North-Western European Journal of Mathematics

Algebraic cycles on certain hyperkähler fourfolds with an order 3 non-symplectic automorphism

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Received: December 25, 2016/Accepted: December 5, 2017/Online: April 12, 2018

Abstract

Let X be a hyperkähler variety, and assume X has a non-symplectic automorphism σ of order $> \frac{1}{2}\dim X$. Bloch's conjecture predicts that the quotient $X/<\sigma>$ should have trivial Chow group of 0-cycles. We verify this for Fano varieties of lines on certain special cubic fourfolds having an order 3 non-symplectic automorphism.

Keywords: Algebraic cycles, Chow groups, motives, Bloch's conjecture, Bloch–Beilinson filtration, hyperkähler varieties, Fano varieties of lines on cubic fourfolds, multiplicative Chow-Künneth decomposition, splitting property, finite-dimensional motive.

MSC: Primary 14C15, 14C25, 14C30.

1 Introduction

Let X be a smooth projective variety over $\mathbb{C}$, and let $A^i(X) := CH^i(X)_{\mathbb{Q}}$ denote the Chow groups of X (i.e. the groups of codimension i algebraic cycles on X with $\mathbb{Q}$-coefficients, modulo rational equivalence). Let $A^i_{hom}(X)$ denote the subgroup of homologically trivial cycles. It does not seem an exaggeration to say that the field of algebraic cycles is filled with open questions[2]. Among these open questions, a prominent position is occupied by Bloch's conjecture, proudly and sturdily overtowering the field like an unscalable mountain top.

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[2]Bloch, 1980, *Lectures on algebraic cycles*;
Jannsen, 1994, "Motivic sheaves and filtrations on Chow groups";
Jannsen, 2007, "On finite–dimensional motives and Murre's conjecture";
Murre, Nagel, and Peters, 2013, *Lectures on the theory of pure motives*;
Voisin, 2014b, *Chow rings, decomposition of the diagonal, and the topology of families*.

Conjecture 1 (Bloch 1980) – *Let X be a smooth projective variety of dimension n. Let $\Gamma \in A^n(X \times X)$ be such that*

$$\Gamma_* = 0\colon H^i(X,\mathcal{O}_X) \to H^i(X,\mathcal{O}_X) \quad \forall i > 0 .$$

Then

$$\Gamma_* = 0\colon A^n_{\mathrm{hom}}(X) \to A^n(X) .$$

A particular case of Conjecture 1 is the following:

Conjecture 2 (Bloch 1980) – *Let X be a smooth projective variety of dimension n. Assume that*

$$H^i(X,\mathcal{O}_X) = 0 \quad \forall i > 0 .$$

Then

$$A^n(X) \cong \mathbb{Q} .$$

The "absolute version" (Conjecture 2) is obtained from the "relative version" (Conjecture 1) by taking Γ to be the diagonal. Conjecture 2 is famously open for surfaces of general type (cf. Pedrini and Weibel 2015; Voisin 2014a for some recent progress).

Let us now suppose that X is a hyperkähler variety (i.e., a projective irreducible holomorphic symplectic manifold[3]), say of dimension $2m$. Suppose there exists a non-symplectic automorphism $\sigma \in \mathrm{Aut}(X)$ of order $k > m$. This implies that

$$\left(\sigma + \sigma^2 + \ldots + \sigma^k\right)_* = 0\colon H^i(X,\mathcal{O}_X) \to H^i(X,\mathcal{O}_X) \quad \forall i > 0 .$$

Conjecture 1 (applied to the correspondence $\Gamma = \sum_{j=1}^k \Gamma_{\sigma^j} \in A^{2m}(X \times X)$, where Γ_f denotes the graph of an automorphism $f \in \mathrm{Aut}(X)$) then predicts the following:

Conjecture 3 – *Let X be a hyperkähler variety of dimension $2m$. Let $\sigma \in \mathrm{Aut}(X)$ be an order k non-symplectic automorphism, and assume $k > m$. Then*

$$\left(\sigma + \sigma^2 + \ldots + \sigma^k\right)_* = 0\colon A^{2m}_{\mathrm{hom}}(X) \to A^{2m}(X) .$$

The main result of this note is that Conjecture 3 is true for a certain family of hyperkähler fourfolds:

Theorem (= Theorem 6) – *Let $Y \subset \mathbb{P}^5(\mathbb{C})$ be a smooth cubic fourfold defined by an equation*

$$f(X_0,X_1,X_2,X_3) + g(X_4,X_5) = 0 ,$$

[3] Beauville, 1983a, "Some remarks on Kähler manifolds with $c_1 = 0$";
Beauville, 1983b, "Variétés Kähleriennes dont la première classe de Chern est nulle".

where f and g are homogeneous polynomials of degree 3. Let $X = F(Y)$ be the Fano variety of lines in Y. Let $\sigma \in \mathrm{Aut}(X)$ be the order 3 automorphism induced by

$$\begin{aligned} \mathbb{P}^5(\mathbb{C}) &\to \mathbb{P}^5(\mathbb{C})\,, \\ [X_0 : \ldots : X_5] &\mapsto [X_0 : X_1 : X_2 : X_3 : \nu X_4 : \nu X_5] \end{aligned}$$

(where ν is a primitive 3rd root of unity).

Then

$$(\mathrm{id} + \sigma + \sigma^2)_* \, A^4_{\mathrm{hom}}(X) = 0\,.$$

As an immediate consequence of Theorem 6, we find that Bloch's Conjecture 2 is verified for the quotient:

Corollary (= Corollary 2) – *Let X and σ be as in Theorem 6, and let $Z := X/<\sigma>$ be the quotient. Then*

$$A^4(Z) \cong \mathbb{Q}\,.$$

Another consequence (corollary 3) is that a certain instance of the generalized Hodge conjecture is verified.

The proof of Theorem 6 relies on the theory of finite-dimensional motives[4], combined with the Fourier decomposition of the Chow ring of X constructed by Shen and Vial[5].

Convention 1 – In this article, the word *variety* will refer to a reduced irreducible scheme of finite type over $\mathbb{C}$. A *subvariety* is a (possibly reducible) reduced subscheme which is equidimensional.

All Chow groups will be with rational coefficients: we will denote by $A_j(X)$ the Chow group of j-dimensional cycles on X with $\mathbb{Q}$-coefficients; for X smooth of dimension n the notations $A_j(X)$ and $A^{n-j}(X)$ are used interchangeably.

The notations $A^j_{\mathrm{hom}}(X)$, $A^j_{AJ}(X)$ will be used to indicate the subgroups of homologically trivial, resp. Abel–Jacobi trivial cycles. For a morphism $f : X \to Y$, we will write $\Gamma_f \in A_*(X \times Y)$ for the graph of f. The contravariant category of Chow motives (i.e., pure motives with respect to rational equivalence as in Murre, Nagel, and Peters 2013; Scholl 1994) will be denoted $\mathcal{M}_{\mathrm{rat}}$.

We will write $H^j(X)$ to indicate singular cohomology $H^j(X, \mathbb{Q})$.

[4] Kimura, 2005, "Chow groups are finite dimensional, in some sense".
[5] Shen and Vial, 2016a, "The Fourier transform for certain hyperKähler fourfolds".

2 Preliminaries

2.1 Quotient varieties

Definition 1 – A *projective quotient variety* is a variety

$$Z = X/G\,,$$

where X is a smooth projective variety and $G \subset \mathrm{Aut}(X)$ is a finite group.

Proposition 1 (Fulton 1998) – *Let Z be a projective quotient variety of dimension n. Let $A^*(Z)$ denote the operational Chow cohomology ring. The natural map*

$$A^i(Z) \;\to\; A_{n-i}(Z)$$

is an isomorphism for all i.

Proof. This is Fulton (1998, Example 17.4.10). □

Remark 1 – It follows from Proposition 1 that the formalism of correspondences goes through unchanged for projective quotient varieties (this is also noted in Fulton 1998, Example 16.1.13). We may thus consider motives $(Z,p,0) \in \mathcal{M}_{\mathrm{rat}}$, where Z is a projective quotient variety and $p \in A^n(Z\times Z)$ is a projector. For a projective quotient variety $Z = X/G$, one readily proves (using Manin's identity principle) that there is an isomorphism of motives

$$h(Z) \cong h(X)^G := (X, \Delta_G, 0) \quad \text{in } \mathcal{M}_{\mathrm{rat}}\,,$$

where Δ_G denotes the idempotent $\frac{1}{|G|}\sum_{g\in G}\Gamma_g$.

2.2 Finite-dimensional motives

We refer to Kimura (2005, Definition 3.7) for the definition of finite-dimensional motive (cf. also André 2004; Ivorra 2011; Jannsen 2007 and Murre, Nagel, and Peters 2013, Chapters 4 and 5 for further context and applications). The following two results provide a lot of examples:

Theorem 1 (Kimura 2005) – *Let X be a smooth projective variety, and assume X is dominated by a product of curves. Then X has finite-dimensional motive.*

Proof. A smooth projective curve has finite-dimensional motive[6]. Since finite-dimensionality is stable under taking products of varieties[7], a product of curves has finite-dimensional motive. Applying Kimura (2005, Proposition 6.9), this implies that X has finite-dimensional motive. □

[6] Kimura, 2005, "Chow groups are finite dimensional, in some sense", Corollary 4.4.
[7] Ibid., Corollary 5.11.

Theorem 2 – *Let X be a smooth projective variety, and let $\widetilde{X}$ be the blow-up of X with smooth center $Y \subset X$. If X and Y have finite-dimensional motive, then also $\widetilde{X}$ has finite-dimensional motive.*

Proof. This is well-known, and follows from the blow-up formula for Chow motives[8]. □

An essential property of varieties with finite-dimensional motive is embodied by the nilpotence theorem:

Theorem 3 (Kimura 2005) – *Let X be a smooth projective variety of dimension n with finite-dimensional motive. Let $\Gamma \in A^n(X \times X)$ be a correspondence which is numerically trivial. Then there is $N \in \mathbb{N}$ such that*

$$\Gamma^{\circ N} = 0 \in A^n(X \times X) .$$

Proof. This is Kimura (2005, Proposition 7.5). □

Actually, the nilpotence property (for all powers of X) could serve as an alternative definition of finite-dimensional motive, as shown by Jannsen[9].

Conjecturally, any variety has finite-dimensional motive[10]; we are still far from knowing this.

2.3 MCK decomposition

Definition 2 (Murre 1993) – Let X be a projective quotient variety of dimension n. We say that X has a *CK decomposition* if there exists a decomposition of the diagonal

$$\Delta_X = \pi_0 + \pi_1 + \cdots + \pi_{2n} \quad \text{in } A^n(X \times X) ,$$

such that the π_i are mutually orthogonal idempotents and $(\pi_i)_* H^*(X) = H^i(X)$.

(NB: “CK decomposition” is shorthand for “Chow–Künneth decomposition”.)

Remark 2 – The existence of a CK decomposition for any smooth projective variety is part of Murre’s conjectures[11].

Definition 3 (Shen and Vial 2016a) – Let X be a projective quotient variety of dimension n. Let $\Delta_X^{sm} \in A^{2n}(X \times X \times X)$ be the class of the small diagonal

$$\Delta_X^{sm} := \Big\{ (x,x,x) \mid x \in X \Big\} \subset X \times X \times X .$$

[8] Scholl, 1994, “Classical motives”, Theorem 2.8.

[9] Jannsen, 2007, “On finite–dimensional motives and Murre’s conjecture”, Corollary 3.9.

[10] Kimura, 2005, “Chow groups are finite dimensional, in some sense”, Conjecture 7.1.

[11] Jannsen, 1994, “Motivic sheaves and filtrations on Chow groups”;
Murre, 1993, “On a conjectural filtration on the Chow groups of an algebraic variety, parts I and II”.

An *MCK decomposition* is a CK decomposition $\{\pi_i^X\}$ of X that is *multiplicative*, i.e. it satisfies

$$\pi_k^X \circ \Delta_X^{sm} \circ (\pi_i^X \times \pi_j^X) = 0 \quad \text{in } A^{2n}(X \times X \times X) \quad \text{for all } i+j \neq k\ .$$

(NB: "MCK decomposition" is shorthand for "multiplicative Chow–Künneth decomposition".)

A *weak MCK decomposition* is a CK decomposition $\{\pi_i^X\}$ of X that satisfies

$$\Big(\pi_k^X \circ \Delta_X^{sm} \circ (\pi_i^X \times \pi_j^X)\Big)_*(a \times b) = 0 \quad \text{for all } a,b \in\ A^*(X)\ .$$

Remark 3 – The small diagonal (seen as a correspondence from $X \times X$ to X) induces the *multiplication morphism*

$$\Delta_X^{sm}\colon\ h(X) \otimes h(X) \ \to\ h(X) \quad \text{in } \mathcal{M}_{\text{rat}}\ .$$

Suppose X has a CK decomposition

$$h(X) = \bigoplus_{i=0}^{2n} h^i(X) \quad \text{in } \mathcal{M}_{\text{rat}}\ .$$

By definition, this decomposition is multiplicative if for any i,j the composition

$$h^i(X) \otimes h^j(X) \ \to\ h(X) \otimes h(X) \ \xrightarrow{\Delta_X^{sm}}\ h(X) \quad \text{in } \mathcal{M}_{\text{rat}}$$

factors through $h^{i+j}(X)$.

If X has a weak MCK decomposition, then setting

$$A^i_{(j)}(X) := (\pi^X_{2i-j})_* A^i(X)\ ,$$

one obtains a bigraded ring structure on the Chow ring: that is, the intersection product sends $A^i_{(j)}(X) \otimes A^{i'}_{(j')}(X)$ to $A^{i+i'}_{(j+j')}(X)$.

It is expected (but not proven !) that for any X with a weak MCK decomposition, one has

$$A^i_{(j)}(X) \stackrel{??}{=} 0 \quad \text{for } j<0\ , \quad A^i_{(0)}(X) \cap A^i_{\text{hom}}(X) \stackrel{??}{=} 0\ ;$$

this is related to Murre's conjectures B and D, that have been formulated for any CK decomposition[12].

The property of having an MCK decomposition is severely restrictive, and is closely related to Beauville's "(weak) splitting property"[13]. For more ample discussion, and examples of varieties with an MCK decomposition, we refer to Shen and Vial (2016a, Section 8), as well as Fu, Tian, and Vial (2016), Shen and Vial (2016b), and Vial (2017).

[12] Murre, 1993, "On a conjectural filtration on the Chow groups of an algebraic variety, parts I and II".

[13] Beauville, 2007, "On the splitting of the Bloch–Beilinson filtration".

2.4 The Fourier decomposition

In what follows, we will make use of the following:

Theorem 4 (Shen and Vial 2016a) – *Let $Y \subset \mathbb{P}^5(\mathbb{C})$ be a smooth cubic fourfold, and let $X := F(Y)$ be the Fano variety of lines in Y. There exists a self-dual CK decomposition $\{\Pi_i^X\}$ for X, and*

$$(\Pi_{2i-j}^X)_* A^i(X) = A^i_{(j)}(X) ,$$

where the right-hand side denotes the splitting of the Chow groups defined in terms of the Fourier transform as in Shen and Vial (2016a, Theorem 2). Moreover, we have

$$A^i_{(j)}(X) = 0 \quad \text{if } j < 0 \text{ or } j > i \text{ or } j \text{ is odd} .$$

In case Y is very general, the Fourier decomposition $A^*_{(*)}(X)$ forms a bigraded ring, and hence $\{\pi_i^X\}$ is a weak MCK decomposition.

Proof. (A matter of notation: what we denote $A^i_{(j)}(X)$ is denoted $CH^i(X)_j$ in Shen and Vial 2016a.)

The existence of a self-dual CK decomposition $\{\Pi_i^X\}$ is Shen and Vial (2016a, Theorem 3.3). (More in detail: Shen and Vial 2016a, Theorem 3.3 applies to any hyperkähler fourfold F of $K3^{[2]}$ type with a cycle class $L \in A^2(F \times F)$ that represents the Beauville–Bogomolov pairing and satisfies Shen and Vial 2016a, equalities (6), (7), (8), (9). For the Fano variety of lines of a cubic fourfold, the cycle L of Shen and Vial 2016a, definition (107) has these properties, as shown in Shen and Vial 2016a, Section 3.)

According to Shen and Vial (2016a, Theorem 3.3), the given CK decomposition agrees with the Fourier decomposition of the Chow groups. The "moreover" part is because the $\{\Pi_i^X\}$ are shown to satisfy Murre's conjecture B[14].

The statement for very general cubics is Shen and Vial (2016a, Theorem 3). □

Remark 4 – Unfortunately, it is not yet known that the Fourier decomposition of Shen and Vial 2016a induces a bigraded ring structure on the Chow ring for *all* Fano varieties X of smooth cubic fourfolds. For one thing, it has not yet been proven that

$$A^2_{(0)}(X) \cdot A^2_{(0)}(X) \overset{??}{\subset} A^4_{(0)}(X)$$

(cf. Shen and Vial 2016a, Section 22.3 for discussion).

[14] Shen and Vial, 2016a, "The Fourier transform for certain hyperKähler fourfolds", Theorem 3.3.

2.5 Refined CK decomposition

Theorem 5 – *Let X be a smooth projective hyperkähler fourfold of $K3^{[2]}$-type. Assume that X has finite-dimensional motive. Then X has a CK decomposition $\{\pi_i^X\}$. Moreover, there exists a further splitting*

$$\pi_2^X = \pi_{2,0}^X + \pi_{2,1}^X \quad \text{in } A^4(X \times X)\,,$$

where $\pi_{2,0}^X$ and $\pi_{2,1}^X$ are orthogonal idempotents, and $\pi_{2,1}^X$ is supported on $C \times D \subset X \times X$, where C and D are a curve, resp. a divisor on X. The action on cohomology verifies

$$(\pi_{2,0}^X)_* H^*(X) = H^2_{tr}(X)\,,$$

where $H^2_{tr}(X) \subset H^2(X)$ is defined as the orthogonal complement of $NS(X)$ with respect to the Beauville–Bogomolov form. The action on Chow groups verifies

$$(\pi_{2,0}^X)_* A^2(X) = (\pi_2^X)_* A^2(X)\,.$$

Proof. It is known[15] that X verifies the Lefschetz standard conjecture $B(X)$. Combined with finite-dimensionality, this implies the existence of a CK decomposition[16].

For the "moreover" statement, one observes that X verifies conditions (*) and (**) of Vial's[17], and so Vial (2013, Theorems 1 and 2) apply. This gives the existence of refined CK projectors $\pi_{i,j}^X$, which act on cohomology as projectors on gradeds for the "niveau filtration" $\widetilde{N}^*$ of loc. cit. In particular, $\pi_{2,1}^X$ acts as projector on $NS(X)$, and $\pi_{2,0}^X$ acts as projector on $H^2_{tr}(X)$. The projector $\pi_{2,1}^X$, being supported on $C \times D$, acts trivially on $A^2(X)$ for dimension reasons; this proves the last equality. □

3 Main result

Theorem 6 – *Let $Y \subset \mathbb{P}^5(\mathbb{C})$ be a smooth cubic fourfold defined by an equation*

$$f(X_0, X_1, X_2, X_3) + g(X_4, X_5) = 0\,,$$

where f and g are homogeneous polynomials of degree 3. Let $X = F(Y)$ be the Fano variety of lines in Y. Let $\sigma \in \mathrm{Aut}(X)$ be the order 3 non-symplectic automorphism induced by

$$\begin{aligned} \sigma_{\mathbb{P}} \colon \mathbb{P}^5(\mathbb{C}) &\to \mathbb{P}^5(\mathbb{C})\,, \\ [X_0 : \ldots : X_5] &\mapsto [X_0 : X_1 : X_2 : X_3 : \nu X_4 : \nu X_5]\,, \end{aligned}$$

where ν is a primitive 3rd root of unity.

[15] Charles and Markman, 2013, "The standard conjectures for holomorphic symplectic varieties deformation equivalent to Hilbert schemes of $K3$ surfaces".

[16] Jannsen, 1994, "Motivic sheaves and filtrations on Chow groups", Lemma 5.4.

[17] Vial, 2013, "Niveau and coniveau filtrations on cohomology groups and Chow groups".

Then

$$(\mathrm{id}+\sigma+\sigma^2)_* \, A^i_{(j)}(X) = 0 \quad \textit{for } (i,j) \in \{(2,2),(4,2),(4,4)\}\,.$$

In particular,

$$(\mathrm{id}+\sigma+\sigma^2)_* \, A^4_{\mathrm{hom}}(X) = 0\,.$$

Proof. (NB: the family of Fano varieties of Theorem 6 is described in Boissière, Camere, and Sarti 2016, Example 6.5, from which I learned that the automorphism σ is non-symplectic.)

The last phrase of the theorem follows from the one-but-last phrase, since[18]

$$A^4_{\mathrm{hom}}(X) = A^4_{(2)}(X) \oplus A^4_{(4)}(X)\,.$$

In a first step of the proof, let us show that the automorphism σ respects (most of) the Fourier decomposition of the Chow ring:

Proposition 2 – *Let X and σ be as in Theorem 6. Let $A^*_{(*)}(X)$ be the Fourier decomposition (Theorem 4). Then*

$$\sigma_* A^i_{(j)}(X) \subset A^i_{(j)}(X) \quad \forall (i,j) \neq (2,0)\,.$$

Proof. Here, the alternative description of the Fourier decomposition $A^*_{(*)}(X)$ in terms of a certain rational map $\phi\colon X \dashrightarrow X$ comes in handy.

Let $Y \subset \mathbb{P}^5(\mathbb{C})$ be any smooth cubic fourfold (not necessarily with automorphisms), and let $X = F(Y)$ be the Fano variety of lines in Y. There exists a degree 16 rational map[19][20]

$$\phi\colon X \dashrightarrow X\,.$$

The map ϕ is defined as follows: Let $x \in X$ be a point, and let $\ell \subset Y$ be the line corresponding to x. For a general point $x \in X$, there is a unique plane $H \subset \mathbb{P}^5$ that is tangent to Y along ℓ. Then $\phi(x) \in X$ is defined as the point corresponding to $\ell' \subset Y$, where

$$H \cap Y = 2\ell + \ell'\,.$$

As in Shen and Vial (2016a, Definition 21.8), for any $\lambda \in \mathbb{Q}$ let us consider the eigenspaces

$$V^i_\lambda := \Big\{ c \in A^i(X) \mid \phi^*(c) = \lambda \cdot c \Big\}\,.$$

These eigenspaces are related to the Fourier decomposition of the Chow ring: indeed, Shen–Vial show[21] that there is a decomposition

$$A^i_{(j)}(X) = V^i_{\lambda_1} \oplus \cdots \oplus V^i_{\lambda_r} \quad \forall (i,j) \neq (2,0)\,. \tag{1}$$

Let us now return to X and σ as in Theorem 6, and let us prove Proposition 2. In view of the decomposition (1), we see that to prove Proposition 2, it suffices to prove the following:

Claim 1 – Let X and σ be as in theorem 6. Then

$$\phi^*\sigma^* = \sigma^*\phi^* \colon A^i(X) \ \to \ A^i(X)\ .$$

In order to prove the claim, we first establish a little lemma:

Lemma 1 – *Set-up as above. There is an equality of rational maps*

$$\phi\circ\sigma = \sigma\circ\phi \colon X \ \dashrightarrow \ X\ .$$

Proof. Let $x\in X$ be a point outside of the indeterminacy locus of ϕ, and let $H\subset \mathbb{P}^5$ be the plane tangent to Y along the line ℓ corresponding to x. By definition, $\phi(x)\in X$ is the point corresponding to $\ell'\subset Y$, where

$$H\cap Y = 2\ell + \ell'\ .$$

Let $\sigma_{\mathbb{P}}\colon \mathbb{P}^5\to\mathbb{P}^5$ denote the linear transformation inducing the automorphism σ. The plane $\sigma_{\mathbb{P}}(H)$ is tangent to Y along $\sigma_{\mathbb{P}}(\ell)$, and

$$\sigma_{\mathbb{P}}(H)\cap Y = 2\sigma_{\mathbb{P}}(\ell) + \sigma_{\mathbb{P}}(\ell')\ .$$

It follows that $\phi(\sigma(x)) = \sigma(\phi(x))$. □

Lemma 1 furnishes a commutative diagram

$$\begin{array}{ccc} X & \stackrel{\phi}{\dashrightarrow} & X \\ \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \sigma} \\ X & \stackrel{\phi}{\dashrightarrow} & X\ . \end{array}$$

This can be "resolved" by a commutative diagram

$$\begin{array}{ccccc} X & \stackrel{p'}{\leftarrow} & Z' & \stackrel{q'}{\rightarrow} & X \\ \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \sigma_Z} & & \downarrow{\scriptstyle \sigma} \\ X & \stackrel{p}{\leftarrow} & Z & \stackrel{q}{\rightarrow} & X\ , \end{array}$$

where horizontal arrows are birational morphisms such that $\phi\circ p' = q'$ and $\phi\circ p = q$ (and so $\phi^* = p_\ast q^* = (p')_\ast (q')^*\colon A^i(X)\to A^i(X)$.).

Let us now prove Claim 1. We have equalities

$$\begin{aligned} \phi^*\sigma^* &= (p')_\ast (q')^*\sigma^* \\ &= (p')_\ast (\sigma_Z)^* q^* \\ &= \sigma^* p_\ast q^* \\ &= \sigma^*\phi^* \colon A^i(X) \ \to\ A^i(X)\ . \end{aligned}$$

Here, in the third equality we have used the following:

Sublemma 1 – *Set-up as above. There is equality*

$$(p')_*(\sigma_Z)^* = \sigma^* p_* \colon A^i(Z) \to A^i(X)\,.$$

Proof. Since σ_Z is a birational morphism, there is equality

$$\sigma_*(p')_*(\sigma_Z)^* = p_*(\sigma_Z)_*(\sigma_Z)^* = p_* \colon A^i(Z) \to A^i(X)\,.$$

Composing on the left with σ^*, this implies

$$\sigma^*\sigma_*(p')_*(\sigma_Z)^* = \sigma^* p_* \colon A^i(Z) \to A^i(X)\,. \qquad \square$$

But $\sigma^* = (\sigma^2)_*$ and so the left-hand side simplifies to $(p')_*(\sigma_Z)^*$, proving the sublemma. □

For later use, we recast Proposition 2 as follows:

Corollary 1 – *Set-up as above. Let $\{\Pi_j^X\}$ be a CK decomposition as in Theorem 4. Then*

$$\sigma_*(\Pi_j^X)_* = (\Pi_j^X)_*\sigma_*(\Pi_j^X)_* \colon A^i(X) \to A^i(X) \quad \forall (i,j) \neq (2,4)\,.$$

Proof. This is just a translation of Proposition 2, using the fact that Π_j^X acts on $A^i(X)$ as projector on $A^i_{(2i-j)}(X)$. □

The second step of the proof is to ascertain that X has finite-dimensional motive:

Proposition 3 – *Let $Y \subset \mathbb{P}^5(\mathbb{C})$ and $X = F(Y)$ be as in Theorem 6. Then Y and X have finite-dimensional motive.*

Proof. To establish finite-dimensionality of Y is an easy exercice in using what is known as the "Shioda inductive structure"[22]. Indeed, applying Katsura and S. Shioda (1979, Remark 1.10), we find there exists a dominant rational map

$$\phi \colon Y_1 \times Y_2 \dashrightarrow Y\,,$$

where $Y_1 \subset \mathbb{P}^3(\mathbb{C})$ is the smooth cubic threefold defined as

$$f(X_0, X_1, X_2, X_3) + V^3 = 0\,,$$

and $Y_2 \subset \mathbb{P}^2(\mathbb{C})$ is the smooth cubic curve defined as

$$g(X_0, X_1) + W^3 = 0\,.$$

The indeterminacy locus of ϕ is resolved by blowing up the locus $S \times P \subset Y_1 \times Y_2$, where $S \subset Y_1$ is a cubic surface, and $P \subset Y_2$ is a set of points. Let us call this blow-up $\hat{Y}$. Using Theorems 1 and 2 and an induction on the dimension, we find that $\hat{Y}$ has finite-dimensional motive. Since $\hat{Y}$ dominates Y, it follows from Kimura (2005, Proposition 6.9) that the cubic Y has finite-dimensional motive.

Finally, Laterveer (2017, Theorem 4) states that for any cubic Y with finite-dimensional motive, the Fano variety $X = F(Y)$ also has finite-dimensional motive. □

The third step of the proof is to show the desired statement for $A^2_{(2)}(X)$, i.e. we now prove that

$$(\mathrm{id}+\sigma+\sigma^2)_\ast\, A^2_{(2)}(X)=0\ . \qquad (2)$$

In order to do so, let us abbreviate

$$\Delta_G:=\frac{1}{3}\Big(\Delta_X+\Gamma_\sigma+\Gamma_{\sigma\circ\sigma}\Big)\in\ A^4(X\times X)\ .$$

Since the action of σ is non-symplectic[23], we have that

$$(\Delta_G)_\ast=0\colon\ H^2(X,\mathcal{O}_X)\ \to\ H^2(X,\mathcal{O}_X)\ .$$

Using the Lefschetz $(1,1)$-theorem, we see that

$$\Delta_G\circ\Pi_2^X=\gamma\quad\text{in } H^8(X\times X)\ ,$$

where γ is some cycle supported on $D\times D\subset X\times X$, for some divisor $D\subset X$. In other words, the correspondence

$$\Gamma:=\Delta_G\circ\Pi_2^X-\gamma\in A^4(X\times X)$$

is homologically trivial. But then (since X has finite-dimensional motive) there exists $N\in\mathbb{N}$ such that

$$\Gamma^{\circ N}=0\quad\text{in } A^4(X\times X)\ .$$

Upon developing this expression, one finds an equality

$$\Gamma^{\circ N}=(\Delta_G\circ\Pi_2^X)^{\circ N}+\gamma'=0\quad\text{in } A^4(X\times X)\ ,$$

where γ' is supported on $D\times D\subset X\times X$. In particular, γ' acts trivially on $A^2_{(2)}(X)\subset A^2_{AJ}(X)$, and so

$$\Big((\Delta_G\circ\Pi_2^X)^{\circ N}\Big)_\ast=0\colon\ A^2_{(2)}(X)\ \to\ A^2(X)\ .$$

Corollary 1 (combined with the fact that Δ_G and Π_2^X are idempotents) implies that

$$\Big((\Delta_G\circ\Pi_2^X)^{\circ N}\Big)_\ast=(\Delta_G\circ\Pi_2^X)_\ast\colon\ A^i(X)\ \to\ A^i(X)\ ,$$

and so we find that

$$\Big(\Delta_G\circ\Pi_2^X\Big)_\ast=(\Delta_G)_\ast=0\colon\ A^2_{(2)}(X)\ \to\ A^2(X)\ .$$

This proves equality (2).

The argument for $A^4_{(2)}(X)$ is similar: the correspondence Γ being homologically trivial, its transpose

$$ {}^t\Gamma = \Pi^X_6 \circ \Delta_G - \gamma'' \in A^4(X \times X) $$

is also homologically trivial (where γ'' is supported on $D \times D$). Using nilpotence and Lemma 1, this implies (just as above) that

$$ (\Pi^X_6 \circ \Delta_G)_* = \big(\Delta_G \circ \Pi^X_6\big)_* = (\Delta_G)_* = 0 \colon A^4_{(2)}(X) \ \to \ A^4(X) \, . $$

In the final step of the proof, it remains to consider the action on $A^4_{(4)}(X)$. Ideally, one would like to use Vial's projector $\pi^X_{4,0}$ of Vial (2013) (mentioned in the proof of Theorem 5). Unfortunately, this approach runs into problems (cf. Remark 5). We therefore proceed somewhat differently: to establish the statement for $A^4_{(4)}(X)$, we use the following proposition:

Proposition 4 – *Notation as above. One has*

$$ \Delta_G \circ \Pi^X_4 - R = 0 \quad \textit{in } H^8(X \times X) \, , $$

where $R \in A^4(X \times X)$ is a correspondence with the property that

$$ R_* = 0 \colon A^4(X) \ \to \ A^4(X) \, . $$

Obviously, this proposition clinches the proof: using the nilpotence theorem, one sees that there exists $N \in \mathbb{N}$ such that

$$ \big(\Delta_G \circ \Pi^X_4 + R\big)^{\circ N} = 0 \quad \text{in } A^4(X \times X) \, . $$

Developing, and applying the result to $A^4(X)$, one finds that

$$ \big((\Delta_G \circ \Pi^X_4)^{\circ N}\big)_* = 0 \colon A^4(X) \ \to \ A^4(X) \, . $$

Corollary 1 (combined with the fact that Δ_G and Π^X_4 are idempotents) implies that

$$ \big((\Delta_G \circ \Pi^X_4)^{\circ N}\big)_* = (\Delta_G \circ \Pi^X_4)_* \colon A^i(X) \ \to \ A^i(X) \quad \forall i \neq 2 \, . $$

Therefore, we conclude that

$$ \big(\Delta_G \circ \Pi^X_4\big)_* = (\Delta_G)_* = 0 \colon A^4_{(4)}(X) \ \to \ A^4(X) \, . $$

It only remains to prove Proposition 4. Here, we use the fact that X is of $K3^{[2]}$-type and so there is an isomorphism[24]

$$ H^4(X) = \mathrm{Sym}^2 H^2(X) \, . $$

Using the truth of the standard conjectures for X[25], and the semi-simplicity of motives for numerical equivalence[26], this means that the map

$$\Delta^{sm}\colon h^2(X)\otimes h^2(X) \ \to\ h^4(X) \quad \text{in } \mathcal{M}_{\text{hom}}$$

admits a right-inverse, where $\Delta^{sm}\in A^8((X\times X)\times X)$ is as before the "small diagonal" (cf. definition 3). Let $\Psi\in A^4(X\times(X\times X))$ denote this right-inverse.

Using the splitting $\pi_2^X=\pi_{2,0}^X+\pi_{2,1}^X$ in $A^4(X\times X)$ of Theorem 5, one obtains a splitting modulo homological equivalence of Π_4^X in 4 components

$$\begin{aligned}\Pi_4^X &= \Pi_4^X\circ\Delta^{sm}\circ(\pi_2^X\times\pi_2^X)\circ\Psi\circ\Pi_4^X\\ &= \Pi_4^X\circ\Delta^{sm}\circ\Big((\pi_{2,0}^X+\pi_{2,1}^X)\times(\pi_{2,0}^X+\pi_{2,1}^X)\Big)\circ\Psi\circ\Pi_4^X\\ &= \sum_{k,\ell\in\{0,1\}}\Pi_4^X\circ\Delta^{sm}\circ(\pi_{2,k}^X\times\pi_{2,\ell}^X)\circ\Psi\circ\Pi_4^X\\ &=: \sum_{k,\ell\in\{0,1\}}\Pi_{4,k,\ell}^X \quad \text{in } H^8(X\times X)\,.\end{aligned}$$

We note that (by construction) $\Pi_{4,0,0}^X$ acts as a projector on

$$\mathrm{Sym}^2H^2_{tr}(X)\subset \mathrm{Sym}^2H^2(X)=H^4(X)\,.$$

Also, we recall that $\pi_{2,1}^X$ is supported on $C\times D\subset X\times X$ (Theorem 5), which implies that $\Pi_{4,k,\ell}^X\in A^4(X\times X)$ is supported on $X\times D$ for $(k,\ell)\neq(0,0)$.

It will be convenient to consider the transpose decomposition

$$\Pi_4^X={}^t\Pi_4^X={}^t\Pi_{4,0,0}^X+{}^t\Pi_{4,1,0}^X+{}^t\Pi_{4,0,1}^X+{}^t\Pi_{4,1,1}^X \quad \text{in } H^8(X\times X)$$

(where we have used that Π_4^X is transpose-invariant, cf. Theorem 4).

This decomposition induces in particular a decomposition

$$\Delta_G\circ\Pi_4^X=\Delta_G\circ{}^t\Pi_{4,0,0}^X+\Delta_G\circ{}^t\Pi_{4,1,0}^X+{}^t\Delta_G\circ\Pi_{4,0,1}^X+{}^t\Delta_G\circ\Pi_{4,1,1}^X \quad \text{in } H^8(X\times X)\,.$$

The last 3 summands in this decomposition act trivially on $A^4(X)$ (indeed, the correspondence ${}^t\Pi_{4,k,\ell}^X$ is supported on $D\times X\subset X\times X$ for $(k,\ell)\neq(0,0)$, and hence acts trivially on $A^4(X)$). These last 3 summands will form the correspondence called R in Proposition 4. To prove Proposition 4, it remains to establish that

$$\Delta_G\circ{}^t\Pi_{4,0,0}^X=0 \quad \text{in } H^8(X\times X)\,. \tag{3}$$

Taking transpose, one sees this is equivalent to proving that

$$\Pi_{4,0,0}^X\circ\Delta_G=0 \quad \text{in } H^8(X\times X)\,,$$

which in turn (since applying σ^* and projecting to $\mathrm{Sym}^2 H^2_{tr}(X)$ commute) is equivalent to proving that

$$\Delta_G \circ \Pi^X_{4,0,0} = 0 \quad \text{in } H^8(X \times X)\,.$$

Invoking Manin's identity principle, it suffices to prove that

$$\sigma^*(c_1 \cup c_2) + (\sigma^2)^*(c_1 \cup c_2) + c_1 \cup c_2 = 0 \quad \text{in } H^4(X) \quad \forall c_1, c_2 \in H^2_{tr}(X)\,.$$

Thanks to the equality

$$c_1 \cup c_2 = \frac{1}{2}\Big((c_1 + c_2) \cup (c_1 + c_2) - c_1 \cup c_1 - c_2 \cup c_2 \Big)\,,$$

it suffices to prove that

$$\sigma^*(c \cup c) + (\sigma^2)^*(c \cup c) + c \cup c = 0 \quad \text{in } H^4(X) \quad \forall c \in H^2_{tr}(X)\,. \tag{4}$$

We now make the following claim:

Claim 2 – Set-up as above. Let $c \in H^2_{tr}(X)$. Then

$$(\sigma^*)(c) \cup (\sigma^2)^*(c) = c \cup c \quad \text{in } H^4(X).$$

It is readily checked that Claim 2 implies equality (4) (and hence equality (3) and hence also Proposition 4): We have

$$\begin{aligned} \sigma^*(c \cup c) + (\sigma^2)^*(c \cup c) + c \cup c &= \Big(\sigma^*(c) + (\sigma^2)^*(c) \Big)^{\cup 2} - 2\sigma^*(c) \cup (\sigma^2)^*(c) + c \cup c \\ &= 2c \cup c - 2\sigma^*(c) \cup (\sigma^2)^*(c) \\ &= 0 \quad \text{in } H^4(X)\,, \end{aligned}$$

proving equality (4). (Here, the second equality is because $\sigma^*(c) + (\sigma^2)^*(c) = -c$, and the third equality is the claim.)

Let us now prove Claim 2. The point is that the subgroup

$$H := \Big\{ c \in H^2(X) \mid (\sigma^*)(c) \cup (\sigma^2)^*(c) = c \cup c \text{ in } H^4(X) \Big\} \subset \ H^2(X)\,,$$

together with its complexification $H_{\mathbb{C}}$, defines a sub-Hodge structure of $H^2(X)$. Let $\omega \in H^{2,0}(X)$ be a generator. Then ω is in $H_{\mathbb{C}}$ (since $\sigma^*\omega = \nu \cdot \omega$, with $\nu^3 = 1$, ν primitive). But $H^2_{tr}(X) \subset H^2(X)$ is the smallest sub-Hodge structure containing ω, and so we must have

$$H^2_{tr}(X) \subset H\,,$$

which proves Claim 2. □

Remark 5 – To prove the statement for $A^4_{(4)}(X)$ in the final step of the above proof, it would be natural to try and use Vial's projector $\pi^X_{4,0}$ of Vial (2013, Theorems 1 and 2) (mentioned in the proof of Theorem 5). However, this approach is difficult to put into practice: the problem is that it seems impossible to prove that

$$\Delta_G \circ \pi^X_{4,0} = 0 \quad \text{in } H^8(X \times X)\,,$$

short of knowing that (1) $H^4(X) \cap F^1 = N^1H^4(X)$, and (2) $N^1H^4(X) = \widetilde{N}^1H^4(X)$, where N^* is the usual coniveau filtration and $\widetilde{N}^*$ is Vial's niveau filtration. Both (1) and (2) seem difficult.

4 Some corollaries

Corollary 2 – *Let X and σ be as in Theorem 6. Let $Z := X/{<}\sigma{>}$ be the quotient. Then*

$$A^4(Z) \cong \mathbb{Q}\,.$$

Proof. We have a natural isomorphism $A^4(Z) \cong A^4(X)^\sigma$. But Theorem 6 (combined with the fact that $\sigma^* A^4_{(j)}(X) \subset A^4_{(j)}(X)$ for all j, cf. proposition 2) implies that

$$A^4(X)^\sigma \subset A^4_{(0)}(X)\,.$$

Since there exists a σ-invariant ample divisor $L \in A^1(X)$, and L^4 generates the 1-dimensional $\mathbb{Q}$-vector space $A^4_{(0)}(X)$, there is equality

$$A^4(X)^\sigma = A^4_{(0)}(X)\,.$$

□

Corollary 3 – *Let X and σ be as in Theorem 6. Then the invariant part of cohomology*

$$H^4(X)^\sigma \subset H^4(X)$$

is supported on a divisor.

[18] Shen and Vial, 2016a, "The Fourier transform for certain hyperKähler fourfolds", Theorem 4.
[19] Voisin, 2004, "Intrinsic pseudo–volume forms and K–correspondences".
[20] Shen and Vial, 2016a, "The Fourier transform for certain hyperKähler fourfolds", Section 18.
[21] Ibid., Theorem 21.9 and Proposition 21.10.
[22] Katsura and S. Shioda, 1979, "On Fermat varieties";
T. Shioda, 1979, "The Hodge conjecture for Fermat varieties".
[23] Boissière, Camere, and Sarti, 2016, "Classification of automorphisms on a deformation family of hyperkähler fourfolds by p–elementary lattices", Example 6.5 and Lemma 6.2.
[24] Beauville and Donagi, 1985, "La variété des droites d'une hypersurface cubique de dimension 4", Proposition 3.
[25] Charles and Markman, 2013, "The standard conjectures for holomorphic symplectic varieties deformation equivalent to Hilbert schemes of $K3$ surfaces", Theorem 1.1.
[26] Jannsen, 1992, "Motives, numerical equivalence, and semi-simplicity", Theorem 1.

Proof. This follows from Theorem 6 by applying the Bloch–Srinivas "decomposition of the diagonal" argument[27]. For the benefit of readers not familiar with Bloch and Srinivas (1983), we briefly resume this argument.

Let $k \subset \mathbb{C}$ be a subfield such that X and Δ_G are defined over k, and such that k is finitely generated over $\mathbb{Q}$. Let $k(X)$ denote the function field of X_k. Since there is an embedding $k(X) \subset \mathbb{C}$, there is a natural homomorphism

$$A^*(X_{k(X)}) \to A^*(X_{\mathbb{C}})$$

that is injective[28]. In particular, there is an injective homomorphism

$$A^4(X_{k(X)})^\sigma \hookrightarrow A^4(X_{\mathbb{C}})^\sigma .$$

As the right-hand side has dimension 1 (Theorem 6), it follows that also

$$\dim A^4(X_{k(X)})^\sigma = 1 .$$

We now consider the image of $\Delta_G \in A^4(X_k \times X_k)^{\sigma\times\sigma}$ under the restriction homomorphism

$$A^4(X_k \times X_k)^{\sigma\times\sigma} \to \varinjlim A^4(X_k \times U)^\sigma \cong A^4(X_{k(X)})^\sigma = \mathbb{Q}$$

(here the limit is over Zariski opens $U \subset X_k$, and the isomorphism follows from Bloch 1980, Appendix to Lecture 1). This gives a decomposition

$$\Delta_G = x \times X + \gamma \quad \text{in } A^4(X_k \times X_k) ,$$

where γ is supported on $X \times D$ for some divisor $D \subset X$. Considering this decomposition for $X = X_{\mathbb{C}}$, and looking at the action of correspondences on cohomology, we find that

$$H^4(X)^\sigma = (\Delta_G)_* H^4(X) = \gamma_* H^4(X) ,$$

and thus $H^4(X)^\sigma$ is supported on the divisor D. □

Acknowledgments

Thanks to all participants of the Strasbourg 2014/2015 "groupe de travail" based on the monograph Voisin (2014b) for a very stimulating atmosphere. Thanks to the referee for genuinely helpful comments. Many thanks to Kai and Len who accepted to be junior members of the Schiltigheim Math Research Institute.

[27] Bloch and Srinivas, 1983, "Remarks on correspondences and algebraic cycles".
[28] Bloch, 1980, *Lectures on algebraic cycles*, Appendix to Lecture 1.

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