\mathfrak{D}(t,\sigma)$ as a second order polynomial in $\theta-\sigma$ with coefficients that are functions of the parameters t and κ. We have that the leading coefficient $\frac{1}{4\kappa}(1-e^{-2\kappa t})$ is always positive as $t>0$ (for all $\kappa\in\mathbb{R}$), so we only need to show that the discriminant of $\mathfrak{D}(t,\sigma)$ is <0 for $t>0$. We let $f(t)$ be the discriminant of $\mathfrak{D}(t,\sigma)$ (regarded as a second-order polynomial in σ, as mentioned above), so that

$$f(t) = \frac{\theta^2}{2\kappa^2}\Big[(2+\kappa t)e^{-2\kappa t} - 4e^{-\kappa t} + 2 - \kappa t\Big]. \tag{20}$$

We then have:

$$f'(t) = -\frac{\theta^2}{2\kappa}\Big[(3+2\kappa t)e^{-2\kappa t} - 4e^{-\kappa t} + 1\Big] \quad \text{and}$$
$$f''(t) = 2\theta^2 e^{-2\kappa t}\Big[1+\kappa t - e^{+\kappa t}\Big] < 0 \quad \text{for } t\neq 0\,.$$

It follows that $f'(t)$ is decreasing, and hence $f'(t) < f'(0) = 0$ for $t>0$. Consequently, $f(t)$ is also decreasing, which gives $f(t) < f(0) = 0$ for positive t. □

This lemma allows us to define $e^{\mathfrak{D}(t)B}$, with $\mathfrak{D}$ as in Equation (19), if I is bounded or if $\lambda=0$. We let then

$$S(t) := e^{\mathfrak{D}(t)B}e^{tA}. \tag{21}$$

Then $S(t)$ is a bounded operator, since it is the composition of bounded operators. We will establish that $S(t)$ is a c_0 semi-group generated by L_0 by splitting the proof in a few lemmas, for convenience. Recall that $\mathcal{K}_1 = H^2_\lambda(I\times\mathbb{R})$.

Lemma 7 – *For all t, $s \geq 0$, the family of operators $S(t)$, defined in Equation* (21)*, satisfies:*

1. $S(t)S(s) = S(t+s)$*;*
2. $S(t)\mathcal{K}_1 \subset \mathcal{K}_1$.

Proof. We first notice that $\mathfrak{D}(t) + \mathfrak{D}(s) \circ \delta_t = \mathfrak{D}(t+s)$, which is easy to check by direct calculation. By definition, using also Lemma 6 on p. 131, we have

$$\begin{aligned} S(t)S(s) &= e^{\mathfrak{D}(t)B} e^{tA} e^{\mathfrak{D}(s)B} e^{sA} = e^{\mathfrak{D}(t)B} e^{(\mathfrak{D}(s)\circ\delta_t)B} e^{tA} e^{sA} \\ &= e^{(\mathfrak{D}(t)+\mathfrak{D}(s)\circ\delta_t)B} e^{(t+s)A} = e^{\mathfrak{D}(t+s)B} e^{(t+s)A} = S(t+s). \end{aligned} \tag{22}$$

This calculation completes the proof of the first part. The last part follows from Corollary 5 on p. 131 and Corollary 4 on p. 129. □

We recall that we assume σ is in a bounded interval $I \subset (0, \infty)$.

Lemma 8 – *We have that for all $j \geq 0$,*

$$\|\partial_\sigma^j \mathfrak{D}(t)/t - \sigma^2/2\|_{L^\infty(I)} \to 0 \quad \text{as } t \to 0\,,\ t > 0\,.$$

Proof. We observe that the function $\partial_\sigma^j \mathfrak{D}(t)/t$, defined on $I \times (0,1]$, extends to a continuous function on $\overline{I} \times [0,1]$. Since I is a bounded interval, this fact is enough to provide the result. □

Lemma 9 – *The following limits in L^2_λ hold for the operators $S(t)$ introduced in Equation* (21)*:*

(i) $\lim_{t\searrow 0} S(t)\xi = \xi$ *for all $\xi \in L^2_\lambda$ and, similarly,*

(ii) $\lim_{t\searrow 0} t^{-1}(S(t)\xi - \xi) = L_0\xi$ *for all $\xi \in \mathcal{K}_1$.*

Proof. By the semi-group property, the operators e^{tB} and e^{tA} are uniformly bounded if $0 \leq t \leq \epsilon$, for any fixed $\epsilon > 0$. Since I is a bounded interval, the functions $\mathfrak{D}(t)$ are uniformly bounded for $t \leq \epsilon$. Moreover, $\|\mathfrak{D}(t)\|_{L^\infty(I)} \to 0$ as $t \searrow 0$. By the definition of $S(t)$, the first part of the lemma follows.

The second part of the lemma is proved in a similar fashion. Indeed, the relations $S(t)\mathcal{K}_1 \subset \mathcal{K}_1$ (see Lemma 7), $\mathfrak{D}'(0) = \sigma^2/2$ (see Lemma 8), the fact that e^{tA} is a c_0 semi-group that leaves $\mathcal{K}_1$ invariant (Corollary 5 on p. 131), and Lemma 4 on p. 128 give that

$$\begin{aligned} \partial_t\Big(S(t)\xi\Big)\Big|_{t=0} &= \partial_t\Big(e^{\mathfrak{D}(t)B} e^{tA}\xi\Big)\Big|_{t=0} = \lim_{t\to 0} t^{-1}\Big(e^{\mathfrak{D}(t)B} e^{tA}\xi - \xi\Big) \\ &= \lim_{t\to 0} t^{-1}\Big(e^{\mathfrak{D}(t)B} e^{tA}\xi - e^{tA}\xi\Big) + \lim_{t\to 0} t^{-1}\Big(e^{tA}\xi - \xi\Big) \\ &= \frac{\partial \mathfrak{D}}{\partial t}(0) B\xi + A\xi = L_0\xi\,, \end{aligned}$$

whenever $\xi \in \mathcal{K}_1$. □

We have the following similar result for $\mathcal{K}_1$, using also Corollary 4 on p. 129.

Lemma 10 – *The following limits in $\mathcal{K}_1$ hold:*

(i) $\lim_{t\searrow 0} S(t)\xi = \xi$ *for all* $\xi \in \mathcal{K}_1$ *and, similarly,*

(ii) $\lim_{t\searrow 0} t^{-1}(S(t)\xi - \xi) = L_0\xi$ *for all* $\xi \in \mathcal{K}_1$ *such that* $L_0\xi \in \mathcal{K}_1$.

In particular, the second limit is valid if $\xi \in H^4(I \times \mathbb{R})$.

Lemma 7 on the previous page, Lemma 9 on the previous page, and Lemma 10, finally imply the generation of the semi-group $S(t)$ on both L^2_λ and $\mathcal{K}_1$.

Theorem 4 – *Let* $\kappa > 0$ *and* $I = (\alpha, \beta)$, *with* $0 < \alpha < \theta < \beta < \infty$, *as before. Then,* $S(t) := e^{\mathfrak{D}(t)B} e^{tA}$ *defines a* c_0 *semi-group* e^{tL_0} *on* L^2_λ, *the generator of which coincides with* L_0 *on* $\mathcal{K}_1$. *Moreover,* $S(t)$ *defines a* c_0 *semi-group on* $\mathcal{K}_1$.

Proof. The first part is an immediate consequence of Lemma 7 on the previous page and Lemma 9 on the previous page. The second part uses Lemma 10 instead. □

We are now in the position to obtain an explicit formula for the kernel of the semi-group $S(t)$ using formula (21). Obtaining explicit formulas is important in practice because it allows for very fast methods. This is one of the reasons Heston's method[17] is so popular. Explicit formulas lead also to faster methods in solving the inverse problem of determining the implied volatility from option prices (see Bellassoued et al. 2013, for instance), and generally in model calibration using inference methods.

Corollary 6 – *Under the assumptions of Theorem 4, let* $h = h(\sigma, x) \in L^2_\lambda(I \times \mathbb{R}) = e^{\lambda\langle x\rangle} L^2(I \times \mathbb{R})$ *and set* $u(t) := S(t)h$. *Then, for almost all* $\sigma \in I$:

$$u(t,\sigma,x) = \frac{1}{\sqrt{4\pi\mathfrak{D}(t)}} \int e^{-\frac{|x-y-\mathfrak{D}(t)|^2}{4\mathfrak{D}(t)}} h(\delta_t(\sigma), y)\, dy \tag{23}$$

and u *is a mild solution of the Initial Value Problem:*

$$\partial_t v - L_0 v = 0, \qquad v(0) = h.$$

If $h \in \mathcal{K}_1$, *then* u *is a strong solution, and a classical solution provided that* $h \in \mathcal{C}^{1,2}(I \times \mathbb{R}) \cap L^2_\lambda(I \times \mathbb{R})$.

[17] Heston, 1993, "A Closed-Form Solution for Options with Stochastic Volatility, with Applications to Bond and Currency Options".

Remark 4 – The operators e^{tA} amounts to a change of coordinates. By another change of coordinates, on can write $e^{\mathfrak{D}(t)B}$ in terms of the standard one-dimensional heat equation. By following through, one can see that the explicit formula of Corollary 6 on the preceding page can also be obtained by reducing the equation $\partial_t - L_0$ to the heat equation. This seems difficult to check directly, though, and even more difficult to guess. Moreover, the change of variables yields a non-standard (non-cylindrical) domain and it does not provide the several mapping properties that we have proved and which will be used below.

Remark 5 – Corollary 3 on p. 128, the local nature in σ of the semi-group generated by $\sigma^2 B$, and the hyperbolic nature (in σ) of A, as well as the explicit formula (21) show that L_0 defines a c_0 semi-group on the spaces $H_\lambda^{i,j}(\Omega)$ introduced in Equation (9). In the same way, we obtain that the operators $S(t)$ act continuously on $\mathcal{C}^\infty(\mathbb{R}_+; H_\lambda^j(\mathbb{R}))$, for any $t > 0$. This action depends continuously on t in the strong topology. We thus obtain the existence of solutions for $\partial_t u - L_0 u = 0$ with some rather general, globally defined initial data.

3.2 Lie algebra identities and semi-groups

In the previous subsection we used implicitly commutator estimates between the operators A and B. Lie algebra ideas can be exploited to derive another formula for the distributional kernel of e^{tL_0}, which in turn will prove useful in Section 4 on p. 140. We collect in this subsection results pertaining to a general class of operators with properties similar to the operators A and B, which, with abuse of notation, we continue to denote by A and B.

Remark 6 – Let V be a finite dimensional real vector space of (possibly unbounded) operators acting on some Banach space X and let A be a closed operator on X with domain $D(A)$. We make the following assumptions

(i) All operators in V have the same domain $\mathcal{K}$, which is endowed with a Banach space norm such that, for any $B \in V$, $B\colon \mathcal{K} \to X$ is continuous.

(ii) The operator A generates a c_0 semi-group e^{tA}, $t \geq 0$, of operators on X that leaves $\mathcal{K}$ invariant and induces a c_0 semi-group on $\mathcal{K}$.

(iii) The space

$$\mathcal{W} := \{\xi \in \mathcal{K} \cap D(A) \cap D(ABe^{tA}),\ (\forall) B \in V, t \geq 0 \text{ and } A\xi \in \mathcal{K}\}$$

is dense in $\mathcal{K}$ in its Banach space norm.

(iv) If $B \in V$, the operator $[A, B]$ with domain $\mathcal{W}$ is the restriction to $\mathcal{W}$ of an operator in V, unique by Item (iii), and denoted $\mathrm{ad}_A(B)$.

Then, denoting by $e^{t\,\mathrm{ad}_A}\colon V \to V$ the exponential of the endomorphism $\mathrm{ad}_A\colon V \to V$ of the finite dimensional space V, we obtain the following Hadamard type formula

$$e^{tA}B\xi = e^{t\,\mathrm{ad}_A}(B)e^{tA}\xi\,, \quad (\forall)\,B \in V\,,\ \xi \in \mathcal{K}\,,\ t \geq 0. \tag{24}$$

This relation can be proved by considering the function $F\colon [0,\infty) \to X$

$$F(t) := e^{tA}B\xi - e^{t\,\mathrm{ad}_A}(B)e^{tA}\xi\,, \quad B \in V \text{ and } \xi \in \mathcal{W}.$$

Our assumptions imply that $F(t) \in D(A)$ for all t, that $F(t)$ is differentiable, and that $F'(t) = AF(t)$. By the uniqueness of strong solutions to this evolution equation[18], it follows that $F(t) = 0$ for all $t \geq 0$, since $F(0) = 0$. This fact proves Formula (24) for $\xi \in \mathcal{W}$. Since $F(t) \in X$ depends continuously on $\xi \in \mathcal{K}$, Formula (24) for $\xi \in \mathcal{K}$ follows from the density of $\mathcal{W}$ in $\mathcal{K}$. By replacing B with $e^{-t\,\mathrm{ad}_A}(B)$ in Formula (24), we obtain

$$e^{tA}e^{-t\,\mathrm{ad}_A}(B)\xi = Be^{tA}\xi\,, \quad (\forall)\,B \in V\,,\ \xi \in \mathcal{K}\,,\ t \geq 0. \tag{25}$$

Let us assume that (the closures of) $B \in V$ and $B_1 := e^{t\,\mathrm{ad}_A}(B)$ generate c_0 semi-groups of operators on X denoted e^{sB} and e^{sB_1}, respectively. The, we also obtain the formula

$$e^{tA}e^{sB} = e^{sB_1}e^{tA}\,, \quad \text{where } B_1 := e^{t\,\mathrm{ad}_A}(B) \in V\,, \quad t,s \geq 0\,, \tag{26}$$

as bounded operators on X. Indeed, for $\xi \in \mathcal{K}$, the equality $e^{tA}e^{sB}\xi = e^{sB_1}e^{tA}\xi$ is obtained by differentiating $F(s) := e^{tA}e^{sB}\xi - e^{sB_1}e^{tA}\xi$ with respect to s (see Equation (26)) and using Equation (24) (which takes care also that all the terms be defined) to obtain that $F(s) \in D(\overline{B})$ and that $F'(s) = BF(s)$. Since $F(0) = 0$, we obtain, by the uniqueness of solutions of $u' = \overline{B}u$, $u(s) \in D(\overline{B})$, that $F(s) = 0$. By density, we then obtain the result for $\xi \in X$.

We shall use the above remark in the following setting.

Remark 7 – We keep the notation of Remark 6 on the previous page. Let $X := L^2_\lambda(I \times \mathbb{R})$, $V = \mathbb{R}\partial_\sigma$ with ∂_σ acting on the first variable, and with domain $\mathcal{K} := \mathcal{K}_1$, where, we recall, $\mathcal{K}_1 := H^2_\lambda(I \times \mathbb{R})$. As before, we let $A := \kappa(\theta - \sigma)\partial_\sigma$ and consider the adjoint action of A on V. We have that e^{tA} maps $H^3_\lambda(I \times \mathbb{R})$ to itself and $H^3_\lambda(I \times \mathbb{R})$ is contained in the domains of A and AB. Hence $H^3_\lambda(I \times \mathbb{R}) \subset \mathcal{W}$, by the definition of $\mathcal{W}$, and hence

$$A\partial_\sigma - \partial_\sigma A = [A, \partial_\sigma] = [\kappa(\theta - \sigma)\partial_\sigma, \partial_\sigma] = [\kappa(\theta - \sigma)\partial_\sigma, \partial_\sigma] = \kappa\partial_\sigma \in V$$

on $\mathcal{W}$. It follows that $e^{tA}\partial_\sigma = e^{\kappa t}\partial_\sigma e^{tA}$.

[18] Amann, 1995, *Linear and quasilinear parabolic problems. Vol. I*; Pazy, 1983, *Semigroups of linear operators and applications to partial differential equations.*

In the same spirit, we have the following remark.

Remark 8 – We keep the same notation and assumptions as in Remark 6 on p. 135, in particular, the space $\mathcal{W}$ has the same meaning. We now present a situation for which we can compute $e^{t(A+B)}$ in terms of A and B and the semi-groups that they generate, using the ideas of Remark 6 on p. 135. I addition to the four assumptions of Remark 6 on p. 135, we consider the following conditions:

(v) There exists a closed cone $C_+ \subset V$ that is invariant with respect to $e^{t\,\mathrm{ad}_A}$, for all $t \in \mathbb{R}$, such that the closure of every $B \in C_+$ generates a c_0 semi-group;

(vi) $e^{tB_1}e^{sB_2} = e^{sB_2}e^{tB_1}$ for all $B_1, B_2 \in C_+$ and $s, t \geq 0$.

(vii) The function $C_+ \ni B \to e^B\xi \in X$ is continuous on C_+ for all $\xi \in X$.

Let $B \in C_+ \subset V$. We will show that if the four conditions above are satisfied (in addition to the four assumptions of Remark 6 on p. 135), then

$$e^{t(A+B)} = e^{tA}e^{b(t)}, \tag{27}$$

where $b\colon [0,\infty) \to C_+$ is a suitable differentiable function with $b(0) = 0$. The proof of this result will be completed in the proof of Theorem 5 on the next page.We first comment briefly on the assumptions above. Let $B_i \in C_+$. It is known, for instance, that $e^{tB_1}e^{sB_2} = e^{tB_1+sB_2}$, which is an instance of the *Trotter's product formula*[19]. Moreover, by results of Hille[20], B_1 and B_2 commute in an obvious sense. We also have that for $\xi \in \mathcal{K}$, the function $C_+ \ni B \to e^B\xi \in X$ is differentiable. More precisely, if $B(t) \in C_+$ depends differentiably on t, then $(e^{B(t)}\xi)' = e^{B(t)}B'(t)\xi = B'(t)e^{B(t)}\xi$. This follows from Trotter's product formula already mentioned. See Hille and Phillips (1957) for a comprehensive introduction to the subject. See also Bobrowski and Bogucki (2008) for related results.

We turn now to Equation (27). By differentiating the right hand side with respect to t and evaluating at $\xi \in X$, we *formally* obtain:

$$\left(e^{tA}e^{b(t)}\xi\right)' = Ae^{tA}e^{b(t)}\xi + e^{tA}b'(t)e^{b(t)}\xi = \left[A + e^{t\,\mathrm{ad}_A}(b'(t))\right]e^{tA}e^{b(t)}\xi\,.$$

Motivated by Equation (27), we then set $\left(e^{tA}e^{b(t)}\xi\right)' = (A+B)e^{tA}e^{b(t)}\xi$, which gives $b'(t) = e^{-t\,\mathrm{ad}_A}(B) \in C_+$. This condition can be verified, at least formally, by integrating this last formula to first see that $b(t) \in C_+$ for all t, since C_+ is closed and convex. Explicitly, let $\mathcal{E}(s) := (e^s - 1)/s$, which is an entire function on $\mathbb{C}$. Then $\mathcal{E}(-t\,\mathrm{ad}_A)$ is defined by holomorphic functional calculus and $b(t) = t\mathcal{E}(-t\,\mathrm{ad}_A)(B)$, which is, as yet, just a formal result. Here $\mathcal{E}(-t\,\mathrm{ad}_A)$ is simply the power series: $\mathcal{E}(-t\,\mathrm{ad}_A) = \sum c_n(-t\,\mathrm{ad}_A)^n$, where $c_n(n+1)! = 1$ are the Taylor coefficients of $\mathcal{E}(s) := (e^s - 1)/s = \sum_n c_n s^n$.

Of course, this procedure has to be justified independently or one has to make sense of all the steps in its derivation. In the previous subsections, we have chosen to verify independently Formula (21) for $X = L^2_\lambda(I \times \mathbb{R})$, V the space $\{pB\}$, with p a polynomial of order ≤ 2, $\mathcal{K} = L^2(I; H^2_\lambda(\mathbb{R}))$, and C_+ the set of polynomials that are ≥ 0 on $\mathbb{R}$.

It is convenient to first prove the following Lemma.

Lemma 11 – *Assume that conditions (i-vii) in Remark 6 on p. 135 and Remark 8 on the previous page are satisfied. Let $B \in C_+$, then $B_1 := t\mathcal{E}(-t\,\mathrm{ad}_A)(B)$ and $B_2 := t\mathcal{E}(t\,\mathrm{ad}_A)(B)$ are in C_+ and $e^{tA}e^{B_1} = e^{B_2}e^{tA}$.*

Proof. We have already seen that $B_1 \in C_+$, since $B_1 = \int_0^t e^{-s\,\mathrm{ad}_A}(B)$ and C_+ is a closed, convex cone invariant for $e^{-s\,\mathrm{ad}_A}$. Similarly $B_2 = \int_0^t e^{s\,\mathrm{ad}_A}(B) \in C_+$ In view of formula (26), it is enough to prove that $e^{t\,\mathrm{ad}_A}(B_1) = B_2$. Indeed, in view of the properties of the functional calculus, it is enough to check that $e^{tz}t(e^{-tz}-1)(-tz)^{-1} = t(e^{tz}-1)(tz)^{-1}$, which is obviously true. □

We summarize the above discussion in a formal result. Recall the function $\mathcal{E}(x) := (e^x - 1)/x$.

Theorem 5 – *Assume the notation and the assumptions of Remark 6 on p. 135 and Remark 7 on p. 136. If $B \in C_+$, then $e^{t(A+B)} = e^{tA}e^{b(t)}$, for $b(t) := t\mathcal{E}(-t\,\mathrm{ad}_A)(B)$.*

Proof. We have that $b(t) \in C_+$ by Lemma 11. Let $S(t) := e^{tA}e^{b(t)}$. Then

$$S(t)S(s) = e^{tA}e^{b(t)}e^{sA}e^{b(s)} = e^{tA}e^{sA}e^{b_1(t)}e^{b(s)} = e^{(t+s)A}e^{b_1(t)+b(s)}$$

where $b_1(t) = e^{-s\,\mathrm{ad}_A}(b(t))$, by formula (26) and by Trotter's product formula. We then compute

$$\begin{aligned} b_1(t) + b(s) &= e^{-s\,\mathrm{ad}_A}\Big(t\mathcal{E}(-t\,\mathrm{ad}_A)(B)\Big) + s\mathcal{E}(-s\,\mathrm{ad}_A)(B) \\ &= (t+s)\mathcal{E}\Big(-(t+s)\,\mathrm{ad}_A\Big)(B) = b(s+t), \end{aligned}$$

by the properties of the functional calculus, since

$$\begin{aligned} e^{-sz}t\mathcal{E}(-tz) + s\mathcal{E}(-sz) &= e^{-sz}t(e^{-tz}-1)(-tz)^{-1} + s(e^{-sz}-1)(-sz)^{-1} \\ &= (-z)^{-1}\Big[e^{-sz}(e^{-tz}-1) + e^{-sz} - 1\Big] \\ &= (s+t)\Big(e^{-(s+t)z} - 1\Big)\Big(-(s+t)z\Big)^{-1} \\ &= (s+t)\mathcal{E}\Big(-(s+t)z)\Big). \end{aligned}$$

[19]Trotter, 1959, "On the product of semi-groups of operators".
[20]Hille, 1950, "Lie theory of semi-groups of linear transformations".

Therefore $S(t)S(s) = S(t+s)$, for all $t, s \geq 0$. To prove that $S(t) = e^{t(A+B)}$, it is enough to check that $S(t)\xi$ is differentiable for $\xi \in \mathcal{W}$ (see Item (iii) on p. 135 for the definition of $\mathcal{W}$) and that $(S(t)\xi)' = S(t)(A+B)\xi$, since $S(t)$ is a semi-group consisting of uniformly bounded operators on compact subsets. Indeed, the relation $(S(t)\xi)' = S(t)(A+B)\xi$ would prove that $S(t)\xi$ is continuous for $\xi \in \mathcal{W}$ (and hence everywhere, since $S(t)$ consists of uniformly bounded operators on compact subsets and $\mathcal{W}$ is dense in X) and that the generator of $S(t)$ is $A+B$, by setting $t = 0$. Now, for $\xi \in \mathcal{W}$, using the notation of Lemma 11 on the preceding page, we have

$$\begin{aligned}\Big(S(t)\xi\Big)' &= \Big(e^{b_1(t)}e^{tA}\xi\Big)' = e^{b_1(t)}e^{tA}A\xi + e^{b_1(t)}b_1'(t)e^{tA}\xi \\ &= S(t)\Big[A + e^{-t\,\mathrm{ad}_A}\Big(b_1'(t)\Big)\Big]\xi = S(t)\Big[A + e^{-t\,\mathrm{ad}_A}\Big(e^{t\,\mathrm{ad}_A}(B)\Big)\Big]\xi\,.\end{aligned}$$

This completes the proof. □

Remark 9 – We use the notation of Theorem 5 on the preceding page and assume that $V = \sum_{a\in\mathcal{I}} V_a$ (for some finite index $\mathcal{I} \subset \mathbb{R}$), where

$$[A, B_a] := AB_a - B_aA = aB_a\,, \quad \text{for any } B_a \in V_a,\ a \in \mathcal{I}.$$

We can simplify the formula $e^{t(A+B)} = e^{tA}e^{b(t)}$, $b(t) = t\mathcal{E}(-t\,\mathrm{ad}_A)(B)$, even further, as follows. Let us write $B = \sum_{a\in\mathbb{R}} B_a$, with $B_a \in V_a$. (Of course, $V_a = 0$, except for finitely many values $a \in \mathbb{R}$, since V is assumed finite dimensional, so the sum $\sum_{a\in\mathbb{R}} B_a$ is actually a finite sum). Then $b(t) = t\mathcal{E}(-t\,\mathrm{ad}_A)(B) = \sum_a f_a(t)B_a$, where $f_a(t) = (1 - e^{-at})/a = t\mathcal{E}(-at)$. Hence, this procedure gives the result

$$e^{t(A+B)} = e^{tA}e^{\sum_a t\mathcal{E}(-at)B_a} = e^{\sum_a t\mathcal{E}(at)B_a}e^{tA}\,. \tag{28}$$

We close by using the results just proved to derive an equivalent formula for $S(T)$, which, by the smoothing properties of e^{tB}, $t > 0$, in x, can also be used to show that $u(t) = S(t)\xi$ defines a classical solution of $\partial_t u - L_0 u = 0$ for $t > 0$, when $\xi \in \mathcal{C}^1(I; L^2_\lambda(\mathbb{R}))$. For this purpose, we introduce the function:

$$\mathfrak{C}(t) := \mathfrak{C}(t,\sigma) := \frac{(\theta-\sigma)^2}{4\kappa}(e^{2\kappa t} - 1) - \frac{\theta(\theta-\sigma)}{\kappa}(e^{\kappa t} - 1) + \frac{1}{2}\theta^2 t. \tag{29}$$

We notice that $\mathfrak{C}(t)$ is obtained from $\mathfrak{D}(t)$ (where $\mathfrak{D}$ is introduced in Equation (19)) by replacing κ with $-\kappa$, therefore it retains its positivity (see Proposition 4 on p. 132 or Lemma 11 on the preceding page). Applying the reasoning in the previous remark, we obtain the following alternative expression for $S(t)$:

$$S(t) := e^{\mathfrak{D}(t)B}e^{tA} = e^{tA}e^{\mathfrak{C}(t)B}. \tag{30}$$

Indeed, we check that Conditions (i)–(vii) are satisfied in Remarks Remark 6 on p. 135 and Remark 8 on p. 137. For (i), we take $V := \{pB\}$ with p a second order polynomial in σ, $\mathcal{K} = \mathcal{K}_1 := H^2_\lambda(I \times \mathbb{R})$, and $A = \kappa(\theta - \sigma)\partial_\sigma$, $\kappa > 0$. We have $H^3_\lambda(I \times \mathbb{R}) \subset \mathcal{W}$ as above and then conditions (i)–(iv) follow easily. The rest of the conditions are also satisfied immediately if on takes C_+ to correspond to the polynomials that are > 0 on I.

4 Mapping properties, asymptotic expansion, and error estimates

In this section, we prove mapping properties between weighted spaces for the semi-groups we constructed. We then use these results to compare the semi-groups e^{tL_0} and e^{tL}. We continue to assume that $I = (\alpha, \beta)$, $0 < \alpha < \theta < \beta < \infty$, and that $\kappa > 0$.

4.1 Mapping properties

We shall need certain mapping properties for the semi-groups e^{tL} and e^{tL_0}. Most of these results are consequences of the properties of analytic semi-groups. We begin with a preliminary lemma.

Lemma 12 – *Assume that $I := (\alpha, \beta)$ is bounded and that $\alpha > 0$. Then there exists $\epsilon > 0$ such that $\mathfrak{D}(t, \sigma) \geq \epsilon t$ for $\sigma \in I$ and $t \in [0, 1]$.*

Proof. We consider the function $h(t, \sigma) := \mathfrak{D}(t, \sigma)/t$ for $\sigma \in [\alpha, \beta]$ and $t \in (0, 1]$. By Proposition 4 on p. 132, h extends to a continuous function on $[\alpha, \beta] \times [0, 1]$. By the assumption that $\alpha > 0$ and by Proposition 4 on p. 132 again, we have that $h > 0$ on $[\alpha, \beta] \times [0, 1]$. Therefore $\epsilon := \inf h > 0$. □

We recall also the following general fact.

Remark 10 – If T generates a c_0 semi-group e^{tT} on a Banach space X, then $(e^{tT})^*$ will also be a semi-group (but the strong continuity property may fail). However, if X is *reflexive*, then $(e^{tT})^*$ is strongly continuous and, in fact, $(e^{tT})^*$ is a c_0 semi-group with generator T^* (see Pazy 1983, Corollary 1.10.6). In other words, $(e^{tT})^* = e^{tT^*}$, if X is reflexive. Moreover, if e^{tT} is an analytic semi-group, then $(e^{tT})^*$ is also analytic since the function $(e^{\bar{z}T})^*$ is holomorphic in a sector Δ_δ, $\delta > 0$.

We first discuss mapping properties of e^{tL_0}. From (30) and the analyticity of e^{tB}, one expect e^{tL_0} to be smoothing in x. The spaces $H_\lambda^{i,j}(I \times \mathbb{R}) := H^i(I, H_\lambda^j(\mathbb{R}))$, used below, are discussed in more detail in (9).

Lemma 13 – *Let $s \geq 0$. There exists $C_s > 0$ such that, for all $h \in L_\lambda^2(I \times \mathbb{R})$,*

$$\|e^{\mathfrak{D}(t)B} h\|_{H_\lambda^{0,s}(I \times \mathbb{R})} \leq C_s t^{-s/2} \|h\|_{L_\lambda^2(I \times \mathbb{R})}, \quad \textit{for } t \in (0, 1].$$

Consequently, $\|\partial_x^k e^{tL_0}\| \leq C t^{-k/2}$, where $t \in (0, 1]$ and C is independent of t. Moreover, $\partial_x^k e^{tL_0} \xi$ is continuous in t.

Whenever not explicitly noted, *all the norms $\| \ \|$ below refer to the norm of vectors in $L_\lambda^2(I \times \mathbb{R})$ or of bounded operators on that space.*

Proof. Let us assume first $s = 2n$, for some positive integer n. The norm $\|g\|_{H^{0,2n}_\lambda(I\times\mathbb{R})}$ is equivalent to the norm $\|g\| + \|B^n g\|$ (Corollary 1 on p. 126). It is therefore enough to show that there exists C'_s such that

$$\|e^{\mathfrak{D}(t)B}h\| + \|B^n e^{\mathfrak{D}(t)B}h\| \leq C'_s t^{-n}\|h\|. \tag{31}$$

since then the desired relation follows with $C_s = CC'_s$. Lemma 12 on the preceding page gives

$$\begin{aligned}\|e^{\mathfrak{D}(t)B}h\| + \|B^n e^{\mathfrak{D}(t)B}h\| &= \|e^{\mathfrak{D}(t)B}h\| + \|e^{(\mathfrak{D}(t)-\epsilon t)B}B^n e^{\epsilon t B}h\| \\ &\leq C\Big(\|h\| + \|B^n e^{\epsilon t B}h\|\Big) \leq C(\epsilon t)^{-n}\|h\|,\end{aligned}$$

since e^{gB} is bounded on $L^2_\lambda(I\times\mathbb{R})$, if $g \geq 0$ is bounded measurable, and $t^n B^n e^{tB}$ is also bounded on the same space (by Equation (15) for $T = B$). Here, we have used the assumption that I is bounded. This argument establishes the result for $s = 2n$. For general $s \geq 0$, the result follows by complex interpolation.

To prove the last part, we write

$$\partial_x^{2k} e^{tL_0} = \partial_x^{2k}(\mu_0 - B)^{-k}(\mu_0 - B)^k e^{\mathfrak{D}(t)B} e^{tA},$$

where μ_0 is large. We have that $\partial_x^{2k}(\mu_0 - B)^{-k}$ is bounded (Theorem 1 on p. 125). Remark 3 on p. 129, Lemma 4 on p. 128 and Lemma 12 on the preceding page show that $(\mu_0 - B)^k e^{\mathfrak{D}(t)B}$ depends smoothly on t. Then, $\partial_x^k e^{tL_0}$ depends continuously on t, as e^{tA} does. Remark 3 on p. 129 also gives that $\|(\mu_0 - B)^k e^{\mathfrak{D}(t)B}\| \leq Ct^{-k}$. This implies that $\|\partial_x^{2k} e^{tL_0}\| \leq Ct^{-k}$, and the desired estimate for all $k > 0$ follows by interpolation. □

In the same way, we obtain the following result.

Lemma 14 – *If $h \in L^2_\lambda(I\times\mathbb{R})$, then*

$$\|e^{tL}h\|_{H^s_\lambda(I\times\mathbb{R})} \leq Ct^{-s/2}\|h\|_{L^2_\lambda(I\times\mathbb{R})}.$$

If P is a differential operator of order k with totally bounded coefficients on $I\times\mathbb{R}$, then Pe^{tL} and $e^{tL}P$ extend to bounded operators on $L^2_\lambda(I\times\mathbb{R})$ of norm $\leq Ct^{-k/2}$ that depend smoothly on $t > 0$.

Proof. The first part of the Lemma follows from (15), using that $(L-\mu_0)^{-n}\colon L^2_\lambda(I\times\mathbb{R}) \to H^{2n}_\lambda(I\times\mathbb{R})$ continuously for μ_0 large enough, using interpolation, and using the analyticity of e^{tL}.

Let P now be as in the statement of the lemma. Then $P\colon H^k_\lambda(I\times\mathbb{R}) \to L^2_\lambda(I\times\mathbb{R})$ is bounded. This implies the result for Pe^{tL}. The result for $e^{tL}P$ is obtained by taking adjoints, since L^* is uniformly strongly elliptic with totally bounded coefficients and generates an analytic semi-group. □

In what follows, we will need the following result. All norms of operators are on $L^2_\lambda(I \times \mathbb{R})$.

Lemma 15 – *The operator $F(s) := e^{(t-s)L}\partial_\sigma e^{sL}$ extends, for each $s \in [0,t]$, to a bounded operator on $L^2_\lambda(I \times \mathbb{R})$, and the resulting function is continuous in $s \in [0,t]$ and differentiable for $s \in (0,t)$. Its derivative is the function*

$$F'(s) = e^{(t-s)L}[\partial_\sigma, L]e^{sL},$$

which satisfies $\|F'(s)\| \le Ct^{-1}$, with C independent of $0 < s < t \le 1$.

Proof. Lemma 14 on the previous page gives that both functions $e^{(t-s)L}$ and $\partial_\sigma e^{sL}$ are continuous on $(0,T]$ and infinitely many times differentiable on $(0,t)$ as functions with values in the space of bounded operators; therefore, $F(s)$ is continuous on $[0,t)$. The formula for the derivative follow from the standard formula $(e^{sL})' = Le^{sL}$, which we note to be valid in norm, since L generates an analytic semi-group and $s > 0$. The continuity on $[0,t)$ follows in the same way by considering $e^{(t-s)L}\partial_\sigma$ and e^{sL}.

If $s \le t/2$, since $[\partial_\sigma, L]$ is a second order differential operator, Lemma 14 on the previous page implies that $e^{(t-s)L}[\partial_\sigma, L]$ is bounded with norm $\le C(t-s)^{-1} \le 2Ct^{-1}$. Hence, $\|F'(s)\| \le Ct^{-1}$. The case $s \ge t/2$ is completely analogous using the bounds for $[\partial_\sigma, L]e^{sL}$ provided by Lemma 14 on the previous page. □

4.2 A comparison of e^{tL} and e^{tL_0}

In this last section, we compare the semi-groups e^{tL_0} and e^{tL}, by regarding L as a perturbation of L_0 for ν sufficiently small. The motivation for this approach is that, while e^{tL} is better behaved as a semi-group, we lack an explicit formula for its distributional kernel.

We recall that we set $L = L_0 + V$, where $V = \nu L_1 + \nu^2 L_2 = \nu\rho\sigma^2\partial_x\partial_\sigma + \frac{\nu^2\sigma^2}{2}\partial_\sigma^2$ We also recall that $\mathcal{K}_1 = H^2_\lambda(I \times \mathbb{R})$ and $\mathcal{K}_0 = H^2_\lambda(I \times \mathbb{R}) \cap \{u(\alpha, x) = u(\beta, x) = 0\}$, where $I = (\alpha, \beta)$ is a fixed bounded interval containing θ.

Lemma 16 – *Let $\xi \in \mathcal{K}_1$. Then $F(s) := e^{(t-s)L}e^{sL_0}\xi$ is continuous on* $[0,t]$ *and differentiable on* $(0,t)$ *with values in $L^2_\lambda(I \times \mathbb{R})$), with $F'(s) = -e^{(t-s)L}Ve^{sL_0}\xi$.*

Proof. Since ξ is in the domain of L_0 (which contains $\mathcal{K}_1$, by Theorem 4 on p. 134), the function $\zeta(s) := e^{sL_0}\xi$ is differentiable for $s \ge 0$. But e^{tL} is a c_0 semi-group, therefore Lemma 4 on p. 128 gives that $F(s) = e^{(t-s)L}\zeta(s)$ is continuous on $[0,t]$. Since e^{tL} is an analytic semi-group, it follows in addition that $F(s)$ is differentiable for $s \in (0,t)$, by Lemma 5 on p. 129, and its derivative is $F'(s) = -e^{(t-s)L}Ve^{sL_0}\xi$. □

We continue to assume that $\|\cdot\|$ refers to the norm in $L^2_\lambda(I \times \mathbb{R})$ or the operator norm of bounded operators on this space.

Lemma 17 – *Let $\xi \in \mathcal{K}_1$, then $e^{(t-s)L}L_1 e^{sL_0}\xi$ depends continuously on s and*

$$(\rho\nu)^{-1}\|e^{(t-s)L}L_1 e^{sL_0}\xi\| = \|e^{(t-s)L}\sigma^2\partial_\sigma\partial_x e^{sL_0}\xi\| \le C(t-s)^{-1/2}s^{-1/2}\|\xi\|.$$

Consequently, $\left\|\int_0^t e^{(t-s)L}L_1 e^{sL_0}\,ds\right\| \le C\rho\nu.$

Proof. Lemma 13 on p. 140 and Lemma 14 on p. 141 show that $e^{(t-s)L}\sigma^2\partial_\sigma$ and $\partial_x e^{s(L_0-\kappa)}\xi$ satisfy the assumptions of Lemma 4 on p. 128, so $e^{(t-s)L}\sigma^2\partial_\sigma\partial_x e^{s(L_0-\kappa)}\xi$ is continuous in s. Similarly, Lemma 13 on p. 140 and Lemma 14 on p. 141 give

$$\|e^{(t-s)L}\sigma^2\partial_\sigma\partial_x e^{s(L_0-\kappa)}\xi\| \le \|e^{(t-s)L}\sigma^2\partial_\sigma\|\|\partial_x e^{s(L_0-\kappa)}\xi\| \le C(t-s)^{-1/2}s^{-1/2}\|\xi\|.$$

The integral can be estimated by splitting the interval $[0,t]$ in two halves. □

To estimate the terms involving L_2, we exploit the next result.

Lemma 18 – *Let $\xi \in \mathcal{K}_1$, then $\partial_\sigma e^{tL_0}\xi = e^{t(L_0-\kappa)}\partial_\sigma\xi + \frac{\partial\mathfrak{D}(t,\sigma)}{\partial\sigma}Be^{tL_0}\xi$.*

Proof. The main calculation is contained in Remark 7 on p. 136. More precisely, this is a direct calculation using Equation (21), together with Lemma 5 on p. 129, with Hadamard's theorem (see Remark 6 on p. 135 and Remark 7 on p. 136), and with the fact that $\mathrm{ad}_{L_0}(\partial_\sigma)\,\mathrm{ad}_A(\partial_\sigma) = \kappa\partial_\sigma$. □

However, the error terms containing L_2 present some additional challenges, since L_0 is not elliptic.

Lemma 19 – *Let $\xi \in \mathcal{K}_1$, then $e^{(t-s)L}L_2 e^{sL_0}\xi$ depends continuously on s and the following estimate holds:*

$$\frac{2}{\nu^2}\|e^{(t-s)L}L_2 e^{sL_0}\xi\| = \|e^{(t-s)L}\sigma^2\partial_\sigma^2 e^{sL_0}\xi\| \le C(t-s)^{-1/2}\left(\|\partial_\sigma\xi\| + \|\xi\|\right).$$

Consequently, $\left\|\int_0^t e^{(t-s)L}L_2 e^{sL_0}\xi\,ds\right\| \le C\nu^2\sqrt{t}\left(\|\partial_\sigma\xi\| + \|\xi\|\right).$

Proof. Lemma 18 gives

$$e^{(t-s)L}\sigma^2\partial_\sigma^2 e^{sL_0}\xi = e^{(t-s)L}\sigma^2\partial_\sigma\left(e^{s(L_0-\kappa)}\partial_\sigma\xi + \frac{\partial\mathfrak{D}(s,\sigma)}{\partial\sigma}Be^{sL_0}\xi\right). \tag{32}$$

As in the proof of Lemma 17, Lemma 14 on p. 141 and Lemma 13 on p. 140 give that both $e^{(t-s)L}\sigma^2\partial_\sigma e^{sL_0}$ and $e^{(t-s)L}\sigma^2\partial_\sigma\frac{\partial\mathfrak{D}}{\partial\sigma}Be^{sL_0}$ define bounded operators that depend continuously on $s \in (0,t)$ in the strong operator topology. We estimate separately the norm of each of them. Again from Lemma 14 on p. 141, we obtain

$$\|e^{(t-s)L}\sigma^2\partial_\sigma e^{s(L_0-\kappa)}\| \le \|e^{(t-s)L}\sigma^2\partial_\sigma\|\|e^{s(L_0-\kappa)}\| \le C(t-s)^{-1/2}.$$

For the estimate of the second term, we first notice that $\|\frac{\partial \mathfrak{D}(t,\sigma)}{\partial \sigma}\|_{L^\infty(I)} \leq Ct$, since the function $\frac{\partial \mathfrak{D}(t,\sigma)}{t\partial \sigma}$ extends to a continuous function on $\overline{I}\times[0,1]$. Hence, $\|\frac{\partial \mathfrak{D}(s,\sigma)}{\partial \sigma}Be^{sL_0}\| \leq \|sBe^{sL_0}\| \leq C$ by Lemma 13 on p. 140, and

$$\left\|e^{(t-s)L}\sigma^2\partial_\sigma \frac{\partial \mathfrak{D}(s,\sigma)}{\partial \sigma}Be^{sL_0}\right\| \leq \|e^{(t-s)L}\sigma^2\partial_\sigma\| \left\|\frac{\partial \mathfrak{D}(s,\sigma)}{\partial \sigma}Be^{sL_0}\right\| \leq C(t-s)^{-1/2}.$$

The last two displayed equations and Equation (32) then combine to give the first part of the statement. The last relation in the statement follows directly by integrating the first one. □

Combining the previous two lemmas we obtain the following corollary.

Corollary 7 – *The family $G(s) := e^{(t-s)L}Ve^{sL_0}$ consists of bounded operators on L^2_λ. Moreover, for any $\xi \in \mathcal{K}_1$, $G(s)\xi$ is continuous and integrable in $s \in (0,t)$ and we have:*

$$\left\|\int_0^t G(s)\xi\,ds\right\| := \left\|\int_0^t e^{(t-s)L}Ve^{sL_0}\xi\,ds\right\| \leq C\Big(\rho\nu\|\xi\| + \nu^2\sqrt{t}\big(\|\partial_\sigma\xi\| + \|\xi\|\big)\Big).$$

Lemma 16 on p. 142 and Corollary 7 then give:

$$e^{tL}\xi - e^{tL_0}\xi = F(0) - F(t) = \int_0^t e^{(t-s)L}Ve^{sL_0}\xi\,ds.$$

The final estimate is for $\xi \in H^1(I, L^2_\lambda(\mathbb{R})) := \{\zeta \in L^2_\lambda(I\times\mathbb{R}),\ \partial_\sigma\zeta \in L^2_\lambda(I\times\mathbb{R})\}$.

Theorem 6 – *There is $C > 0$ such that*

$$\|e^{tL}\xi - e^{tL_0}\xi\| \ \leq\ C\nu\big(\|\xi\| + \nu\|\partial_\sigma\xi\|\big),$$

for $\xi \in H^1(I, L^2_\lambda(\mathbb{R}))$ and $0 \leq t \leq T$. The bound C depends on T, but not on ξ.

Proof. The statement was proved for $\xi \in \mathcal{K}_1$. For general ξ, it follows from the density of $\mathcal{K}_1 := H^2_\lambda(I\times\mathbb{R})$ in $H^1(I, L^2_\lambda(\mathbb{R}))$ and the continuity on $H^1(I, L^2_\lambda(\mathbb{R}))$ of all the operators appearing on the left and right sides of the inequality. □

The approach presented in this subsection can be iterated to derive higher-order approximate solutions in the parameter ν by applying Duhamel's formula repeatedly, provided the data is sufficiently smooth. Numerical and real data tests[21] show that approximations of the form Theorem 6 work well even for ν in the range that arises in applications, that is $\nu \in [0,2]$ and $\sigma \leq .5$, but for *second order* approximations in ν. This and further research are part of an article in preparation, where we will

[21] Grishchenko, Han, and Nistor, n.d., "A Volatility-of-Volatility Expansion of the Option Prices in the SABR Stochastic Volatility".

prove a higher order version of the approximation in Theorem 6 on the preceding page with computable coefficients. We close by observing that similar commutator estimates were obtained in Cheng, Costanzino, et al. (2011), Cheng, Mazzucato, and Nistor (n.d.), Constantinescu et al. (2010), and Grishchenko, Han, and Nistor (n.d.). The main difficulty addressed in this work is that L_0 is not an elliptic operator.

A Semi-groups and solutions of evolution equations

This section is devoted to briefly review known facts about abstract evolution equations and semi-groups of operators needed for the analysis. We also review needed facts about the function spaces we employ, in particular *exponentially weighted Sobolev spaces*. As remarked in the Introduction, these spaces are needed to handle initial conditions of the form $h(\sigma, x) := |e^x - K|_+$, $(\sigma, x) \in (0, \infty) \times \mathbb{R}$, which arise in applications. We follow primarily Amann (1995), Lunardi (1995), and Pazy (1983).

A.1 Unbounded operators and c_0 semi-groups

We begin by recalling the notion of a semi-group generated by a linear operator. Throughout, $\mathcal{L}(X)$ *will denote the space of bounded linear operators on a Banach space* X, which is a Banach algebra using the operator norm.

Definition 4 – Let X be a Banach space. A *strongly continuous* or c_0 *semi-group of operators on* X is a family of bounded operators $S(t)\colon X \to X$, $t \geq 0$, satisfying:

(i) $S(t_1 + t_2) = S(t_1)S(t_2)$, for all $t_i \geq 0$,

(ii) $S(0) = I$, where I represent the identity operator on X,

(iii) $\lim_{t\to 0} S(t)x = x$, for all $x \in X$, where the limit is taken with respect to the topology of X.

It follows from Item (iii) that $S(t)$ is strongly continuous in t, that is, the map $S(\cdot)x\colon [0, \infty) \to X$ is continuous for every $\xi \in X$, hence the name.

We will need also the notion of analytic semi-groups. To this end, for a given $\delta > 0$, we let Δ_δ denote the sector:

$$\Delta_\delta := \{z = re^{\imath\theta},\ -\delta < \theta < \delta,\ r > 0\}. \tag{33}$$

Also, for any Banach space X, let $\mathcal{L}(X)$ denote the Banach algebra of bounded operators on X.

Definition 5 – Let X be a Banach space. An *analytic semi-group of operators on X* is a function $S\colon \Delta_\delta \cup \{0\} \to \mathcal{L}(X)$, $\delta > 0$, with the properties

(i) S is analytic in Δ_δ;

(ii) $S(z_1 + z_2) = S(z_1)S(z_2)$, if $z_i \in \Delta_\delta \cup \{0\}$;

(iii) $S(0) = I$, the identity operator on X;

(iv) $\lim_{z\to 0} S(z)x = x$, for all $x \in X$.

The limit $\lim_{z\to 0} S(z)x$ is computed for $z \in \Delta_\delta$. An analytic semi-group is, in particular, a c_0 semi-group.

Definition 6 – The *generator* T of a c_0 semi-group $S(t)$ on X is the unbounded operator T defined by:

$$T\xi := \lim_{t\searrow 0} t^{-1}\Big(S(t)\xi - \xi,\Big)$$

for every $\xi \in X$ for which the limit exists. The collection of such vectors forms the domain of the operator.

It is known that the generator of a c_0 semi-group is closed and densely defined. We next review criteria for an unbounded operator T to generate a c_0 semi-group $S(t)$. When this is the case, then $u(t) := S(t)h$ is a (suitable) solution of $u' - Tu = 0$, $u(0) = h$. A useful criterion for T to generate a c_0 semi-group is provided by the Lumer-Phillips theorem, which we discuss next. Since two c_0 semi-groups with the same generator coincide (see e.g. Amann 1995; Pazy 1983), we shall write $S(t) = e^{tT}$ for the semi-group generated by T, if such a semi-group exists.

A.2 Dissipativity

In the following, $\mathfrak{Re}(z) = \mathfrak{Re}\, z$ will denote the real part of $z \in \mathbb{C}$. Let X be a Banach space and let X^* denote its dual. If $x \in X$, the Hahn-Banach theorem implies, in particular, that the set

$$\mathcal{F}(x) := \{f \in X^*, f(x) = \|x\|^2 = \|f\|^2\}$$

is not empty.

Definition 7 – A (possibly unbounded) operator T on a Banach space X is called *quasi-dissipative* if there exists $\mu \geq 0$ such that, for every $x \in D(T)$, there exists an $f \in \mathcal{F}(x) \subset X^*$ with the property that and $\mathfrak{Re}\Big(f(Tx - \mu x)\Big) \leq 0$.

This definition is simply saying that for some $\mu > 0$, the operator $Tx - \mu x$ is dissipative. The *numerical range* of T, denoted $\mathfrak{N}(T)$, is the set

$$\mathfrak{N}(T) := \{f(Tx),\ \|x\| = 1, f \in \mathcal{F}(x)\}. \tag{34}$$

A quasi-dissipative operator T is thus one that has the property that

$$\mathfrak{N}(T) \subset \{z \in \mathbb{C},\ \mathfrak{Re}(z) \leq \mu\} = \mu + \Delta_{\pi/2}^c \tag{35}$$

with Δ_δ defined in Equation (33) and $\Delta_\delta^c := \mathbb{C} \smallsetminus \Delta_\delta$ its complement.

Quasi-dissipativity, together with some mild conditions on the operator T stated below, is sufficient for the generation of a c_0 semi-group, by the celebrated Lumer-Phillips theorem, which we now recall, in the form that we are going to use, for the benefit of the reader[22].

Theorem 7 (Lumer-Phillips) – *Let X be a Banach space and let T be a densely defined, quasi-dissipative operator on X such that $T - \lambda$ is invertible for λ large. Then T generates a c_0 semi-group on X.*

By strengthening condition (35), we obtain the following similar theorem that yields generators of *analytic semi-groups*. The proof of this theorem is contained in the proof of Pazy (1983, Theorem 7.2.7).

Theorem 8 – *Let X be a Banach space and let T be a densely defined operator on X such that $\mathfrak{N}(T) \subset \mu + \Delta_\vartheta^c$ for some $\mu \in \mathbb{R}$ and some $\vartheta > \pi/2$. Assume also that $T - \lambda$ is invertible for λ large. Then T generates an analytic semi-group.*

We note that the assumption that $T - \lambda$ be invertible in Theorem 7 and Theorem 8 implies that T is closed. The theorem above is especially useful when T is a uniformly strongly elliptic operator (see Definition 2 on p. 125) in view of the following Lemma, the proof of which is again contained in the proof of Pazy (1983, Theorem 7.2.7). See also Lions (1961).

Lemma 20 – *Let P be an order $2m$ differential operator on some domain $\Omega \subset \mathbb{R}^n$, regarded as an unbounded operator on $L^2(\Omega)$ with domain $D(P) \subset H^{2m}(\Omega)$. We assume that there exists $C > 0$ such that*

$$\mathfrak{Re}(Pv, v) \leq -C^{-1}\|v\|_{H^m(\Omega)} \quad \textit{and} \quad |(Pv, v)| \leq C\|v\|_{H^m(\Omega)}, \quad (\forall)\, v \in D(P).$$

Then $\mathfrak{N}(P) \subset \Delta_\vartheta^c$ for some $\vartheta > \pi/2$.

From Theorem 8 and Lemma 20, we get the following corollary.

Corollary 8 – *Let P be as in Lemma 20 and assume that $D(P)$ is dense in $L^2(\Omega)$ and that $P - \lambda$ is invertible for λ large. Then P generates an analytic semi-group on X.*

[22] Amann, 1995, *Linear and quasilinear parabolic problems. Vol. I*;
Pazy, 1983, *Semigroups of linear operators and applications to partial differential equations.*

A.3 Classical and other types of solutions

We consider the initial-value problem for abstract parabolic equations of the form (10) (that is $\partial_t u - Pu = F$, $u(0) = h \in X$) where P is a (usually unbounded) operator on a Banach space X and with domain $D(P)$. In our applications, X will be a space of functions on Ω, but it is convenient to consider this equation also abstractly, from the point of view of semi-groups of operators.

Definition 8 – A function $u\colon [0,T] \to X$ is a *strong solution* of the initial value problem (10) for $F \in \mathcal{C}([0,T];X)$ if

(i) u is continuous for the norm topology on X and $u(0) = h$;

(ii) $\partial_t u = u'$ is defined and continuous as a function $(0,T] \to X$;

(iii) $u(t) \in D(P)$ for $t \in (0,T]$; and

(iv) u satisfies the equation $\partial_t u(t) - Pu(t) = F(t) \in X$, for $t \in (0,T]$.

We shall also need the following weaker form of a solution.

Definition 9 – A function $u\colon [0,T] \to X$ is called a *mild solution* of the initial-value problem (10) if $h \in X$, $F \in L^1([0,T],X)$, and

$$u(t) = e^{tP}h + \int_0^t e^{(t-\tau)P} F(\tau)\, d\tau,$$

with equality as elements of X pointwise in time $t \in (0,T)$.

The following remark recalls the connection between semi-groups and the various types of solutions of the Initial Value Problem (10).

Remark 11 – For the applications of interest in this work, we can reduce to homogeneous equations, that is $F(0) = 0$, as we assume now. We also assume that the operator P generates a c_0 semi-group e^{tP} on X. Then $u(t) := e^{tP}h$ is a mild solution for any $h \in X$. If, moreover, $h \in D(P)$ or if P generates an analytic semi-group, then $u(t) := e^{tP}h$ is also a strong solution of Equation (10) (see Amann 1995; Lunardi 1995; Pazy 1983, for instance).

We specialize to the case when P is a m-th order partial differential operator defined on a domain $\Omega \subset \mathbb{R}^d$:

$$P := \sum_{|\alpha| \le m} a_\alpha \partial^\alpha, \tag{36}$$

with coefficients $a_\alpha \in C^\infty(\overline{\Omega})$, and assume that X is a space of functions on Ω, that is, $X \subset L^1_{loc}(\Omega)$. We also assume that the domain of P contains the space of smooth functions with compact support in Ω, and hence the same is satisfied by its adjoint.

We next recall the notion of classical and weak solutions. We use the convenient notation:

$$u(t)(q) := u(t,q), \quad t \geq 0 \text{ and } q \in \Omega,$$

which is in agreement with (10).

Definition 10 – A function $u\colon [0,T] \times \Omega \to \mathbb{C}$ is a *classical solution* of the initial value problem (10) if

(i) u is continuous on $[0,T] \times \Omega$ and $u(0,q) = h(q)$, for all $q \in \Omega$;

(ii) $\partial_t u = u'$ and $\partial^\alpha u$, $|\alpha| \leq m$, are defined and continuous on $(0,T] \times \Omega$; and

(iii) u satisfies the equation $\partial_t u - Pu = F$ pointwise in $(0,T] \times \Omega$.

If boundary conditions for u on $\partial\Omega$ are given, we require them to be satisfied as equalities of continuous functions.

We note that in the abstract setting, strong solutions are often referred to as classical or strict solutions (see e.g. Lunardi 1995; Pazy 1983. The following lemma follows from known results (see Lunardi 1995, Section 4.3, Chapter 5).

Lemma 21 – *Assume that there exists $n \geq 0$ such that $D(P^n) \ni f \to \partial^\alpha f \in \mathcal{C}(\overline{\Omega})$ is continuous for all $|\alpha| \leq m$. In addition, assume that P generates a c_0 semi-group on X and that $F = 0$. Then $u(t) := e^{tP}h$ is a classical solution of Equation* (10) *for all $h \in D(P^{n+1})$.*

We denote by

$$P^t v := \sum_{|\alpha| \leq m} (-1)^{|\alpha|} \partial^\alpha (a_\alpha v) \tag{37}$$

be the *transpose* of P (so that $\int_\Omega (Pu)v\,dx = \int_\Omega u(P^t v)dx$ whenever u and v are compactly supported in Ω).

Definition 11 – A function $u\colon [0,T) \times \Omega \to \mathbb{C}$ is a *weak solution* of the initial value problem (10) if $u, F \in L^1_{loc}([0,T) \times \Omega)$ and, for all $\phi \in \mathcal{C}_c^\infty([0,T) \times \Omega)$,

$$\int_\Omega \Big[\phi(0,x)h(x) + \int_0^T \Big(\partial_t \phi + P^t \phi\Big) u\,dt + \int_0^T \phi F\,dt\Big] dx = 0. \tag{38}$$

If, moreover, u is also a classical solution on $[\delta, T]$ for all $\delta > 0$, $[T < R$, we shall say that v is a *classical* solution on $(0,R)$.

Again, the following lemma is well-known (see e.g. Lunardi 1995; Pazy 1983).

Lemma 22 – *Assume that P generates a c_0 semi-group on X. Then $u(t) := e^{tP}h$ is a weak solution of the homogeneous Initial-Value Problem* (10) *with $F = 0$ for all $h \in X$.*

Combining the two lemmas above we obtain.

Proposition 5 – *Assume that $D(P^n) \ni f \to \partial^\alpha f \in \mathcal{C}(\overline{\Omega})$ is continuous for all $|\alpha| \leq m$, for some $n \geq 0$. Assume in addition that P generates an analytic semi-group on X and that $F = 0$. Then, for all $h \in X$, $u(t) := e^{tP}h$ is a classical solution on $(0, \infty)$ of the IVP* (10).

After this paper was first circulated, a related interesting preprint[23] also appeared.

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