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Qualitative results for parabolic equations involving the p-Laplacian under dynamical boundary conditions

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Abstract

We discuss comparison principles, the asymptotic behaviour, and the occurrence of blow up phenomena for nonlinear parabolic problems involving the p-Laplacian operator of the form

$$\begin{cases} \partial_t u = \Delta_p u + f(t,x,u) & \text{in } \Omega \text{ for } t > 0, \\ \sigma \partial_t u + |\nabla u|^{p-2} \partial_\nu u = 0 & \text{on } \partial\Omega \text{ for } t > 0, \\ u(0,\cdot) = u_0 & \text{in } \overline{\Omega}, \end{cases}$$

where Ω is a bounded domain of $\mathbb{R}^N$ with Lipschitz boundary, and where

$$\Delta_p u := \operatorname{div}\left(|\nabla u|^{p-2} \nabla u\right)$$

is the *p-Laplacian* operator for $p > 1$. As for the *dynamical* time lateral boundary condition $\sigma \partial_t u + |\nabla u|^{p-2} \partial_\nu u = 0$ the coefficient σ is assumed to be a nonnegative constant. In particular, the asymptotic behaviour in the large for the parameter dependent nonlinearity $f(\cdot,\cdot,u) = \lambda |u|^{q-2} u$ will be investigated by means of the evolution of associated norms.

Keywords: Nonlinear degenerate parabolic problems, p-Laplacian, dynamical boundary conditions, blow up, comparison principles.

MSC: Primary 35K92, 35B40, 35K10; Secondary 35B44, 35K57, 35K65, 35R45.

1 Introduction

This paper deals with the behaviour of solutions of nonlinear parabolic problems of the form

$$(P_{\sigma,f}) \quad \begin{cases} \partial_t u = \Delta_p u + f(t,x,u) & \text{in } \Omega \text{ for } t > 0, \\ \sigma \partial_t u + |\nabla u|^{p-2} \partial_\nu u = 0 & \text{on } \partial\Omega \text{ for } t > 0, \end{cases}$$

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where Ω is a bounded domain of $\mathbb{R}^N$, $N \geq 1$, with Lipschitz boundary, and where

$$\Delta_p u := \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)$$

is the well known p-Laplacian operator defined in $W^{1,p}(\Omega)$ in a weak setting in the usual way for any real number $p > 1$. Another distinctive feature in the present context is the dynamical boundary condition imposed on the time lateral boundary relating the outer normal derivative to the time derivative. For the sake of simplicity, the dynamical coefficient σ is assumed to be a nonnegative constant.

Of particular interest will be the Cauchy problem

$$(P_{\sigma,f_{\lambda,q},u_0}) \quad \begin{cases} \partial_t u = \Delta_p u + \lambda |u|^{q-2}u & \text{in } \Omega \text{ for } t > 0, \\ \sigma \partial_t u + |\nabla u|^{p-2}\partial_\nu u = 0 & \text{on } \partial\Omega \text{ for } t > 0, \\ u(0,\cdot) = u_0 & \text{in } \overline{\Omega}, \end{cases}$$

where λ is a real parameter and $q > 1$.

The classical heat equation, i.e. $p = 2$, and reaction-diffusion equations under dynamical boundary conditions have been intensively studied on L^q-spaces or spaces of continuous functions, see e.g. Bandle, Below, and Reichel (2006), Below and De Coster (2000), Below and Pincet Mailly (2003), Hintermann (1989), Pincet (2001), and Vázquez and Vitillaro (2009) and the references therein. We refer also to Escher (1993), where Escher proves that the heat equation generates a strongly continuous analytic semigroup on $L^q(\Omega) \times W^{1-1/q,q}(\partial\Omega)$ for $q > N$ for more general quasilinear equations.

Some special cases of $(P_{\sigma,f})$ for $p > 1$ have been considered recently by Gal[2], Gal and Warma[3] and Showalter[4], where the generation of the corresponding C^0-semigroups is shown. In fact, generation of C^0-semigroups in $L^2(\Omega)$ of the p-Laplacian heat equation under Dirichlet, Neumann and Robin boundary conditions was already studied by J.L. Lions[5]. Boundedness and higher Hölder-regularity have been extensively treated by DiBenedetto[6] in the homogeneous case, i.e. $f \equiv 0$. Recently, Cipriani and Grillo[7] have shown generation of C^0-semigroups for the p-Laplacian under Dirichlet boundary conditions on L^q-spaces and have obtained some ultracontractivity properties of the associated semigroups.

Beyond the approaches to local existence and higher regularity of weak solutions, the existence or exclusion of global solutions, as well as the occurrence of blow up

[2] Gal, 2012, "On a class of degenerate parabolic equations with dynamic boundary conditions".

[3] Gal and Warma, 2010, "Well posedness and the global attractor of some quasi-linear parabolic equations with nonlinear dynamic boundary conditions".

[4] Showalter, 1997, *Monotone operators in Banach space and nonlinear partial differential equations*.

[5] Lions, 1969, *Quelques méthodes de résolution des problèmes aux limites non linéaires*.

[6] DiBenedetto, 1993, *Degenerate parabolic equations*.

[7] Cipriani and Grillo, 2001, "Uniform bounds for solutions to quasilinear parabolic equations".

phenomena for problem $(P_{\sigma,f})$ are of particular interest. In the works[8] the authors dealt with the linear principal part, i.e. $p = 2$, and with different nonlinearities f. A main aim in the present paper is to generalize some of the results from these references to the p-Laplacian heat equation under dynamical boundary conditions. We also prove some ultra conductivity bounds of the solutions for problem $(P_{\sigma,0})$ that seem not to be available in the literature yet. Moreover, the absence of general boundedness result for weak solutions of $(P_{\sigma,f})$ is one of the major difficulties in establishing the qualitative properties that we present in this work. Unfortunately, the existing results as e.g. the aforementioned ones[9] do not apply to the equations considered here. More recent existence results by Li and You, see K. Li and You (2013) and the references therein, can be applied to some of the problems $(P_{\sigma,f})$. However, as it stands, the aim of the present paper is not to deal with existence results, but to detail qualitative properties of solutions.

The present paper is organized as follows. In Section 2 the notion of weak solutions and of upper and lower weak solutions is made precise. Moreover, weak comparison principles are shown under a generalized one-sided Lipschitz condition imposed to f given in Definition 4 and involving a Lipschitz constant that can depend on the solutions. In particular, it will be shown in Theorem 2 that a weak lower solution u_1 of (P_{σ,f,u_0^1}) and a weak upper solution u_2 of (P_{σ,f,u_0^2}) whose initial data satisfy $u_0^1 \le u_0^2$ a.e. in Ω and on $\partial\Omega$, maintain the same inequality for $t > 0$. We also compare solutions under different boundary conditions, namely homogeneous Dirichlet boundary conditions vs. dynamical ones, see e.g. Theorem 4.

Sections 3 and 4 are devoted to the evolution of associated norms and energy functionals. For time independent nonlinearities f under a specific Lipschitz condition (11), the energy of weak solutions u of (P_{σ,f,u_0}), $E_F(u) = \frac{1}{p}\|\nabla u\|_p^p - \int_\Omega \int_0^u f(\cdot,z)\,dz\,dx$ will be shown to fulfil an identity of the form

$$-\frac{d}{dt}E_F(u(t,\cdot)) = \|\partial_t u(t,\cdot)\|_2^2 + \sigma\|\partial_t u_{|\partial\Omega}(t,\cdot)\|_{2,\partial\Omega}^2. \tag{12}$$

For the special case $f = f_{\lambda,q} = \lambda|u|^{q-2}u$ with $\lambda < 0$, the evolution of the L^2-norms in Ω and on $\partial\Omega$, can be completely controlled leading to $\lim_{t\to\infty}\|u(t,\cdot)\|_{\mathcal{X}^2} = 0$. In particular, for $p < 2$, the solutions vanish to 0 in finite time, see Theorem 2.

In Section 5 the behaviour in the large of weak solutions u of $(P_{\sigma,f_{\lambda,q},u_0})$ with $\lambda \le 0$ is investigated. For $\lambda < 0$ and $p > 2$, it turns out that an analogue to Berryman

[8] Bandle, Below, and Reichel, 2006, “Parabolic problems with dynamical boundary conditions: eigenvalue expansions and blow up”;
Below and Pincet Mailly, 2003, “Blow up for reaction diffusion equations under dynamical boundary conditions”;
Pincet, 2001, “EDP sous des conditions de bords dynamiques”.

[9] Escher, 1993, “Quasilinear parabolic systems with dynamical boundary conditions”;
Gal, 2012, “On a class of degenerate parabolic equations with dynamic boundary conditions”;
Gal and Warma, 2010, “Well posedness and the global attractor of some quasi-linear parabolic equations with nonlinear dynamic boundary conditions”.

and Holland's asymptotic result for the porous media equation[10] holds, i.e. there is a sequence of time steps $(t_n)_{n\in\mathbb{N}}$ tending to ∞ such that

$$\lim_{n\to\infty} \|(1+(p-2)t_n)^{\frac{2}{p-2}} u(t_n,\cdot) - w\|_{\mathcal{X}^2} = 0, \qquad \text{(Theorem 6)}$$

where $w \in W^{1,p}(\Omega)$ is the solution of the elliptic equation $-\Delta_p w - \lambda |w|^{p-2} w = w$ under the Robin–Steklov boundary condition stemming from the dynamical one. Another distinctive asymptotic result in this section deals with the solutions for the homogeneous problem $(P_{\sigma,0,u_0})$ and says that

$$\lim_{t\to\infty} u(t,\cdot) = \frac{\int_\Omega u_0\, dx + \sigma \oint_{\partial\Omega} u_0\, d\rho}{|\Omega| + \sigma|\partial\Omega|} \qquad \text{(Theorem 7)}$$

in $W^{1,p}(\Omega)$. Note that this asymptotic formula is exactly the same as for the classical Laplacian[11], i.e. $p = 2$, established via Fourier expansion of the initial data, that, however, does not apply for $p \neq 2$. For initial data belonging to $W^{1,p}(\Omega)$ this holds also with respect to the L^∞-norm, see Theorem 8, and will be very useful in order to guarantee strict positivity a.e. in finite time, e.g. when showing the exclusion of global existence, see Theorem 10 without using a strong parabolic minimum principle. It turns out that $\lambda = 0$ plays the same role under dynamical boundary conditions as the first eigenvalue λ_1 does for homogeneous Dirichlet boundary conditions as it has been shown in Y. Li and Xie (2003). For short, for $\lambda \leq 0$ the weak solutions are bounded for each $t > 0$, while for $\lambda > 0$ blow up occurs for $q = p > 2$ and for $q > \max\{2, p\}$. Closing this section, we deduce global existence close to an equilibrium fulfilling $f'(B) < 0$ in the general autonomous case $\partial_t u = \Delta_p u + f(u)$, see Theorem 9.

Section 6 deals with the occurrence of blow up phenomena with respect to the L^∞-norm. First, we generalize the result on the non existence of global solutions from Bandle, Below, and Reichel (2006) to the p-Laplacian with nonlinearities $f(t,x,u) = m(t,x)g(u)$ under the same assumptions as in Bandle, Below, and Reichel (2006), see Theorem 10. As for Problem $(P_{\sigma,f_{\lambda,q},u_0})$ with $\lambda > 0$ and $q > 2$, it remains to determine the behaviour in the large. In fact, for $p = q$ and for initial energy $\frac{1}{p}\int_\Omega (|\nabla u_0|^p - |u_0|^p)\, dx < 0$ it will be shown that solutions blow up in finite time, see Theorem 11. For $q > \max\{2, p\}$ we first adopt a technique developed in Ball (1977) and Below and Pincet Mailly (2003) in order to establish an upper bound for the blow up time under homogeneous Dirichlet boundary conditions for initial data $0 \neq u_0 \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ with energy $\frac{1}{p}\int_\Omega |\nabla u_0|^p\, dx - \frac{\lambda}{q}\int_\Omega |u_0|^q\, dx \leq 0$, see Theorem 12. Then the comparison techniques from Section 2 apply in order to

[10] Berryman and Holland, 1980, "Stability of the separable solution for fast diffusion".

[11] Bandle, Below, and Reichel, 2006, "Parabolic problems with dynamical boundary conditions: eigenvalue expansions and blow up".

establish the occurrence of blow up under dynamical boundary conditions too, see Theorems 13 and Corollary 5 for nonlinearities satisfying $f(\cdot,\cdot,z) \geq \lambda|z|^{q-2}z$. In the latter one it will be shown that for initial data $u_0 \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega;[0,\infty))$ with nonpositive energy the L^2-norm of the weak solution of (P_{σ,f,u_0}) blows up at the latest at time T with

$$T_{\max}(u) \leq T \leq \frac{q}{\lambda(q-2)(q-p)}|\Omega|^{\frac{q-2}{2}}\left(\int_\Omega |u_0|^2\,dx\right)^{\frac{2-q}{2}}, \qquad \text{(Corollary 5)}$$

where $T_{\max}(u)$ is the maximal existence time with respect to the L^∞-norm. Finally, we present an optimal upper bound for the blow up time under the Neumann boundary condition based on the evolution of the L^1-norm.

We close this introduction with some notations. We shall denote by $d\rho$ the restriction to $\partial\Omega$ of the $(N-1)$-dimensional Hausdorff measure, which coincides with the usual Lebesgue hyper-surface measure, since $\partial\Omega$ is supposed to be Lipschitz. Moreover, we shall denote by $\nu = \nu(x)$ its outer normal vector field at $x \in \partial\Omega$ defined ρ-a.e. in $\partial\Omega$.

The Lebesgue norm of $L^q(\Omega)$ will be denoted by $\|\cdot\|_q$, and the Lebesgue norm of $L^q(\partial\Omega,\rho)$ by $\|\cdot\|_{q,\partial\Omega}$, for $q \in [1,\infty]$. The scalar product of $L^2(\Omega)$ will be denoted by $\langle\cdot,\cdot\rangle$ and the scalar product of $L^2(\partial\Omega,\rho)$ will be denoted by $\langle\cdot,\cdot\rangle_0$:

$$\langle u,v\rangle = \int_\Omega uv\,dx, \quad \langle u,v\rangle_0 = \oint_{\partial\Omega} uv\,d\rho.$$

The conjugate of any $r \in [1,\infty]$ will be denoted by r'. The critical Sobolev exponent for the embedding $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ will be denoted by $p^* := \frac{pN}{N-p}$ if $1 < p < N$, and by $p^* = \infty$ otherwise. It is worth noting that

$$W^{1,p}(\Omega) \hookrightarrow L^2(\Omega) \Longleftrightarrow p \geq p_0 := \frac{2N}{N+2}.$$

The *trace* $u_{|\partial\Omega}$ of any function $u \in W^{1,p}(\Omega)$ is well defined since $\partial\Omega$ is regular enough. We recall that, if γ denotes the trace operator, then $\gamma(W^{1,p}(\Omega)) = W^{1-1/p,p}(\partial\Omega,\rho)$.

Moreover, the trace operator $W^{1,p}(\Omega) \to L^q(\partial\Omega,\rho)$ is continuous if and only if $1 \leq q \leq p_*$ if $p \neq N$ and for $1 \leq q < \infty$ if $p = N$. Recall that $p_* := \frac{p(N-1)}{N-p}$ if $1 < p < N$, and that $p_* = \infty$ if $p \geq N$. Note that for $q = 2$, the trace operator is well-defined and continuous under the following condition:

$$W^{1,p}(\Omega) \to L^2(\partial\Omega,\rho) \iff p \geq p_1 := \frac{2N}{N+1}.$$

Finally, for a reflexive Banach space $(V,\|\cdot\|_V)$ and $r \in [1,\infty)$, the classical Bochner space $L^r((0,T);V)$ will be endowed with the norm

$$\|u\|_{L^r((0,T);V)} := \left(\int_0^T \|u\|_V^r\,dt\right)^{1/r}.$$

2 Weak solutions, comparison principles and uniqueness results

Throughout this paper, unless otherwise stated, we shall assume that $p > p_1$ and $p \neq 2$. The case $1 < p < p_1$ is a bit more involved, as one has to work with functions belonging to $W^{1,p}(\Omega) \cap L^2(\Omega)$ having $L^2(\partial\Omega, \rho)$-trace instead of $W^{1,p}(\Omega)$. The case $p = p_1$ should also be treated separately, as the Sobolev embedding $W^{1,p}(\Omega) \to L^{p_*}(\partial\Omega, \rho)$ is not compact bearing in mind that here $p_* = 2$.

Set $\mathcal{X}^q = L^q(\Omega) \times L^q(\partial\Omega, \rho)$, for $1 \leq q \leq \infty$, and

$$U = (u, \varphi) \in \mathcal{X}^q, \quad \|U\|_{\mathcal{X}^q} := \left(\|u\|_q^q + \sigma\|\varphi\|_{q,\partial\Omega}^q\right)^{1/q},$$

and for $q = 2$ and $U = (u, \varphi), V = (v, \psi) \in \mathcal{X}^2$

$$\langle U, V\rangle_{\mathcal{X}^2} := \langle u, v\rangle + \sigma\langle \varphi, \psi\rangle_0.$$

Identifying each element $u \in W^{1,p}(\Omega)$ with the vector $U = (u, u_{|\partial\Omega})$, the space $W^{1,p}(\Omega)$ can be regarded as a subspace of $\mathcal{X}^s$ for any $1 \leq s < p_*$. Moreover, $\mathcal{X}^s$ and $L^s(\overline{\Omega}, d\tau)$ can be identified in a natural way for $s \geq 1$, where the measure $d\tau = dx_{|\Omega} \oplus d\rho_{|\partial\Omega}$ is defined for any measurable set $A \subset \overline{\Omega}$ by $\tau(A) = |A| + \rho(A \cap \partial\Omega)$. We will agree that A is *measurable* if $A \cap \Omega$ is Lebesgue measurable and $A \cap \partial\Omega$ is measurable with respect to the $(N-1)$-Hausdorff measure ρ.

For any $T > 0$, let us denote $\Omega_T = (0, T) \times \Omega$. Let us recall the definition of a local weak solution of the evolution problem $(P_{\sigma,f})$. Let $f : \Omega_T \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function.

Definition 1 – A function $u : \overline{\Omega}_T \to \mathbb{R}$ is called a *weak solution of problem* $(P_{\sigma,f})$ if

(i) $u \in L^p((0, T); W^{1,p}(\Omega)) \cap C([0, T]; \mathcal{X}^2)$,

(ii) $\partial_t u \in L^2((0, T); L^2(\Omega))$; $\partial_t u_{|\partial\Omega} \in L^2((0, T); L^2(\partial\Omega, \rho))$,

(iii) $\tilde{f} := f(\cdot, \cdot, u(\cdot, \cdot)) \in L^2((0, T); L^2(\Omega))$,

(iv) for any $\varphi \in W^{1,p}(\Omega)$ and for almost all $t \in [0, T]$ it holds

$$\langle \partial_t u, \varphi\rangle + \langle |\nabla u|^{p-2}\nabla u, \nabla\varphi\rangle - \langle f(t, x, u), \varphi\rangle + \sigma\langle \partial_t u_{|\partial\Omega}, \varphi\rangle_0 = 0.$$

Clearly, in (ii) the notation $\partial_t u$ (resp. $\partial_t u_{|\partial\Omega}$) stands for the weak temporal derivative of u(resp. of $u_{|\partial\Omega}$).

By writing $u \in \mathcal{X}^q$ we mean that $u : \overline{\Omega} \to \mathbb{R}$ is such that $u_{|\Omega} \in L^q(\Omega)$ and also $u_{|\partial\Omega} \in L^q(\partial\Omega, \rho)$.

Definition 2 – For given $u_0 \in \mathcal{X}^2$ a function $u\colon \overline{\Omega}_T \to \mathbb{R}$ is called a weak solution of the Cauchy problem (P_{σ,f,u_0})

$$(P_{\sigma,f,u_0}) \quad \begin{cases} \partial_t u = \Delta_p u + f(t,x,u) & \text{in } \Omega \text{ for } t>0, \\ \sigma \partial_t u + |\nabla u|^{p-2}\partial_\nu u = 0 & \text{on } \partial\Omega \text{ for } t>0, \\ u(0,\cdot) = u_0 & \text{in } \overline{\Omega}, \end{cases}$$

if u is a weak solution of $(P_{\sigma,f})$ in the sense of Definition 1 and if, in addition,

$$u(0,\cdot) = u_0 \quad \tau\text{-a.e. in } \overline{\Omega}.$$

Let us also recall the definition of upper and lower solution for our evolution problem. Writing $(u,\varphi) \le (v,\psi)$ for functions belonging to $\mathcal{X}^s$ means that $u \le v$ a.e. in Ω and that $\varphi \le \psi$ ρ-a.e. on $\partial\Omega$.

Definition 3 – For given $u_0 \in \mathcal{X}^2$, a function $u\colon \Omega_T \to \mathbb{R}$ satisfying (i)-(iii) from Definition 1 is called a *weak lower solution* of (P_{σ,f,u_0}) if for all $\varphi \in W^{1,p}(\Omega)$ with $\varphi \ge 0$, and for almost all $t \in [0,T]$

$$\begin{cases} \langle \partial_t u, \varphi\rangle + \langle |\nabla u|^{p-2}\nabla u, \nabla\varphi\rangle - \langle f(t,x,u),\varphi\rangle + \sigma\langle \partial_t u,\varphi\rangle_0 \le 0, \\ u(0,\cdot) \le u_0 \quad \tau\text{-a.e. in } \overline{\Omega}, \end{cases}$$

in its interval of existence. Similarly, a weak upper solution is defined by reversing the last two inequalities.

In order to compare two different solutions of the evolution equations, it is necessary to require some Lipschitz condition on f with respect to the variable u and with respect to the pair $(u_1,u_2) \in L^2(\Omega_T)^2$. This is the motivation of the following definition.

Definition 4 – A function f is said to satisfy the following one-sided Lipschitz condition for the pair $(u_1,u_2) \in L^2(\Omega_T)^2$ if

$$\begin{aligned} &\exists l \in L^1((0,T);[0,\infty)) \text{ such that for a.a. } (t,x) \in \Omega_T, \\ &[f(t,x,u_1(t,x)) - f(t,x,u_2(t,x)) - l(t)(u_1(t,x)-u_2(t,x))]\Big(u_1(t,x) - u_2(t,x)\Big)^+ \le 0. \end{aligned} \tag{1}$$

Observe that no order between u_1 and u_2 is assumed here and that (1) is in fact a condition for the set of $(t,x) \in \Omega_T$ for which $u_1(t,x) \ge u_2(t,x)$. At the end of this section, we shall give some special cases where (1) is satisfied, c.f. Remark 3. Let us prove the following comparison result.

Theorem 1 – *Let u_0^1, $u_0^2 \in \mathcal{X}^2$ be given, let u_1 be a weak lower solution of (P_{σ,f,u_0^1}) and let u_2 be a weak upper solution of (P_{σ,f,u_0^2}). Assume that the pair (u_1,u_2) satisfies (1) and let us denote $v := u_1 - u_2$, $v_0 := u_0^1 - u_0^2$. Then*

(i) $\|v^+(t,\cdot)\|^2_{\mathcal{X}^2} \le e^{2L(t)}\|v_0^+\|^2_{\mathcal{X}^2}$ *for all* $t \in [0,T]$, *with* $L(t) := \int_0^t l(s)\,ds$.

(ii) *Moreover, there exists a constant* $C = C(p,N,T,\|l\|_1) > 0$ *such that*

$$\|\nabla v^+\|^p_{L^p(\Omega_T)} \le \begin{cases} C\|v_0^+\|^2_{\mathcal{X}^2} & \text{if } p \ge 2,\\ C\|v_0^+\|^p_{\mathcal{X}^2}\big(\|\nabla u_1\|_{L^p(\Omega_T)} + \|\nabla u_2\|_{L^p(\Omega_T)}\big)^{p(1-\frac{p}{2})} & \text{if } p < 2.\end{cases}$$

Proof. (i) Fixing $t \in [0,T]$ and taking $\varphi = v^+(t,\cdot)$ as a test function in the differential inequalities satisfied by u_1 and u_2, subtracting them and integrating over Ω yield

$$\begin{aligned}&\langle \partial_t v, v^+\rangle + \sigma\langle \partial_t v, v^+\rangle_0 \le\\ &\qquad -\langle |\nabla u_1|^{p-2}\nabla u_1 - |\nabla u_2|^{p-2}\nabla u_2, \nabla(u_1-u_2)^+\rangle + \langle f(t,x,u_1) - f(t,x,u_2), v^+\rangle.\end{aligned} \tag{2}$$

Here Conditions (i)-(iv) from Definition 1 assure that all the above integrals are finite. By convexity of the function $z \mapsto |z|^p$ in $\mathbb{R}^N$ the first term of the r.h.s. is negative, while the second term is bounded from above by $l(t)\langle v(t,\cdot), v^+(t,\cdot)\rangle$ using (1). Moreover, we deduce for almost all t that

$$\begin{aligned}\langle \partial_t v, v^+\rangle &= \frac{1}{2}\frac{d}{dt}\|v^+\|_2^2,\\ \langle \partial_t v, v^+\rangle_0 &= \frac{1}{2}\frac{d}{dt}\|v^+\|^2_{2,\partial\Omega}.\end{aligned}$$

These identities and the existence of the derivatives on the r.h.s. are justified, see e.g. Showalter (1997, Chapter III, Proposition 1.2). It follows that

$$\frac{d}{dt}\|v^+\|_2^2 + \sigma\frac{d}{dt}\|v^+\|^2_{2,\partial\Omega} \le 2l(t)\|v^+\|_2^2,$$

and therefore, by the Bellman–Gronwall Inequality[12]

$$\|v^+(t,\cdot)\|^2_{\mathcal{X}^2} \le e^{2L(t)}\|v_0^+\|^2_{\mathcal{X}^2}$$

for all $t \in [0,T]$.

(ii) We apply the following inequality from Simon (1978): $\exists C = C(p,N) > 0\,\forall z_1, z_2 \in \mathbb{R}^N$:

$$\begin{cases}\langle |z_1|^{p-2}z_1 - |z_2|^{p-2}z_2, z_1 - z_2\rangle_{\mathbb{R}^N} \ge C|z_1 - z_2|^p_{\mathbb{R}^N}, & \text{if } p \ge 2,\\ \langle |z_1|^{p-2}z_1 - |z_2|^{p-2}z_2, z_1 - z_2\rangle_{\mathbb{R}^N} \ge C(|z_1|_{\mathbb{R}^N} + |z_2|_{\mathbb{R}^N})^{p-2}|z_1 - z_2|^2_{\mathbb{R}^N}, & \text{if } 1 < p \le 2\end{cases}$$

In order to bound from above the first term of the r.h.s. of (2) by $C\|\nabla v^+(t,\cdot)\|_p^p$ if $p \ge 2$ or by $C\int_\Omega (|\nabla u_1(t,x)| + |\nabla u_2(t,x)|)^{p-2}|\nabla v^+(t,x)|^2\,dx$ if $p < 2$. Therefore we deduce for almost all t that

$$\frac{1}{2}\frac{d}{dt}\|v^+(t,\cdot)\|^2_{\mathcal{X}^2} + Ca(t) \le l(t)\|v^+(t,\cdot)\|^2_{\mathcal{X}^2},$$

where we abbreviate

$$a(t) = \begin{cases} \|\nabla v^+(t,\cdot)\|_p^p, & \text{if } p \geq 2, \\ \int_\Omega (|\nabla u_1(t,\cdot)| + |\nabla u_2(t,\cdot)|)^{p-2} |\nabla v^+(t,\cdot)|^2 \, dx, & \text{if } 1 < p \leq 2. \end{cases}$$

Integrating over $[0,T]$ and using the estimate (i) we obtain

$$\frac{1}{2}(\|v^+(T,\cdot)\|^2_{\mathcal{X}^2} - \|v^+(0,\cdot)\|^2_{\mathcal{X}^2}) + C\int_0^T a(t)\,dt \leq \|l\|_1 e^{2L(T)} \|v_0^+\|^2_{\mathcal{X}^2}$$

i.e.

$$\int_0^T a(t)\,dt \leq \frac{1}{C}\left(\|l\|_1 e^{2L(T)} + \frac{1}{2}\right)\|v_0^+\|^2_{\mathcal{X}^2}.$$

If $p \geq 2$, then the statement (ii) is clear. Consider $p < 2$. First, Hölder's inequality leads to

$$\begin{aligned} \|\nabla v^+\|_p^p &\leq a(t)^{\frac{p}{2}} \left(\int_\Omega (|\nabla u_1| + |\nabla u_2|)^p \, dx\right)^{1-\frac{p}{2}} \\ &\leq 2^{(p-1)(1-\frac{p}{2})} a(t)^{p/2} \left(\|\nabla u_1\|_p^p + \|\nabla u_2\|_p^p\right)^{1-\frac{p}{2}}, \end{aligned}$$

since $1 < p < 2$. Then, we use Hölder's inequality with respect to the dependence on t and obtain

$$\begin{aligned} \|\nabla v^+\|^p_{L^p(\Omega_T)} &\leq 2^{(p-1)(1-\frac{p}{2})} \left(\int_0^T a(t)\,dt\right)^{p/2} \left(\|\nabla u_1\|^p_{L^p(\Omega_T)} + \|\nabla u_2\|^p_{L^p(\Omega_T)}\right)^{1-\frac{p}{2}} \\ &\leq 2^{(p-1)(1-\frac{p}{2})} \left(\int_0^T a(t)\,dt\right)^{p/2} \left(\|\nabla u_1\|_{L^p(\Omega_T)} + \|\nabla u_2\|_{L^p(\Omega_T)}\right)^{p(1-\frac{p}{2})}. \qquad \square \end{aligned}$$

Immediate consequences of the previous theorem are given by the following corollaries.

Corollary 1 – *Let $f_1, f_2 \colon \Omega_T \to \mathbb{R}$ be two measurable functions such that $f_1 \leq f_2$ a.a. $(t,x) \in \Omega_T$. Let $u_0^1, u_0^2 \in \mathcal{X}^2$ be given, let u_1 be a weak lower solution of (P_{σ,f_1,u_0^1}) and let u_2 be a weak upper solution of (P_{σ,f_2,u_0^2}). Then the conclusions (i) and (ii) hold with $l \equiv 0$.*

[12] Bellman, 1953, *Stability theory of differential equations.*

Corollary 2 – *Let $u_0 \in \mathcal{X}^2$ be given. Let u_1 and u_2 be two weak solutions of (P_{σ,f,u_0}). Assume that the pair (u_1,u_2) satisfies the Lipschitz condition:*

$$\begin{aligned}&\exists l \in L^1(0,T),\, l \geq 0, \text{ such that for a.a. } (t,x) \in \Omega_T,\\ &[f(t,x,u_1(t,x)) - f(t,x,u_2(t,x)) - l(t)(u_1(t,x) - u_2(t,x))]\Big(u_1(t,x) - u_2(t,x)\Big) \leq 0.\end{aligned} \tag{3}$$

Then for all $t \in [0,T]$, $u_1(t,\cdot) = u_2(t,\cdot)$ a.e. in Ω.

Note that (3) implies (1) for both $(u_1 - u_2)^+$ and $(u_1 - u_2)^-$, which means that uniqueness is plain. For the Lipschitz case we can state the following

Corollary 3 – *Assume that f satisfies the following Lipschitz condition* (4) *with respect to the variable u:*

$$\begin{aligned}&\exists l \in L^1(0,T),\, l \geq 0, \text{ such that for a.a. } (t,x) \in \Omega_T, \text{ for all } (u_1,u_2) \in \mathbb{R}^2,\\ &\quad [f(t,x,u_1) - f(t,x,u_2) - l(t)(u_1 - u_2)]\Big(u_1 - u_2\Big) \leq 0.\end{aligned} \tag{4}$$

Let $u_0 \in \mathcal{X}^2$. Let $\left(u_n^0\right)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{X}^2$ such that $\lim_{n\to\infty} \|u_n^0 - u_0\|_{\mathcal{X}^2} = 0$. Let u_n be a weak solution of (P_{σ,f,u_n^0}) and u a weak solution of (P_{σ,f,u_0}). Then $u_n(t,\cdot) \to u(t,\cdot)$ in $\mathcal{X}^2$ and $u_n \to u$ in $L^p((0,T);W^{1,p}(\Omega))$.

Proof. The convergence in $\mathcal{X}^2$ follows readily from (i) of Theorem 1. From (ii), if $p > 2$, we have directly that there exists a constant $C = C(p,N,T,\|l\|_1) > 0$ such that

$$\|\nabla u_n - \nabla u\|^p_{L^p(\Omega_T)} \leq C\|u_n^0 - u_0\|^2_{\mathcal{X}^2}.$$

If $p < 2$ we have

$$\|\nabla u_n - \nabla u\|^p_{L^p(\Omega_T)} \leq C\|u_n^0 - u_0\|^p_{\mathcal{X}^2}\Big(\|\nabla u_n\|_{L^p(\Omega_T)} + \|\nabla u\|_{L^p((\Omega_T)}\Big)^{p(1-\frac{p}{2})}.$$

Since $\|u_n^0 - u_0\|_{\mathcal{X}^2} \to 0$, the latter inequality yields that the sequence $\|\nabla u_n\|_{L^p(\Omega_T)}$ is bounded and therefore $\|\nabla u_n - \nabla u\|^p_{L^p(\Omega_T)} \to 0$. □

Another consequence of Theorem 1 is the following standard comparison principle.

Theorem 2 (Weak comparison principle with dynamical boundary condition) – *Let $u_0^1, u_0^2 \in \mathcal{X}^2$ satisfy $u_0^1 \leq u_0^2$ τ-a.e. in $\overline{\Omega}$. Let u_1 be a weak lower solution of (P_{σ,f,u_0^1}) and let u_2 be a weak upper solution of (P_{σ,f,u_0^2}). Assume that the pair (u_1,u_2) satisfies* (1). *Then*

$$u_1(t,\cdot) \leq u_2(t,\cdot) \quad \tau\text{-a.e. in } \overline{\Omega}.$$

Proof. By hypothesis

$$\|(u_1(0,\cdot)-u_2(0,\cdot))^+\|_2 = \|(u_0^1-u_0^2)^+\|_2 = 0$$

and

$$\|(u_1(0,\cdot)-u_2(0,\cdot))^+\|_{2,\partial\Omega}^2 = \|(u_0^1-u_0^2)^+\|_{2,\partial\Omega} = 0.$$

Hence, from Theorem 1 (i) it follows that $(u_1(t,\cdot)-u_2(t,\cdot))^+ = 0$ a.e. in Ω, which permits to conclude. □

In Section 5 we need to compare a weak solution of a parabolic problem with dynamical boundary condition with a solution of the analogous parabolic problem under homogeneous Dirichlet boundary condition for the forcing term

$$f(t,x,u) = f_{\lambda,q}(u) := \lambda|u|^{q-2}u$$

with $\lambda > 0$ and $q > \max\{2,p\}$. Let us consider here a more general parabolic problem for the p-Laplacian under Dirichlet boundary condition

$$\begin{cases} \partial_t v = \Delta_p v + f(t,x,v) & \text{in } \Omega \text{ for } t>0, \\ v = 0 & \text{on } \partial\Omega \text{ for } t>0, \\ v(0,\cdot) = v_0 & \text{in } \Omega, \end{cases} \tag{5}$$

for an initial value $v_0 \in L^2(\Omega)$. A function $v\colon [0,T]\times\Omega \to \mathbb{R}$ is called a weak solution of (5) if $v \in L^p((0,T);W_0^{1,p}(\Omega)) \cap C([0,T];L^2(\Omega))$, $\partial_t v \in L^2(\Omega_T)$, $v(0,\cdot) = v_0$ a.e. in Ω and for any test function $\varphi \in W_0^{1,p}(\Omega)$ we have

$$\langle \partial_t v, \varphi\rangle + \langle |\nabla v|^{p-2}\nabla v, \nabla\varphi\rangle - \langle f(t,x,v),\varphi\rangle = 0$$

for a.a. $t \in (0,T)$. Correspondingly, we define upper and lower solutions for Problem (5). For the sake of completeness let us state the following Weak Comparison Principle for Problem (5), though the result is partially known, namely for the special case $f \equiv 0$, c.f. DiBenedetto (1993).

Theorem 3 (Weak Comparison Principle with Dirichlet boundary condition) – *Let v_1 be a lower weak solution of* (5) *with initial data $v_0^1 \in L^2(\Omega)$ and let v_2 be an upper weak solution of* (5) *with initial data $v_0^2 \in L^2(\Omega)$, satisfying $v_0^1 \le v_0^2$ a.e. in Ω. Assume further that the pair (v_1,v_2) satisfies the one-sided Lipschitz condition* (1). *Then $v_1(t,\cdot) \le v_2(t,\cdot)$ a.e. in Ω, for all $t \in [0,T]$.*

Proof. Fix $t \in [0,T]$. The choice of $\varphi \equiv (v_1(t,\cdot) - v_2(t,\cdot))^+$ leads to

$$\langle \partial_t(v_1 - v_2), (v_1 - v_2)^+ \rangle = -\int_\Omega (|\nabla v_1|^{p-2}\nabla v_1 - |\nabla v_2|^{p-2}\nabla v_2) \cdot \nabla(v_1 - v_2)^+ \, dx + \int_\Omega (f(t,x,v_1) - f(t,x,v_2))(v_1 - v_2)^+ \, dx.$$

Thus, by convexity of the function $z \in \mathbb{R}^N \mapsto |z|^p$ and by Condition (1),

$$\langle \partial_t(v_1 - v_2), (v_1 - v_2)^+ \rangle \leq l(t) \|(v_1 - v_2)^+\|_2^2.$$

As in the proof of the Theorem 2 we can conclude

$$2\int_0^s l(t) \|(v_1(t,\cdot) - v_2(t,\cdot))^+\|_2^2 \, dt \geq \int_0^s \frac{d}{dt} \|(v_1(t,\cdot) - v_2(t,\cdot))^+\|_2^2 \, dt = \|(v_1(s,\cdot) - v_2(s,\cdot))^+\|_2^2 - \|(v_1(0,\cdot) - v_2(0,\cdot))^+\|_2^2.$$

By the Bellman–Gronwall Inequality and that $(v_1(0,\cdot) - v_2(0,\cdot))^+ = (v_0^1 - v_0^2)^+ = 0$ a.e. in Ω, it follows that $\|(v_1(s,\cdot) - v_2(s,\cdot))^+\|_2^2 \leq e^{2L(s)} \|(v_1(0,\cdot) - v_2(0,\cdot))^+\|_2^2 = 0$ for all $s \in [0,T]$, which permits to conclude. □

As a consequence of the previous theorem we have

Proposition 1 (Weak Maximum Principle with Dirichlet boundary condition) – *Let v be a weak upper solution of* (5) *with initial data $v_0 \in L^2(\Omega)$ satisfying $v_0 \geq 0$ a.e. in Ω. Assume further that $f(\cdot,\cdot,0) \geq 0$ and that the pair $(0,v)$ satisfies the one-sided Lipschitz condition* (1). *Then v satisfies $v(t,\cdot) \geq 0$ a.e. in Ω, for all $t \in [0,T]$.*

Finally, we compare solutions of parabolic problems under dynamical boundary conditions and nonnegative solutions of parabolic problems under Dirichlet boundary conditions. For that purpose, the solution v of (5) is required to fulfil $v(t,\cdot) \in C^1(\overline{\Omega})$, since we shall need some estimates of the gradient of the solution $v(t,\cdot)$ on $\partial\Omega$. See also the Remark 1 below about the regularity of weak solutions of Problem (5).

Theorem 4 – *Assume that $\partial\Omega$ is of class C^2. Let u be a weak lower solution of Problem (P_{σ,f,u_0}) with initial data $u_0 \in \mathcal{X}^2$. Let v be the weak solution of Problem* (5) *with initial data $v_0 \in L^2(\Omega)$, satisfying $v_0 \geq 0$ a.e. in Ω. Assume that*

(i) $f(\cdot,\cdot,0) \geq 0$,

(ii) *the pair $(0,v)$ satisfies the one-sided Lipschitz condition* (1),

(iii) *the pair (v,u) satisfies the one-sided Lipschitz condition* (1),

(iv) *for all $t \in (0,T)$, $v(t,\cdot) \in C^1(\overline{\Omega})$.*

If, in addition,

$$u_0 \geq v_0 \geq 0 \text{ a.e. in } \Omega, \quad u_0 \geq 0 \;\; \rho\text{-a.e. in } \partial\Omega, \tag{6}$$

then $u(t,\cdot) \geq v(t,\cdot)$ *a.e. in* Ω, *for all* $t \in [0,T]$.

Proof. By Theorem 3 and hypotheses (i) and (ii), the solution v is nonnegative in $[0,T]\times\Omega$. Since $v(t,\cdot) \in W_0^{1,p}(\Omega) \cap C(\overline{\Omega})$, and since $\partial\Omega$ is of class C^1, the solution v has to vanish on $\partial\Omega$ for all $t \in (0,T)$. Thus, $\partial_\nu v(t,\cdot) \leq 0$ on $\partial\Omega$ for all $t \in (0,T)$, and multiplying the differential equation of Problem (5) by any nonnegative function $\varphi \in W^{1,p}(\Omega)$ and integrating over Ω yield

$$\begin{aligned}\langle \partial_t v, \varphi\rangle &= -\langle |\nabla v|^{p-2}\nabla v, \nabla\varphi\rangle + \langle |\nabla v|^{p-2}\partial_\nu v, \varphi\rangle_0 + \langle f(t,x,v), \varphi\rangle \\ &\leq -\langle |\nabla v|^{p-2}\nabla v, \nabla\varphi\rangle + \langle f(t,x,v), \varphi\rangle. \end{aligned}\tag{7}$$

Although the test function φ used in the weak formulation of Problem (5) must belong to $W_0^{1,p}(\Omega)$, the above integration by parts is justified e.g. by Cuesta and Takáč (2000, Lemma A.1). Hence, $\partial_t v(t,\cdot) \equiv 0$ on $\partial\Omega$, (7), and the weak formulation of Problem (P_{σ,f,u_0}) with the test function $w(t,\cdot) = (v(t,\cdot) - u(t,\cdot))^+$ yield that

$$\begin{aligned}\int_\epsilon^s (\langle \partial_t w, w\rangle + \sigma\langle \partial_t w, w\rangle_0)\, dt \leq \int_\epsilon^s &\langle |\nabla v|^{p-2}\nabla v - |\nabla u|^{p-2}\nabla u, \nabla w\rangle\, dt \\ &+ \int_\epsilon^s \langle f(t,x,v) - f(t,x,u), w\rangle\, dt.\end{aligned}$$

for any $0 < \epsilon < s < T$. By convexity of the function $z \mapsto |z|^p$ in $\mathbb{R}^N$, the first integral on the r.h.s. is seen to be positive, which implies in turn that

$$\frac{1}{2}\int_\epsilon^s \frac{d}{dt}\|w(t,\cdot)\|_{\mathcal{X}^2}^2\, dt \leq \int_\epsilon^s \langle f(t,x,v) - f(t,x,u), w\rangle\, dt$$

and

$$\begin{aligned}\|w(s,\cdot)\|_{\mathcal{X}^2}^2 - \|w(\epsilon,\cdot)\|_{\mathcal{X}^2}^2 &\leq 2\int_\epsilon^s \left(\int_\Omega |f(t,x,v) - f(t,x,u)|\, w\, dx\right) dt \\ &\leq 2\int_\epsilon^s l(t)\|w(t,\cdot)\|_2^2\, dt.\end{aligned}\tag{8}$$

As $w(\epsilon,\cdot) = u^-(\epsilon,\cdot)$ ρ-a.e. in $\partial\Omega$,

$$\lim_{\epsilon\to 0}\|w(\epsilon,\cdot)\|_{\mathcal{X}^2}^2 = \|w(0,\cdot)\|_2^2 + \sigma\|u_0^-\|_{2,\partial\Omega}^2 = 0$$

by (6). Thus (8) reduces to

$$\|w(s,\cdot)\|_2^2 \leq 2\int_0^s l(t)\|w(t,\cdot)\|_2^2\, dt.$$

As the function $l(t)$ is integrable and nonnegative, the Bellman–Gronwall Inequality yields

$$\forall s \in [0,T]\colon \|w(s,\cdot)\|_2^2 = 0.$$

Thus, we can conclude that $v(s,\cdot) \leq u(s,\cdot)$ a.e. in Ω for all $s \in [0,T]$. □

Remark 1 – Standard regularity results (c.f. Lieberman 1993, Theorem 01) imply that a weak solution v of Problem (5) satisfies $v \in C^1((0,T]\times\overline{\Omega})$ *provided that* $v \in L^\infty((\epsilon,T)\times\Omega)$ *for all* $\epsilon \in (0,T)$ *and that* $\partial\Omega$ *is of class* $C^{1,\alpha}$ *for some* $0<\alpha<1$.

Remark 2 – In order to assure the global boundedness in the large of any weak solution v of Problem (5), that is, $v \in L^\infty((\epsilon,T)\times\Omega)$, one should impose to f a growth condition with respect to the u-variable, say e.g.

$$|f(t,x,v)| \leq A|v|^{q-1} \quad \text{for a.a. } (t,x) \in (\epsilon,T)\times\Omega, \forall v \in \mathbb{R},$$

with $q < p\frac{N+2}{N}$ (c.f. DiBenedetto 1993, Theorem 3.2 Chapter V).

Remark 3 – In the special case $f(t,x,u) = f_{\lambda,q}(u)$, with $q>1$

(i) Conditions (1) and (3) are trivially satisfied for any pair of functions (u_1,u_2) if $\lambda \leq 0$.

(ii) If $\lambda > 0$ and $q>2$, the one-sided Lipschitz condition (1) is satisfied for any pair and $(u_1,u_2) \in L^1((0,T);L^\infty(\Omega))^2$ because

$$\left|f_{\lambda,q}(u_1(t,\cdot)) - f_{\lambda,q}(u_2(t,\cdot))\right| \leq l(t)|u_1(t,\cdot)-u_2(t,\cdot)|$$

with $l(t) := (q-1)\lambda\max\{\|u_1(t,\cdot)\|_\infty, \|u_2(t,\cdot)\|_\infty\}^{q-2}$.

(iii) Clearly, for $q<2$ Condition (1) does not hold, and Theorem 2 does not apply. In general, the solutions of $(P_{\sigma,f_{\lambda,q},u_0})$ are even not unique. Take e.g. $\sigma=0$ and $q=\frac{3}{2}$, then the continuum of nonnegative solutions of $\dot{z} = \lambda z^{\frac{1}{2}}$ under $z(0)=0$ furnishes also non unique solutions of $(P_{0,f_{\lambda,\frac{3}{2}},0})$.

3 An energy identity

In this section we show an energy estimate for solutions of Problem (P_{σ,f,u_0}) for time independent nonlinearities:

$$\forall (t,x,u) \in \Omega_T \times \mathbb{R}\colon f(t,x,u) = f(x,u). \tag{9}$$

For any weak solution u of Problem (P_{σ,f,u_0}) we introduce the energy E_F by

$$E_F(u(t,\cdot)) = \frac{1}{p}\|\nabla u(t,\cdot)\|_p^p - \int_\Omega F(x,u(t,x))\,dx \tag{10}$$

with

$$F(x,s) := \int_0^s f(x,z)\,dz.$$

The following *energy identity* for weak solutions of (P_{σ,f,u_0}) will be shown as in the case $f \equiv 0$, where the classical theory of maximal monotone operators[13] has been applied.

Theorem 5 – *Let $u\colon \overline{\Omega}_T \to \mathbb{R}$ be a weak solution of Problem (P_{σ,f,u_0}) with $u_0 \in \mathcal{X}^2$. Assume that f satisfies* (9) *and that there exists a constant $l \geq 0$ such that*

$$[f(x,u(t,x)) - f(x,u(s,x)) - l(u(t,x)-u(s,x))](u(t,x)-u(s,x)) \leq 0 \tag{11}$$

for a.a. $t,s \in (0,T]$ and for a.a. $x \in \Omega$. Then for a.a. $t \in (0,T]$, the time derivative of $E_F(u(t,\cdot))$ exists and satisfies the identity

$$-\frac{d}{dt}E_F(u(t,\cdot)) = \|\partial_t u(t,\cdot)\|_2^2 + \sigma\|\partial_t u_{|\partial\Omega}(t,\cdot)\|_{2,\partial\Omega}^2. \tag{12}$$

For simplicity we shall abbreviate

$$\langle \partial_t u(t,\cdot), \varphi\rangle_{\mathcal{X}^2} := \langle \partial_t u(t,\cdot), \varphi\rangle + \sigma\langle \partial_t u_{|\partial\Omega}(t,\cdot), \varphi\rangle_0,$$

for any $\varphi \in W^{1,p}(\Omega)$, and

$$\|\partial_t u\|_{\mathcal{X}^2}^2 := \|\partial_t u(t,\cdot)\|_2^2 + \sigma\|\partial_t u_{|\partial\Omega}(t,\cdot)\|_{2,\partial\Omega}^2.$$

Note that we do not state here that $\partial_t u_{|\partial\Omega}(t,\cdot)$ is the trace of $\partial_t u(t,\cdot)$, since this function is assumed only to belong to $L^2(\Omega)$!

Proof. Recall the following basic convexity inequality, valid for any $p > 1$ and any $x,y \in \mathbb{R}^N$:

$$p|x|^{p-2}x\cdot(y-x) \leq |y|^p - |x|^p. \tag{13}$$

Fix $t \in (0,T)$ and $h > 0$ small and choose $\varphi = u(t+h,\cdot) - u(t,\cdot)$ in the weak formulation of (P_{σ,f,u_0}) at time t. Then, using (13)

$$\begin{aligned}
&\langle \partial_t u(t,\cdot), u(t+h,\cdot) - u(t,\cdot)\rangle_{\mathcal{X}^2}\\
&= -\langle |\nabla u(t,\cdot)|^{p-2}\nabla u(t,\cdot), \nabla u(t+h,\cdot) - \nabla u(t,\cdot)\rangle + \langle f(\cdot,u(t,\cdot)), u(t+h,\cdot)-u(t,\cdot)\rangle\\
&\geq -\frac{1}{p}(\|\nabla u(t+h,\cdot)\|_p^p - \|\nabla u(t,\cdot)\|_p^p) + \langle f(\cdot,u(t,\cdot)), u(t+h,\cdot)-u(t,\cdot)\rangle.
\end{aligned} \tag{14}$$

[13] Brézis, 1973, *Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert*, Theorem 3.2.

Now take the test function φ again in the weak formulation of (P_{σ,f,u_0}) at $t+h$ and get

$$\begin{aligned}
&\langle \partial_t u(t+h,\cdot), u(t+h,\cdot)-u(t,\cdot)\rangle_{\mathcal{X}^2} \\
&= -\langle |\nabla u(t+h,\cdot)|^{p-2}\nabla u(t+h,\cdot), \nabla u(t+h,\cdot)-\nabla u(t,\cdot)\rangle \\
&\quad + \langle f(\cdot, u(t+h,\cdot)), u(t+h,\cdot)-u(t,\cdot)\rangle \\
&\le \frac{1}{p}(\|\nabla u(t,\cdot)\|_p^p - \|\nabla u(t+h,\cdot)\|_p^p) + \langle f(\cdot,u(t+h,\cdot)), u(t+h,\cdot)-u(t,\cdot)\rangle. \qquad (15)
\end{aligned}$$

For $s>0$, let us denote for the sake of simplicity

$$\begin{aligned}
g(s) &= \langle \partial_t u(t,\cdot), u(t+s,\cdot)-u(t,\cdot)\rangle_{\mathcal{X}^2}, \\
e(s) &= \frac{1}{p}(\|\nabla u(t+s,\cdot)\|_p^p - \|\nabla u(t,\cdot)\|_p^p), \\
k(s) &= \langle f(\cdot,u(t,\cdot)), u(t+s,\cdot)-u(t,\cdot)\rangle, \\
d(s) &= \langle \partial_t u(t+s,\cdot)-\partial_t u(t,\cdot), u(t+s,\cdot)-u(t,\cdot)\rangle_{\mathcal{X}^2} \\
&\qquad - \langle f(\cdot,u(t+s,\cdot))-f(\cdot,u(t,\cdot)), u(t+s,\cdot)-u(t,\cdot)\rangle.
\end{aligned}$$

Combining (14) and (15) yields

$$-g(h)+k(h) \le e(h) \le -g(h)+k(h)-d(h). \qquad (16)$$

The assertion of the proposition will follow from dividing (16) by h and passing to the limit as $h \to 0$. Let us study the existence of those limits. Inequality (16) implies in particular that $d(h) \le 0$, which in turn in combination with (11) leads to

$$\frac{1}{2}\frac{d}{dt}\left(\|u(t+h,\cdot)-u(t,\cdot)\|_{\mathcal{X}^2}^2\right) - l\|u(t+h,\cdot)-u(t,\cdot)\|_2^2 \le d(h) \le 0,$$

and, after integrating over any interval $[s_1,s_2] \subset (0,T]$,

$$e^{-2ls_2}\|u(s_2+h,\cdot)-u(s_2,\cdot)\|_{\mathcal{X}^2}^2 - e^{-2ls_1}\|u(s_1+h,\cdot)-u(s_1,\cdot)\|_{\mathcal{X}^2}^2 \le 0.$$

Dividing by h^2 and letting $h \to 0$ we have then

$$e^{-2ls_2}\|\partial_t u(s_2,\cdot)\|_{\mathcal{X}^2}^2 - e^{-2ls_1}\|\partial_t u(s_1,\cdot)\|_{\mathcal{X}^2}^2 \le 0.$$

Thus, we can conclude that there exists a constant $M>0$ depending on u and t such that

$$\beta_n := \|\partial_t u(t+h_n,\cdot)\|_{\mathcal{X}^2}^2 \le M,$$

where $(h_n)_{n\in\mathbb{N}}$ is any sequence of real numbers tending to 0. Since $(\beta_n)_{n\in\mathbb{N}}$ is bounded, there exists some $(\xi_1,\xi_2) \in \mathcal{X}^2$ such that, up to a subsequence,

$(\partial_t u(t+h_n,\cdot), \partial_t u_{|\partial\Omega}(t+h_n,\cdot)) \rightharpoonup (\xi_1,\xi_2)$ in $\mathcal{X}^2$. Hence, by (16) and by $u \in C([0,T];\mathcal{X}^2)$, $g(s)$ and the r.h.s. of (16) are bounded, as well as $(u(t+h_n,\cdot))_{n\in\mathbb{N}}$ in $W^{1,p}(\Omega)$. Therefore there exists some $z \in W^{1,p}(\Omega)$ such that, up to a subsequence, $u(t+h_n,\cdot) \rightharpoonup z$ in $W^{1,p}(\Omega)$, strongly in $\mathcal{X}^2$ and simply a.e. in Ω. By continuity of $t \mapsto u(t,\cdot)$ in $\mathcal{X}^2$ we must have $z = u(t,\cdot)$.

Finally, we conclude that $(\xi_1,\xi_2) = (\partial_t u(t,\cdot), \partial_t u_{|\partial\Omega}(t,\cdot))$ by the following argument. In the weak formulation of (P_{σ,f,u_0}) at time $t+h_n$ we find, passing to the limit

$$\begin{aligned}\langle \xi_1,\varphi\rangle + \sigma\langle \xi_2,\varphi\rangle_0 &= -\langle |\nabla u(t,\cdot)|^{p-2}\nabla u(t,\cdot),\nabla\varphi\rangle + \langle f(\cdot,u(t,\cdot)),\varphi\rangle \\ &= \langle \partial_t u(t,\cdot),\varphi\rangle_{\mathcal{X}^2}\end{aligned}$$

for any $\varphi \in W^{1,p}(\Omega)$. The special choice of $\varphi \in W_0^{1,p}(\Omega)$ implies that $\xi_1 = \partial_t u(t,\cdot)$ and consequently $\xi_2 = \partial_t u_{|\partial\Omega}(t,\cdot)$. Hence,

$$\lim_{h\to 0}\frac{g(h)}{h} = \|\partial_t u(t,\cdot)\|^2_{\mathcal{X}^2} \quad \text{and} \quad \lim_{h\to 0}\frac{k(h)}{h} = \langle f(\cdot,u(t,\cdot)),\partial_t u(t,\cdot)\rangle.$$

This completes the proof. □

Remark 4 – One can readily see that the same arguments apply for the evolution equation under Dirichlet boundary conditions, i.e. for Problem (5) with f satisfying (11), obtaining similarly for a.a. $t \in (0,T]$

$$-\frac{d}{dt}E_F(u(t,\cdot)) = \|\partial_t u(t,\cdot)\|_2^2. \tag{17}$$

Now, let us consider the particular case $f = f_{\lambda,p}$, i.e.

$$\begin{cases}\partial_t u = \Delta_p u + \lambda|u|^{p-2}u & \text{in } \Omega \text{ for } t>0,\\ \sigma\partial_t u + |\nabla u|^{p-2}\partial_\nu u = 0 & \text{on } \partial\Omega \text{ for } t>0,\\ u(0,\cdot) = u_0 & \text{in } \overline{\Omega}.\end{cases} \tag{18}$$

Throughout we shall denote

$$E_\lambda(u) = \frac{1}{p}\int_\Omega (|\nabla u|^p - \lambda|u|^p)\,dx.$$

We can use the energy identity to prove the following result on the *Rayleigh quotient*

$$\mathcal{E}_\lambda[u](t) := \frac{\int_\Omega (|\nabla u(t,\cdot)|^p - \lambda|u(t,\cdot)|^p)\,dx}{\|u\|^p_{\mathcal{X}^2}}. \tag{19}$$

Here, we followed Savaré and Vespri (1994) where a similar result was proved under Dirichlet boundary conditions.

Lemma 1 – *Let* $\lambda \in \mathbb{R}$ *and* $u \not\equiv 0$ *be a solution of* $(P_{\sigma,f_{\lambda,p},u_0})$ *with* $u_0 \in \mathcal{X}^2$*. In the case* $\lambda > 0$ *assume further that* $u \in L^\infty(\Omega_T)$*. Then*

(i) *the function* $\mathcal{E}_\lambda[u](t)$ *defined in* (19) *is non increasing with respect to* $t \in (0,T]$,

(ii) *and the mapping* $t \to \|u(t,\cdot)\|_{\mathcal{X}^2}^{2-p}$ *is concave if* $p > 2$ *and convex if* $p < 2$.

Remark 5 – Since $\mathcal{E}_\lambda[u](t)$ is only defined a.e. in $(0,T)$, one should understand the result of (i) as follows: "there is an integrable function g with $g = 0$ a.e. such that $\mathcal{E}_\lambda[u] + g$ is non increasing with respect to t".

Proof. (i) We apply the energy identity 12 to f. Note that Condition (11) is always satisfied for any function u if $\lambda \le 0$, whereas for $\lambda > 0$ it holds for $u \in L^\infty(\Omega_T)$.

On the one hand, multiplying the differential equation of $(P_{\sigma,f_{\lambda,p},u_0})$ by u and integrating over Ω yield

$$pE_\lambda(u(t,\cdot)) = -\int_\Omega u\partial_t u\,dx - \sigma\oint_{\partial\Omega} u\partial_t u_{|\partial\Omega}\,d\rho = -\frac{1}{2}\frac{d}{dt}\|u(t,\cdot)\|_{\mathcal{X}^2}^2 \tag{20}$$

and using Hölder's inequality in (20),

$$p^2 E_\lambda(u(t,\cdot))^2 \le \|\partial_t u(t,\cdot)\|_{\mathcal{X}^2}^2 \|u(t,\cdot)\|_{\mathcal{X}^2}^2. \tag{21}$$

On the other hand, by Theorem 5

$$\frac{d}{dt}E_\lambda(u(t,\cdot)) = -\|\partial_t u(t,\cdot)\|_{\mathcal{X}^2}^2 \le 0 \tag{22}$$

a.e. in $(0,T)$. Combining (21) et (22) we have

$$\frac{d}{dt}E_\lambda(u(t,\cdot)) \le -p^2\frac{E_\lambda(u(t,\cdot))^2}{\|u(t,\cdot)\|_{\mathcal{X}^2}^2} = \frac{p}{2}E_\lambda(u(t,\cdot))\frac{\frac{d}{dt}\|u(t,\cdot)\|_{\mathcal{X}^2}^2}{\|u(t,\cdot)\|_{\mathcal{X}^2}^2},$$

Hence, if $E_\lambda(u) > 0$,

$$\frac{d}{dt}\Big(\ln(E_\lambda(u(t,\cdot)) - \ln\|u(t,\cdot)\|_{\mathcal{X}^2}^p\Big) \le 0$$

and $t \mapsto \ln E_\lambda(u(t,\cdot)) - \ln\|u(t,\cdot)\|_{\mathcal{X}^2}^p$ is a non increasing function of $t > 0$ and the conclusion follows. If $E_\lambda(u) < 0$, $t \mapsto \ln|E_\lambda(u(t,\cdot))| - \ln\|u(t,\cdot)\|_{\mathcal{X}^2}^p$ is a non decreasing function, which leads to the same conclusion. If $E_\lambda(u(t,\cdot)) = 0$, then, by (22), $E_\lambda(u(s,\cdot)) \le 0$ for $s > t$ and therefore, $\mathcal{E}_\lambda[u](s) \le 0 = \mathcal{E}_\lambda[u](t)$.

(ii) Note that by (20), $\mathcal{E}_\lambda[u] = -\frac{1}{2}\frac{dH}{dt}H^{-\frac{p}{2}}$, where we have set $H(t) = \|u(t,\cdot)\|_{\mathcal{X}^2}^2$ for simplicity. It readily follows that

$$\frac{d\mathcal{E}_\lambda[u]}{dt} \le 0 \Leftrightarrow \left(\frac{dH}{dt}\right)^2 \le \frac{2}{p}H\frac{d^2H}{dt^2} \Leftrightarrow \operatorname{sign}\left(1-\frac{p}{2}\right)\cdot\frac{d^2H^{1-\frac{p}{2}}}{dt^2} \ge 0,$$

which shows (ii). □

4 Estimates of the $\mathcal{X}^2$–norm

In this section we present some estimates of the $\mathcal{X}^2$–norm of the solutions of Problem $(P_{\sigma,f_{\lambda,q},u_0})$ for various cases of q and λ.

Proposition 2 – *Assume $\lambda < 0$ and let u be a weak solution of Problem $(P_{\sigma,f_{\lambda,p},u_0})$ with initial data $u_0 \in \mathcal{X}^2$.*

(i) *Assume $p > 2$. Then there exists a positive constant K depending only on p, Ω, λ, and σ such that for all $t \geq 0$,*

$$\|u(t,\cdot)\|_{\mathcal{X}^2} \leq \left(\|u_0\|_{\mathcal{X}^2}^{2-p} + K(p-2)t\right)^{\frac{1}{2-p}}. \tag{23}$$

In particular, $\lim_{t\to\infty} \|u(t,\cdot)\|_{\mathcal{X}^2} = 0$.

(ii) *In the case $p < 2$, the estimate* (23) *holds for $0 \leq t \leq T^* := \frac{\|u_0\|_{\mathcal{X}^2}^{2-p}}{K(2-p)}$ and $\|u(t,\cdot)\|_{\mathcal{X}^2} = 0$ for all $t \geq T^*$.*

Proof. Consider any $t \geq 0$ in the maximal interval of existence containing 0. Multiplying the differential equation by u yields

$$\frac{1}{2p}\frac{d}{dt}\|u(t,\cdot)\|_{\mathcal{X}^2}^2 = -E_\lambda(u(t,\cdot)).$$

By Sobolev's classical embedding theorem there exists a constant $K > 0$ depending only on p, Ω, λ, and σ such that for any $u \in W^{1,p}(\Omega)$,

$$E_\lambda(u) \geq K\|u\|_{\mathcal{X}^2}^p.$$

Combining these two results leads to

$$\frac{d}{dt}\|u(t,\cdot)\|_{\mathcal{X}^2}^2 \leq -K\|u(t,\cdot)\|_{\mathcal{X}^2}^p$$

and therefore,

$$\frac{1}{2-p}(\|u(t,\cdot)\|_{\mathcal{X}^2}^{2-p} - \|u_0\|_{\mathcal{X}^2}^{2-p}) \leq -Kt. \tag{24}$$

For $p > 2$ it follows that the solution exists for all $t \geq 0$ and

$$\|u(t,\cdot)\|_{\mathcal{X}^2}^{2-p} \geq -K(2-p)t + \|u_0\|_{\mathcal{X}^2}^{2-p}$$

and

$$\|u(t,\cdot)\|_{\mathcal{X}^2}^2 \leq \left(K(p-2)t + \|u_0\|_{\mathcal{X}^2}^{2-p}\right)^{\frac{2}{2-p}}. \tag{25}$$

In the case $p < 2$, the inequalities (24) and (25) hold only for $0 \leq t \leq T^*$. But, as the r.h.s. in (25) vanishes at T^*, we conclude that $\|u(t,\cdot)\|_{\mathcal{X}^2} = 0$ for $t = T^*$, and, thereby, $\|u(t,\cdot)\|_{\mathcal{X}^2} = 0$ for all $t \geq T^*$. □

Remark 6 – For Problem (5) with $f_{\lambda,q} \equiv 0$, DiBenedetto[14] proved that positive solutions extinct in finite time, i.e. there exists $T_* > 0$ such that $u(t,\cdot) = 0$ for $t \geq T_*$ for $1 < p < 2$. Here we proved that the same result holds for the solutions of Problem $(P_{\sigma,f_{\lambda,p},u_0})$ if $\lambda < 0$ and $p_1 < p < 2$.

Next, we give a point-wise one-sided estimate of the solution $u(t,\cdot)$ in the case of a bounded initial data with definite sign:

Proposition 3 – *Suppose $\lambda < 0$ and $u_0 \in \mathcal{X}^\infty \cap W^{1,p}(\Omega)$. Then the unique solution u of $(P_{\sigma,f_{\lambda,p},u_0})$ is globally bounded. Moreover, if $p > 2$ and $\sup_{\overline{\Omega}} u_0 < 0$, then*

$$u(t,x) \leq \eta(t) := -\left((-\sup_{\overline{\Omega}} u_0)^{2-p} + (2-p)\lambda t\right)^{\frac{1}{2-p}}.$$

Correspondingly, if $p > 2$ and $\inf_{\overline{\Omega}} u_0 > 0$, then

$$u(t,x) \geq \tilde{\eta}(t) = \left(\inf_{\overline{\Omega}} u_0^{2-p} + (2-p)\lambda t\right)^{\frac{1}{2-p}}$$

for all $t \geq 0$ and a.e. in Ω.

Proof. The boundedness follows straightforwardly from Remark 3 and Theorem 2 applied to u and the constant $\pm\|u_0\|_{\mathcal{X}}^\infty$. Now suppose that $u_0 \leq 0$ $\tau-$ a.e. in $\overline{\Omega}$. As $\lambda < 0$ and $2 < p$, $\eta(t) < 0$ for $t \geq 0$, and η clearly satisfies the ODE

$$\eta' = \lambda|\eta|^{p-2}\eta.$$

Moreover, observe that $u_0(x) \leq \eta(0)$ τ-a.e. in $\overline{\Omega}$, and $\sigma\eta'(t) \geq 0$ on $\partial\Omega$, since $\sigma \geq 0$. Then, again, Theorem 2 permits to conclude that $u(t,\cdot) \leq \eta(t)$ a.e. in Ω. Thus, we are led to

$$-\|u_0\|_{\mathcal{X}}^\infty = \inf_{\overline{\Omega}} u_0 \leq u(t,x) \leq \eta(t)$$

for all $t \geq 0$ and a.e. in Ω. The case $u_0 \geq 0$ is shown similarly. □

For $1 < q < 2$ and $\lambda \geq 0$ the $\mathcal{X}^2$-norm of a solution remains bounded on bounded time intervals. This is part of the following result bearing in mind that for the present case, the solutions of $(P_{\sigma,f_{\lambda,q},u_0})$ are not unique in general, see Remark 3 (iii).

Proposition 4 – *Suppose $1 < q \leq 2$ and $\lambda \geq 0$. Let $u_0 \in \mathcal{X}^2$ be given. Then for any weak solution of Problem $(P_{\sigma,f_{\lambda,q},u_0})$, the following estimate holds for all $t \in [0,T]$:*

$$\|u(t,\cdot)\|_{\mathcal{X}^2} \leq \begin{cases} \|u_0\|_{\mathcal{X}^2}\, e^{\lambda t} & \text{for } q = 2, \\ \left(\|u_0\|_{\mathcal{X}^2}^{2-q} + (2-q)\lambda|\Omega|^{\frac{2-q}{2}} t\right)^{\frac{1}{2-q}} & \text{for } 1 < q < 2. \end{cases}$$

[14] DiBenedetto, 1993, *Degenerate parabolic equations*, Chapter VII, Proposition 2.1.

Proof. Set $H(t) = \|u(t,\cdot)\|^2_{\mathcal{X}^2}$. Multiplying the differential equation by u yields

$$\frac{dH}{dt} = -2\int_\Omega |\nabla u|^p\,dx + 2\lambda\int_\Omega |u|^q\,dx.$$

In the case $q = 2$, it follows readily that $\frac{dH}{dt} \leq 2\lambda H$. In the case $1 < q < 2$, the inequality

$$\frac{dH}{dt} \leq 2\lambda|\Omega|^{\frac{2-q}{2}}\left(\int_\Omega u^2\,dt\right)^{\frac{q}{2}} \leq 2\lambda|\Omega|^{\frac{2-q}{2}} H^{\frac{q}{2}}$$

leads to the desired estimate. □

5 Behaviour at infinity

In this section we investigate the behaviour and growth order estimates at infinity of solutions of $(P_{\sigma,f_{\lambda,p},u_0})$ in the case $\lambda \leq 0$. The uniqueness of solutions of Problem $(P_{\sigma,f_{\lambda,p},u_0})$ stems from Theorem 2 and Remark 3. Our aim is to prove first some behaviour at infinity in the spaces $\mathcal{X}^2$ and $W^{1,p}(\Omega)$, c.f. Proposition 7. Secondly we shall give a more precise rate of convergence at ∞ of the $\mathcal{X}^\infty$ norms, c.f. Proposition 8. Finally, we give more precise informations about the behaviour of the solution of Problem (18) as $t \to \infty$.

Theorem 6 – *Let $\lambda < 0$ and $p > 2$ and let u be a solution of Problem $(P_{\sigma,f_{\lambda,p},u_0})$ with initial data $u_0 \in \mathcal{X}^2$. Then there exists a sequence $(t_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^+$ tending to ∞ and a solution $w \in W^{1,p}(\Omega)$ of the elliptic problem with Robin–Steklov boundary condition*

$$\begin{cases} -\Delta_p w - \lambda|w|^{p-2}w = w & \text{in } \Omega,\\ |\nabla w|^{p-2}\partial_\nu w = \sigma w & \text{on } \partial\Omega,\end{cases}$$

such that

$$\lim_{n\to\infty} \|(1+(p-2)t_n)^{\frac{2}{p-2}} u(t_n,\cdot) - w\|_{\mathcal{X}^2} = 0.$$

Proof. We can follow the proof of Berryman and Holland (1980). Set $z(t,x) = (1+(p-2)t)^{\frac{1}{p-2}} u(t,x)$ for $t \geq 0, x \in \overline{\Omega}$. A simple calculation shows that z solves

$$\begin{cases} (1+(p-2)t)\partial_t z = z + \Delta_p z + \lambda|z|^{p-2}z & \text{in } \Omega \text{ for } t > 0,\\ \sigma(1+(p-2)t)\partial_t z = -|\nabla z|^{p-2}\partial_\nu z + \sigma z & \text{on } \partial\Omega \text{ for } t > 0,\\ z(0,\cdot) = u_0 & \text{in } \overline{\Omega},\end{cases} \tag{26}$$

Consider the energy functional J associated to (26)

$$J(v) = \frac{1}{p}\int_\Omega (|\nabla v|^p - \lambda |v|^p)\,dx - \frac{1}{2}\left(\int_\Omega |v|^2\,dx + \sigma \oint_{\partial\Omega} |v|^2\,d\rho\right)$$

By Proposition 2, $\|z(t,\cdot)\|_{\mathcal{X}^2}$ is uniformly bounded for $t > 0$ and thereby, using Lemma 1, $\|z(t,\cdot)\|_{W^{1,p}}$ is uniformly bounded for $t \geq t_0 > 0$ for any t_0 as well. A similar reasoning as in the proof of Lemma 1, shows that

$$\frac{d}{dt}J(z(t,\cdot)) = -(1 + (p-2)t)\|\partial_t z(t,\cdot)\|^2_{\mathcal{X}^2} < 0$$

for a.a. $t > 0$. Thus, $t \mapsto J(z(t,\cdot))$ is bounded from below and decreasing, and therefore there exists a sequence $(t_n)_{n\in\mathbb{N}}$ with $t_n \to \infty$ such that

$$\lim_{n\to\infty} \frac{d}{dt}J(z(t_n,\cdot)) = 0.$$

Set $w_n = z(t_n,\cdot)$. By Lemma 1 the sequence $(\|w_n\|_{W^{1,p}})_{n\in\mathbb{N}}$ is bounded as well, thus there exists a subsequence still denoted by $(w_n)_{n\in\mathbb{N}}$, and a function $w \in W^{1,p}(\Omega)$ such that $w_n \rightharpoonup w$ weakly in $W^{1,p}(\Omega)$, strongly in $\mathcal{X}^2$ and simply $\tau-$ a.e. in $\overline{\Omega}$. Then the l.h.s. of the equations in (26) tend weakly to 0 in $\mathcal{X}^2$ as $n \to \infty$, while the r.h.s. tend weakly in $W^{1,p}(\Omega)$ to $\Delta_p w + \lambda |w|^{p-2} w + w$ in Ω and to $-|\nabla w|^{p-2}\partial_\nu w + \sigma w$ on $\partial\Omega$. □

In the case $\lambda = 0$ an asymptotic result holds that is analogous to Proposition 2 and Theorem 6. Note that the asymptotic constant c_0 defined in (27) is exactly the same one as for the classical Laplacian, i.e. $p = 2$, see Bandle, Below, and Reichel (2006).

Theorem 7 – *Assume $\lambda = 0$ and let $u_0 \in \mathcal{X}^2$ be given. Let u be the weak solution of the Cauchy problem $(P_{\sigma,0,u_0})$ with initial data u_0. Then*

$$\lim_{t\to\infty} u(t,\cdot) = c_0 \text{ in } W^{1,p}(\Omega)$$

with

$$c_0 := \frac{\int_\Omega u_0\,dx + \sigma \oint_{\partial\Omega} u_0\,d\rho}{|\Omega| + \sigma|\partial\Omega|}. \tag{27}$$

Moreover, for $p < 2$ there exists $t_* \geq 0$ such that $u(t,x) = c_0$ for all $(t,x) \in [t_*,\infty[\times\overline{\Omega}$. Furthermore

$$t_* = \frac{\|u_0 - c_0\|^{2-p}_{\mathcal{X}^2}}{K(2-p)} > 0$$

with some constant $K > 0$ depending only on N, p, σ and Ω.

Proof. Integrating the differential equation leads to

$$0 = \int_\Omega \partial_t u\, dx - \oint_{\partial\Omega} |\nabla u|^{p-2}\partial_\nu u\, d\rho = \int_\Omega \partial_t u\, dx + \sigma \oint_{\partial\Omega} \partial_t u\, d\rho;$$

that is $\frac{d}{dt}\Big(\int_\Omega u\, dx + \sigma \oint_{\partial\Omega} u\, d\rho\Big) = 0$ for any $t \geq 0$. Writing $\alpha := |\Omega| + \sigma|\partial\Omega|$, we have

$$\int_\Omega u\, dx + \sigma \oint_{\partial\Omega} u\, d\rho = \int_\Omega u_0\, dx + \sigma \oint_{\partial\Omega} u_0\, d\rho = c_0\alpha.$$

Again, integrating the differential equation multiplied by u leads to

$$E(u(t,\cdot)) = -\frac{1}{2p}\frac{d}{dt}\|u(t,\cdot)\|^2_{\mathcal{X}^2},$$

where $E(u) := E_0(u) = \frac{1}{p}\int_\Omega |\nabla u|^p\, dx$. Set $v = u - c_0$ and $v_0 = u_0 - c_0$ and observe that $\int_\Omega v_0\, dx + \sigma \oint_{\partial\Omega} v_0\, d\rho = 0$ and that v is a weak solution of Problem $(P_{\sigma,0,u_0})$ for the initial data v_0. Since $v \mapsto \|\nabla v\|_p$ defines an equivalent norm to the usual $W^{1,p}(\Omega)$-norm on the subspace H_0 defined by

$$H_0 = \left\{ v \in W^{1,p}(\Omega) \,\middle|\, \int_\Omega v\, dx + \sigma \oint_{\partial\Omega} v\, d\rho = 0 \right\},$$

we infer from Sobolev's embedding $W^{1,p}(\Omega) \hookrightarrow \mathcal{X}^2$ that

$$E(v) \geq K\|v\|^p_{\mathcal{X}^2}$$

with some constant $0 < K = K(p,N,\sigma,\Omega)$. Then we proceed as in the proof of Proposition 2 in order to get the bound (23) for the function v. Thus, $\|v\|_{\mathcal{X}^2} = 0$ for all $t \geq t_* := \frac{\|v_0\|^{2-p}_{\mathcal{X}^2}}{K(2-p)}$ if $2 < p$. If $p > 2$, then $\lim_{t\to\infty}\|v(t,\cdot)\|_{\mathcal{X}^2} = 0$. Finally, Lemma 1 implies that the Rayleigh quotient $\mathcal{E}_0[v](t) := \frac{\int_\Omega (|\nabla v(t,\cdot)|^p}{\|v\|^p_{\mathcal{X}^2}}$ is non increasing with respect to t for $t > 0$, that is

$$\|\nabla v(t,\cdot)\|_p \leq \frac{\|\nabla v(t_0,\cdot)\|_p}{\|v(t_0,\cdot)\|_{\mathcal{X}^2}}\|v(t,\cdot)\|_{\mathcal{X}^2} \qquad \text{for } t > t_0 > 0.$$

Then if $p \neq 2$, $\lim_{t\to\infty}\|\nabla v(t,\cdot)\|_p = 0$ and $\lim_{t\to\infty}\|v(t,\cdot)\|_{W^{1,p}(\Omega)} = 0$ since $v \in H_0$, which permits to conclude that $u(t,\cdot) \to c_0$ in $W^{1,p}(\Omega)$ as $t \to \infty$. □

Next, we want to establish the L^∞-convergence for the previous limit result $c_0 = \lim_{t\to\infty} u(t,\cdot)$. The problem of the global boundedness of positive weak solutions for the parabolic equation in $(P_{\sigma,f_{\lambda,p},u_0})$ with Dirichlet or Neumann boundary conditions and bounded initial data has been completely treated e.g. in DiBenedetto (1993, Chapter V). Later F. Cipriani and G. Grillo[15] gave the so called "ultra-conductivity bounds" of the solutions of the parabolic equation of $(P_{\sigma,f_{\lambda,p},u_0})$ in the case $2 < p < N$ under Dirichlet boundary conditions and initial data $u_0 \in L^q(\Omega)$ with q sufficiently large. In the case $\lambda = 0$ under dynamical boundary conditions such results seem to be unavailable yet in the literature. Therefore we present the following one here.

Theorem 8 – *Let us assume $\lambda = 0$ and $u_0 \in W^{1,p}(\Omega)$. Let u be a solution of $(P_{\sigma,0,u_0})$ with initial data u_0 and let c_0 be defined as in (27). Then the following estimates hold.*

(i) *If $p < N$ and $u_0 \in \mathcal{X}^{p_*}$, then there exists $0 < d = d(p_*,\Omega,\sigma)$ such that, for all $t > 0$,*

$$\|u(t,\cdot) - c_0\|_{\mathcal{X}^\infty} \le d\, t^{-\frac{1}{p_*-2}} \|u_0 - c_0\|_{\mathcal{X}^{p_*}}^{\frac{p_*-p}{p_*-2}}. \tag{28}$$

(ii) *If $p \ge N$ and $u_0 \in \mathcal{X}^q$ then for some $q > \max\{p,2\}$, there exists $0 < d = d(q,\Omega,\sigma)$ such that, for all $t > 0$,*

$$\|u(t,\cdot) - c_0\|_{\mathcal{X}^\infty} \le d\, t^{-\frac{1}{q-2}} \|u_0 - c_0\|_{\mathcal{X}^q}^{\frac{q-p}{q-2}}.$$

In particular

$$\lim_{t\to\infty} \|u(t,\cdot) - c_0\|_{\mathcal{X}^\infty} = 0.$$

Proof. We shall give only the proof in the case (i), as the proof of case (ii) is the same provided p^* is replaced by q. Let us assume first that $u_0 \in \mathcal{X}^\infty \cap W^{1,p}(\Omega)$, we shall get rid of this assumption at the end of the proof. As in the proof of Proposition 7, we shall use $v = u - c_0$ and $v_0 = u_0 - c_0$. Since $v_0 \in \mathcal{X}^\infty$, Theorem 2 gives readily the estimates

$$\|v\|_{\infty,\mathbb{R}^+\times\Omega} \le \|v_0\|_{\infty,\Omega}, \quad \|v\|_{\infty,\mathbb{R}^+\times\partial\Omega} \le \|v_0\|_{\infty,\partial\Omega},$$

in particular for all $t \ge 0$, $v(t,\cdot) \in \mathcal{X}^\infty$. We can consider for any $m \ge 2$ the test function $\varphi = |v|^{m-2}v$, yielding

$$\frac{d}{dt}\|v(t,\cdot)\|_{\mathcal{X}^m}^m = -m(m-1)\left(\frac{p}{p+m-2}\right)^p \int_\Omega \left|\nabla |v(t,\cdot)|^{\frac{m-2+p}{p}}\right|^p dx. \tag{29}$$

Note that for any $m \ge 2$, $|v|^{\frac{m-2}{p}} v \in H$, where H is defined in Lemma 2 below. Hence, by combining (32) and trace embeddings we infer the existence of a constant C depending only on p_*, Ω, and σ such that

$$\int_\Omega \left|\nabla |v(t,\cdot)|^{\frac{m-2+p}{p}}\right|^p dx \ge C \left\| |v(t,\cdot)|^{\frac{m-2+p}{p}} \right\|_{\mathcal{X}^{p_*}}^p.$$

[15] Cipriani and Grillo, 2001, "Uniform bounds for solutions to quasilinear parabolic equations".

Then it follows from (29) that

$$\frac{d}{dt}\|v(t,\cdot)\|_{\mathcal{X}^m}^m \le -Cm(m-1)\left(\frac{p}{p+m-2}\right)^p \|v(t,\cdot)\|_{\mathcal{X}^{\frac{p_*}{p}(p+m-2)}}^{p+m-2}.$$

Set $r_0 = \frac{p_*}{p}$, $m_0 = p_*$ and for $k \in \mathbb{N}^*$

$$m_k = r_0(p + m_{k-1} - 2) = r_0^{k+1}\frac{p_* - 2}{r_0 - 1} - \frac{r_0(p-2)}{r_0 - 1}.$$

Now the previous inequality for $m = m_k$ reads

$$\frac{d}{dt}\|v(t,\cdot)\|_{\mathcal{X}^{m_k}}^{m_k} \le -C\theta_k \|v(t,\cdot)\|_{\mathcal{X}^{m_{k+1}}}^{\frac{m_{k+1}}{r_0}}, \tag{30}$$

where $\theta_k := m_k(m_k-1)\left(\frac{p}{p+m_k-2}\right)^p$. Denote for simplicity $Y_k = \|v\|_{\mathcal{X}^{m_k}}^{m_k}$ and fix $t > 0$ and $k \in \mathbb{N}^*$. Define, for $0 \le j \le k$,

$$s_0 = t, \quad s_j - s_{j+1} = \frac{\mu}{\theta_{k-j} r_0^{(p-1)(k-j)}}$$

for some $\mu > 0$ to be chosen later. By integrating (30) between s_1 and s_0 we have, using that $t \mapsto Y_j(t)$ is a positive decreasing function,

$$Y_k(s_0) - Y_k(s_1) \le -C\theta_k(s_0 - s_1) Y_{k+1}^{\frac{1}{r_0}}(t)$$

and

$$Y_{k+1}(t) \le [C\mu r_0^{-(p-1)k}]^{-r_0} Y_k(s_1)^{r_0}.$$

Iterating $k+1$ times leads to

$$Y_{k+1}(t) \le \alpha_k \beta_k Y_0(s_{k+1})^{r_0^{k+1}}$$

with

$$\alpha_k := (\mu C)^{-\sum_{j=0}^k r_0^{j+1}} = (\mu C)^{\frac{r_0}{r_0-1}(1-r_0^{k+1})},$$

$$\beta_k := \prod_{j=0}^k r_0^{(p-1)(k-j)r_0^{j+1}} = r_0^{(p-1)[-k\frac{r_0}{r_0-1} + \frac{r_0^2}{(r_0-1)^2}(r_0^k - 1)]}.$$

Thus,

$$Y_{k+1}^{\frac{1}{m_{k+1}}}(t) \le \alpha_k^{\frac{1}{m_{k+1}}} \beta_k^{\frac{1}{m_{k+1}}} Y_0(s_{k+1})^{\frac{r_0^{k+1}}{m_{k+1}}}. \tag{31}$$

Now choose μ in such a way that $\lim_{k\to\infty} s_k = 0$, i.e.

$$\mu = \frac{t}{\sum_{j=0}^{\infty} r_0^{j(1-p)} \theta_j^{-1}}.$$

Note that the series in the denominator converges, since θ_j scales with $(m_j)^{2-p}$ and $r_0^{(2-p)j}$. Moreover, a simple calculation gives

$$\lim_{k\to\infty} \frac{r_0^{k+1}}{m_{k+1}} = \frac{(r_0-1)}{(p_*-2)r_0}, \quad \lim_{k\to\infty} \alpha_k^{\frac{1}{m_{k+1}}} = (\mu C)^{-\frac{1}{p_*-2}}, \quad \lim_{k\to\infty} \beta_k^{\frac{1}{m_{k+1}}} = r_0^{\frac{p-1}{(p_*-2)(r_0-1)}}.$$

Now, the estimate (28) follows by passing to the limit in (31). Note that the convergence of $Y_0(s_{k+1}) = \|v(s_{k+1},\cdot)\|_{p_*}^{p_*}$ to $Y_0(0) = \|v_0\|_{p_*}^{p_*}$ follows from the fact that $v \in C([0,T];L^2(\Omega))$ by Definition 1 and by the boundedness of v.

Finally assume that $u_0 \in \mathcal{X}^{p_*}$ and take a sequence $\left(u_n^0\right)_{n\in\mathbb{N}}$ in $\mathcal{X}^\infty \cap W^{1,p}(\Omega)$ converging to u_0 in $\mathcal{X}^{p_*}$. In particular $c_n^0 := c_0\left(u_n^0\right) \to c_0$. Let u_n be the unique solution of $(P_{\sigma,f_{\lambda,p},u_0})$ for $\lambda = 0$ and initial data u_n^0. Thus, (28) holds for each u_n. By Corollary 3, for any $t \in \mathbb{R}$, $u_n(t,\cdot) \to u(t,\cdot)$ in $\mathcal{X}^2$ and therefore, for a.a. $x \in \Omega$, $u_n(t,x) \to u(t,x)$, which permits to conclude. □

Lemma 2 – *Let $0 < q < p_*$ be fixed and set*

$$H := \left\{ u \in W^{1,p}(\Omega) \,\middle|\, \exists s \in (0,q]\colon \int_\Omega |u|^{s-1} u \, dx + \sigma \oint_{\partial\Omega} |u|^{s-1} u \, d\rho = 0 \right\}.$$

Then there exists a constant $C = C(q) > 0$ such that for all $u \in H$,

$$\int_\Omega |\nabla u|^p \, dx \geq C \int_\Omega |u|^p \, dx. \tag{32}$$

Proof. Assume by contradiction that there exists a sequence $(u_n)_{n\in\mathbb{N}}$ in $W^{1,p}(\Omega)$ and a sequence $(s_n)_{n\in\mathbb{N}}$ in $(0,q]$ such that

$$\int_\Omega |u_n|^{s_n-1} u_n \, dx + \sigma \oint_{\partial\Omega} |u_n|^{s_n-1} u_n \, d\rho = 0, \tag{33}$$

$\int_\Omega |u_n|^p \, dx = 1$ and $\int_\Omega |\nabla u_n|^p \, dx \leq \frac{1}{n}$. Choose a subsequence of $(s_n)_{n\in\mathbb{N}}$ denoted again by $(s_n)_{n\in\mathbb{N}}$, that is converging to some $s_0 \in [0,q]$. As the sequence $(u_n)_{n\in\mathbb{N}}$ is bounded in $W^{1,p}(\Omega)$, there exists a subsequence of $(u_n)_{n\in\mathbb{N}}$, still denoted by $(u_n)_{n\in\mathbb{N}}$, converging weakly to some $v_0 \in W^{1,p}(\Omega)$, strongly in $L^p(\Omega) \cap \mathcal{X}^{s_0}$ and simply a.e. in Ω. Then v_0 has to be constant, say c since

$$\int_\Omega |\nabla v_0|^p \, dx \leq \lim_{n\to\infty} \int_\Omega |\nabla u_n|^p \, dx = 0.$$

But $v_0 = c \neq 0$ a.e. in Ω by the strong convergence in $L^p(\Omega)$. On the other hand, by letting n tend to ∞ in (33) and by the strong convergence in $\mathcal{X}^{s_0}$, we are led to the contradiction

$$0 = \int_\Omega |c|^{s_0-1} c\, dx + \sigma \oint_{\partial\Omega} |c|^{s_0-1} c\, d\rho. \qquad \square$$

Remark 7 – The existence of a solution of $(P_{\sigma,f_{\lambda,p},u_0})$ in the case $\lambda \leq 0$ in the sense of distributions has been proved for instance in Showalter (1997, p. 141) and in Lions (1969) in the case $\lambda = 0$, that is readily extended to the case $\lambda < 0$. Note that our notion of a weak solution in Definition 1 requires more regularity, since we impose $\partial_t u(t,\cdot) \in L^2(\Omega)$ and $\partial_t u_{|\partial\Omega}(t,\cdot) \in L^2(\partial\Omega, \rho)$ for a.a. $t \in [0,T)$. This higher regularity is not a restriction since distributional solutions of $(P_{\sigma,f_{\lambda,p},u_0})$ *with initial data* $u_0 \in W^{1,p}(\Omega)$ bear this property, see e.g. Showalter (1997, p. 124) or Brézis (1973, Theorem 3.1). Furthermore, following Brézis (1973, Theorem 3.2), $\partial_t u \in L^\infty(\mathbb{R}^+, \mathcal{X}^2)$.

Closing this section we present the occurrence of global solution existence in the presence of a hyperbolic equilibrium for a reaction term $f \in \mathcal{C}^1(\mathbb{R})$. Thus, the Cauchy problem in question reads

$$\begin{cases} \partial_t u = \Delta_p u + f(u) & \text{in } \Omega \text{ for } t > 0, \\ \sigma \partial_t u + |\nabla u|^{p-2} \partial_\nu u = 0 & \text{on } \partial\Omega \text{ for } t > 0, \\ u(0,\cdot) = u_0 \in \mathcal{C}(\overline{\Omega}). \end{cases} \tag{34}$$

We suppose that

$$(34) \text{ defines a local flow in } \mathcal{X}^\infty \cap W^{1,p}(\Omega) \tag{35}$$

and that there are real numbers $-\infty < A < B < C < \infty$ such that

$$f(A) = f(B) = f(C) = 0, \quad f'(B) < 0, \quad f > 0 \text{in } (A,B), \quad f < 0 \text{in } (B,C). \tag{36}$$

Introduce $F(s) = \int_A^s f(\eta)\, d\eta$, and recall that the energy functional $E_F \colon W^{1,p}(\Omega) \to \mathbb{R}$ is defined formally by $E_F(u) = \frac{1}{p}\int_\Omega |\nabla u|^p\, dx - \int_\Omega F(u)\, dx$. Now we can state the following.

Theorem 9 – *Under the conditions* (35), (36), *and* $u_0 \in \mathcal{X}^\infty \cap W^{1,p}(\Omega)$, *any weak solution of Problem* (34) *with initial condition fulfilling* $A \leq u_0 \leq C$ *τ-a.e. in* $\overline{\Omega}$ *exists globally in* $[0,\infty)$ *and satisfies* $A \leq u \leq C$ *τ-a.e. in* $\overline{\Omega}$ *for all* $t \geq 0$. *Moreover, the equilibrium* B *is stable in the class of functions from* $W^{1,p}(\Omega)$ *taking their values in* $[A,C]$ *τ-a.e. in* $\overline{\Omega}$.

Proof. If u_0 takes its values in $[A,B]$ or $[B,C]$ τ-a.e. in $\overline{\Omega}$, then Theorem 2 for the given σ permits to conclude that $A \le u(t,\cdot) \le B$ or $B \le u(t,\cdot) \le C$ respectively τ-a.e. in $\overline{\Omega}$ for all $t \ge 0$. For an initial condition u_0 satisfying $A \le u_0 \le C$ τ-a.e. in $\overline{\Omega}$ Theorem 2 again applies to the solutions $\underline{u}$, u, and $\overline{u}$ with respective initial data

$$\underline{u}_0 := \min\{u_0, B\} \le u_0 \le \max\{u_0, B\} =: \overline{u}_0$$

and yields

$$\forall t \ge 0 \colon A \le \underline{u}(t,\cdot) \le u(t,\cdot) \le \overline{u}(t,\cdot) \le C \quad \tau\text{-a.e. in } \overline{\Omega}.$$

As for the last assertion, note that $E_F(u)$ defines a generalized Lyapunov function for the equilibrium B in the class of functions from $W^{1,p}(\Omega)$ taking their values in $[A,C]$ τ-a.e. As above, the orbital derivative along solutions of (34) is nonpositive, since

$$\frac{d}{dt} E_F(u(t,\cdot)) = -\int_\Omega (\partial_t u(t,\cdot))^2 \, dx - \sigma \oint_{\partial\Omega} (\partial_t u_{|\partial\Omega}(t,\cdot))^2 \, ds \le 0.$$

Moreover, for functions belonging to the mentioned class the potential energy term is bounded from below by $-|\Omega| F(B)$. Thus, the equilibrium B is stable, see e.g. Amann (1990, Section 18). □

We note in passing that by Lasalle's Principle[16], the trajectories of the flow belonging to (34) have their ω-limits in the set of functions satisfying $\frac{d}{dt} E_F(u) = 0$. This yields another global existence proof. Moreover, attractive properties of the equilibrium B might be obtained by using similar arguments as in Below (1994, Theorem 17.31) We omit the details here and mention only the following result under the Neumann boundary condition.

Corollary 4 – *Under the hypotheses of Theorem 9 in the case $\sigma = 0$, any weak solution of*

$$\begin{cases} \partial_t u = \Delta_p u + f(u) & \text{in } \Omega \text{ for } t > 0, \\ \partial_\nu u = 0 & \text{on } \partial\Omega \text{ for } t > 0, \\ u(0,\cdot) = u_0 \in \mathcal{C}(\overline{\Omega}) \setminus \{A, C\}, \end{cases}$$

satisfies

$$\lim_{t \to \infty} \|u(t,\cdot) - B\|_{\mathcal{X}^\infty} = 0.$$

[16] Amann, 1990, *Ordinary differential equations*, Section 18.

Proof. Recall $c_0 := \frac{\int_\Omega u_0\,dx + \sigma \oint_{\partial\Omega} u_0\,d\rho}{|\Omega| + \sigma|\partial\Omega|}$ and get by assumption that $A < c_0 < C$. Using the notations of Theorem 9, by Theorems 8 and 2 there exists $t_c \geq 0$ such that

$$\forall t \geq t_c\colon\ \frac{A+c_0}{2} \leq \underline{v}(t,\cdot) \leq \underline{u}(t,\cdot) \leq u(t,\cdot) \leq \overline{u}(t,\cdot) \leq \overline{v}(t,\cdot) \leq \frac{C+c_0}{2} \quad \tau\text{-a.e. in } \overline{\Omega},$$

where $\underline{v}$ and $\overline{v}$ denote the solutions for $f = 0$ with initial conditions $\underline{u}_0$ and $\overline{u}_0$, respectively. But the solutions z_1 and z_2 of

$$\begin{cases} \dot{z_1} = f(z_1) & \text{for } t \geq t_c, \\ z_1(t_c) = \frac{A+c_0}{2}, \end{cases} \qquad \begin{cases} \dot{z_2} = f(z_2) & \text{for } t \geq t_c, \\ z_2(t_c) = \frac{C+c_0}{2}, \end{cases}$$

tend both uniformly to B, while $z_1(t) \leq u(t,\cdot) \leq z_2(t)$ τ-a.e. in $\overline{\Omega}$ for all $t \geq t_c$ by Theorem 2. □

6 Blow up phenomena

This section is devoted to the occurrence of blow up for the solutions of Problem (P_{σ,f,u_0}). We note in passing, that among others, the recent rather general blow up results by Vulkov[17] complement nicely the ones presented here, but do not include the latter ones. First, we consider a rather general case of nonnegative source terms $f(t,x,u)$ of the form (37). Then we shall study a source term with a sign change of the form $f = f_{\lambda,q}$, with $q \geq p$ and $\lambda > 0$. For the first case, assume that f is of the form

$$f(t,x,u) = m(t,x)g(u), \tag{37}$$

where g and m satisfy

$$g \in C^1(\mathbb{R}),\ g(s) > 0 \quad \text{for all } s > 0, \tag{38}$$

$$g(s) \geq 0 \text{ for all } s \in \mathbb{R}, \quad g'(s) \geq 0 \text{ for all } s > 0 \tag{39}$$

$$\int_{s_0}^{\infty} \frac{d\eta}{g(\eta)} < \infty \quad \text{for some } s_0 > 0, \tag{40}$$

$$m \in L^1_{\text{loc}}(\mathbb{R}^+; L^1(\Omega)), \quad m \geq 0,$$

and

$$\int_0^\tau \left(\int_\Omega m(t,x)\,dx \right) dt \to \infty \quad \text{as} \quad \tau \to \infty \tag{41}$$

[17] Vulkov, 2007, “Blow up for some quasilinear equations with dynamical boundary conditions of parabolic type”.

We follow a technique developed in Bandle, Below, and Reichel (2006) to exclude the existence of global weak solution of Problem (P_{σ,f,u_0}) for positive sources in the case $p = 2$. Unlike in that reference, it is impossible to establish an expansion formula for the solution of the homogeneous linear problem corresponding to (P_{σ,f,u_0}) for $p \neq 2$. However, as a consequence of the asymptotic result Theorem 8 obtained for the homogeneous Problem $(P_{\sigma,0,u_0})$, u is bounded from below after a certain finite time. A main result of this section is the following.

Theorem 10 – *Assume* (37)–(41) *and let* $u_0 \in \mathcal{X}^{p_*}$ *if* $p < N$ *or* $u_0 \in \mathcal{X}^q$ *for some* $q > p$ *if* $p \geq N$. *Assume that* u_0 *fulfils*

$$\int_\Omega u_0\, dx + \sigma \oint_{\partial\Omega} u_0\, d\rho > 0.$$

Then there is no weak solution u *of* (P_{σ,f,u_0}) *existing for all times.*

Proof. Suppose that u is a global weak solution of (P_{σ,f,u_0}). First, we claim that there exists $t_0 > 0$ such that $u(t,\cdot) \geq \frac{c_0}{2}$ a.e. in Ω and for $t \geq t_0$, with c_0 defined in (27). Let v be the unique solution of the homogeneous problem $(P_{\sigma,0,u_0})$ with initial data u_0. Theorem 8 implies that there exists a $t_0 \geq 0$ such that for all $t \geq t_0$, $\frac{c_0}{2} \leq v(t,\cdot)$ a.e. in Ω. As v is a solution of $(P_{\sigma,0,u_0})$ for $t \geq t_0$, Corollary 1 applies to $f_1 := 0 \leq m(t,x)g(u) =: f_2$ by (39) and yields $u(t,\cdot) \geq v(t,\cdot)$ a.e. in Ω for all $t \geq t_0$. Trivially, by (39), we also have $g(u(t,x)) \geq g(\frac{c_0}{2})$ a.e. in Ω for all $t \geq t_0$.

Fix any $t > t_0$ and any $M > \frac{c_0}{2}$ and consider the test function defined by

$$\varphi_M(x) = \begin{cases} \frac{1}{g(u(t,x))} & \text{if } u(t,x) \leq M; \\ \frac{1}{g(M)} & \text{if } u(t,x) > M. \end{cases}$$

Then φ_M is admissible in the formulation of a weak solution of (P_{σ,f,u_0}), i.e. $\varphi_M \in W^{1,p}(\Omega)$. Thus,

$$\int_\Omega \partial_t u \varphi_M(u)\, dx + \sigma \oint_{\partial\Omega} \partial_t u \varphi_M(u)\, d\rho \\ - \underbrace{\int_{\{x\in\Omega | u(t,x)<M\}} |\nabla u|^p \frac{g'(u)}{g(u)^2}\, dx}_{\geq 0} - \int_\Omega m(x,t) g(u) \varphi_M(u)\, dx = 0,$$

so that

$$\int_\Omega \partial_t u \varphi_M(u)\, dx + \sigma \oint_{\partial\Omega} \partial_t u \varphi_M(u)\, d\rho \geq \int_\Omega m(x,t) g(u) \varphi_M(u)\, dx \\ \geq \int_\Omega m(x,t)\, dx.$$

Letting M tend to ∞ implies

$$\int_\Omega \frac{\partial_t u}{g(u)}\,dx + \sigma \oint_{\partial\Omega} \frac{\partial_t u}{g(u)}\,d\rho \geq \int_\Omega m(x,t)\,dx.$$

Setting $h(s) = \displaystyle\int_{s_0}^{s} \frac{d\eta}{g(\eta)}$ and integrating between t_0 and τ lead to

$$\int_\Omega \Big(h(u(\tau,x)) - h(u(t_0,x))\Big)\,dx + \sigma\left(\oint_{\partial\Omega} \Big(h(u(\tau,r)) - h(u(t_0,r))\Big)\,d\rho(r)\right)$$
$$\geq \int_{t_0}^{\tau}\int_\Omega m(x,t)\,dx\,dt.$$

By (41), the r.h.s. tends to infinity as $\tau \to \infty$, whereas the l.h.s. remains bounded by Hypothesis (40). This contradiction permits to conclude. □

Remark 8 – The result in Theorem 10 applies especially to reaction terms of the form $f = \lambda|u|^{q-1}$ with $\lambda > 0$ and $q > 2$ or of the form $f(u) = e^u$.

In the sequel, $T_{\max}(u)$ will denote the maximal existence time of the weak solution of $(P_{\sigma,f})$ with respect to the $L^\infty(\Omega)$-norm, i.e.

$$T_{\max}(u) \stackrel{\text{def}}{=} \inf\left\{ s > 0 \,\middle|\, \limsup_{t \nearrow s} \|u(t,\cdot)\|_\infty = \infty \right\}. \tag{42}$$

In the case $m \equiv 1$ a lower bound for the maximal existence time can be obtained by comparison with the solution of the ODE under an appropriate initial condition.

Proposition 5 – *Let $u_0 \in \mathcal{X}^\infty$. Under* (38) *and* (40), *suppose that z is the solution of the ordinary IVP*

$$\begin{cases} \dot{z} = g(z), & \text{for } 0 < t < t_0 := \displaystyle\int_{\|u_0\|_{\mathcal{X}_\infty}}^{\infty} \frac{d\eta}{g(\eta)} \\ z(0) = \|u_0\|_{\mathcal{X}_\infty}, \end{cases}$$

Then any weak solution u of (P_{σ,g,u_0}) satisfies either $T_{\max}(u) = 0$ or $T_{\max}(u) \geq t_0$ and $u(t,\cdot) \leq z(t)$ for all $t \in [0, T_{\max}(u))$, a.e. in Ω.

Proof. Since g is of class C^1, g and the pair (u,z) satisfy the one-sided Lipschitz condition (1) for $0 < t < T_{\max}(u)$, with $l(t) = \max\{|g'(s)| \mid s \in [0, \max\{z(t), \|u(t,\cdot)\|_\infty\}]\}$ (c.f. Remark 3). Thus, by Theorem 2, for all $t \in [0, T_{\max}(u))$, $u(t,\cdot) \leq z(t)$ a.e. in Ω. Note that by (40), the maximal existence time t_0 of z satisfies $\infty > t_0 = \int_{\|u_0\|_{\mathcal{X}_\infty}}^{\infty} \frac{d\eta}{g(\eta)}$, since by separation of variables, $t = \int_0^t \frac{\dot{z}(s)}{g(z(s))}\,ds = \int_{z(0)}^{z(t)} \frac{d\eta}{g(\eta)}$. □

Let us prove now three different blow up results for nonlinearities of the form $f = f_{\lambda,q}$, always assuming that $\lambda > 0$. We start with the particular case $q = p > 2$, i.e. with Problem $(P_{\sigma,f_{\lambda,p},u_0})$. We recall the definition of $E_\lambda(u) := \frac{1}{p}\int_\Omega(|\nabla u|^p - |u|^p)\,dx$. Note that according to Proposition 4, blow up in finite time cannot occur for $1 < q \leq 2$.

Theorem 11 – *Suppose $u_0 \in W^{1,p}(\Omega)\cap L^\infty(\Omega)$ satisfies $E_\lambda(u_0) < 0$. Let u be a weak solution of Problem $(P_{\sigma,f_{\lambda,p},u_0})$ with $\lambda > 0, p > 2$ and initial data u_0. Then $T_{\max}(u) < \infty$.*

Proof. Assume by contradiction that $T_{\max}(u) = \infty$. Then, for any $s > 0$ there exist $M_s, \delta_s > 0$ such that $\|u(t,\cdot)\|_\infty < M_s$ for all $t \in [s - \delta_s, s]$. Put $\delta_0 = 0, M_0 = \|u_0\|_\infty$. Hence, by a compactness argument, $u \in L^\infty(\Omega_T)$ for any $T > 0$, and the condition (11) of Theorem 5 will be satisfied in $[0,T]$. Using the results and notations of Lemma 1, and the fact that $E_\lambda(u_0) < 0$ and $H(t) \geq 0$, it follows that $E_\lambda(u(t,\cdot)) < 0$ a.e. and thereby, that $H^{1-\frac{p}{2}}$ is decreasing. The continuity of H and the concavity of $H^{1-\frac{p}{2}}$ will imply that $H^{1-\frac{p}{2}}$ vanishes in finite time, which leads to the desired contradiction. □

Next, we shall prove that there is also blow up for solutions of Problem $(P_{\sigma,f_{\lambda,q},u_0})$ for $q > \max\{2,p\}$ and for (sufficiently regular) *positive* initial data u_0. We shall use in this case the following parabolic equation under Dirichlet boundary conditions

$$\begin{cases} \partial_t u = \Delta_p u + \lambda |u|^{q-2}u & \text{in } \Omega \text{ for } t > 0, \\ u(t,\cdot) = 0 & \text{on } \partial\Omega \text{ for } t > 0. \end{cases} \tag{43}$$

The corresponding energy functional E_F defined in (10) related to Problem (43) and to $(P_{\sigma,f_{\lambda,q},u_0})$ will be denoted by $E_{\lambda,q}$:

$$E_{\lambda,q}(u) = \frac{1}{p}\int_\Omega |\nabla u|^p\,dx - \frac{\lambda}{q}\int_\Omega |u|^q\,dx.$$

For any solution u of Problem (43) we will define $T_{\max}(u)$ identically as in (42). We have

Theorem 12 – *Suppose that $\lambda > 0$ and $q > \max\{2,p\}$. Let $u_0 \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ with $E_{\lambda,q}(u_0) \leq 0$. Then any weak solution v of the Cauchy problem* (43) *with initial data u_0 blows up in finite time with respect to $\|\cdot\|_2$ at the latest at time T^* satisfying satisfying*

$$T_{\max}(v) \leq T^* \leq \frac{q}{\lambda(q-2)(q-p)}|\Omega|^{\frac{q-2}{2}}\left(\int_\Omega |u_0|^2\,dx\right)^{\frac{2-q}{2}} =: T_2. \tag{44}$$

Proof. According to the results of Zhao (1993, Theorem 2.1), there exists a local solution of (43) which is bounded. Thus, $T_{\max}(v) > 0$. Introduce

$$N(t) = \|v(t,\cdot)\|_2^2 = \int_\Omega |v(t,x)|^2\,dx.$$

Multiplying the differential equation by v leads to

$$\int_\Omega v\partial_t v\,dx = -\int_\Omega |\nabla v|^p\,dx + \lambda\int_\Omega |v|^q\,dx = -pE_{\lambda,q}(v) + \lambda\frac{q-p}{q}\int_\Omega |v|^q\,dx.$$

Since by hypothesis $q > 2$, the function $f_{\lambda,q}$ satisfies the Lipschitz condition (11) in $t \in [0,T]$, with any $T < T_{\max}(v)$. Hence (17) implies that $E_{\lambda,q}(v(t,\cdot))$ is decreasing in time and, since $E_{\lambda,q}(u_0) \leq 0$, we have $E_{\lambda,q}(v(t,\cdot)) \leq 0$ for all $t \in [0,T]$. Hence

$$\frac{dN(t)}{dt} = 2\int_\Omega v\partial_t v\,dx \geq \frac{2\lambda(q-p)}{q}\int_\Omega |v|^q\,dx$$

and by Hölder's inequality

$$\int_\Omega |v|^2\,dx \leq |\Omega|^{\frac{q-2}{q}}\left(\int_\Omega |v|^q\,dx\right)^{\frac{2}{q}},$$

we obtain

$$\frac{dN(t)}{dt} \geq \frac{2\lambda(q-p)}{q}|\Omega|^{\frac{2-q}{2}}N(t)^{\frac{q}{2}} =: \alpha N(t)^{\frac{q}{2}}.$$

Integration between 0 and $t > 0$ leads to

$$N(t) \geq \left(N(0)^{\frac{2-q}{2}} - \frac{q-2}{2}\alpha t\right)^{\frac{2}{2-q}}.$$

Since $q > 2$, $N(t)$ becomes infinite at $t = T_2$, with T_2 as defined in the assertion. As $L^\infty(\Omega) \subset L^2(\Omega)$, the inequality in (44) is plain. □

By comparing positive solutions of parabolic problem with Dirichlet boundary conditions with those of parabolic problem with dynamical boundary boundary conditions we have the following result:

Theorem 13 – *Assume that $\partial\Omega$ is of class C^2. Suppose that $q > \max\{2,p\}$ and $\lambda > 0$. Let $u_0 \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, $u_0 \geq 0$ in Ω and $E_{\lambda,q}(u_0) \leq 0$. Then any weak solution of the Problem $(P_{\sigma,f_{\lambda,q},u_0})$ u blows up at the latest for $t = T_2$.*

Proof. Let us assume by contradiction that $T_{\max}(u) > T_2$. Let v be the unique weak solution of Problem (43) with initial data u_0 (c.f. **J**). As $u_0 \geq 0$, $v \geq 0$ as well in $[0,T]$ with $T < T_{\max}(v) \leq T_2 < T_{\max}(u)$. In particular, both solutions u and v belong to $L^\infty(\Omega_T)$. Consequently, c.f. Remark 3, the hypotheses (ii) and (iii) of the comparison result Theorem 4 are satisfied. Moreover, the regularity results already cited in Remark 1 imply that $v \in C^1((0,T]\times\Omega)$. Thus, the hypothesis (iv) is satisfied too, and thereby, $u(t,\cdot) \geq v(t,\cdot)$ a.e. in Ω for all $t \in [0,T]$ and $T_2 < T_{\max}(u) \leq T_{\max}(v)$. This contradicts (44) and permits to conclude. □

As for more general nonlinearities, we deduce the following result

Corollary 5 – *Under the hypothesis of Theorem 13, let u be a weak solution of* (P_{σ,f,u_0}) *with f satisfying*

$$f(\cdot,\cdot,z) \geq \lambda|z|^{q-2}z \quad \textit{for all } z \geq 0.$$

Then u blows up at the latest at T_2*:*

$$T_{\max}(u) \leq T_2.$$

Proof. Let $\tilde{u}$ be a solution of Problem $(P_{\sigma,f_{\lambda,q},u_0})$ with initial data u_0. As $\tilde{u}$ is a weak lower solution of Problem (P_{σ,f,u_0}) and $u(t,\cdot)$ is bounded for any $t \in [0, T_{\max}(u))$, then the Lipschitz condition (1) applies to the pair $(\tilde{u},u)$ and Theorem 2 permits to conclude. □

Finally, dealing with nonnegative solutions, we can derive another upper bound for the blow up time under the Neumann boundary condition ($\sigma = 0$) for arbitrary $p > p_1$ and arbitrary $q > 2$. Note that the upper bound T_1 will be optimal, as readily follows by choosing constant positive initial data.

Theorem 14 – *Suppose that* $q > 2$ *and* $\lambda > 0$*. Let* $u_0 \in \mathcal{X}^2$ *and* $u_0 \geq 0$ *a.e. in* Ω*,* $u_0 \not\equiv 0$*. Then a weak solution u of* $(P_{0,f_{\lambda,q},u_0})$ *blows up in finite time with respect the* L^1*-norm at the latest at time* T^{**} *satisfying*

$$T_{\max}(u) \leq T^{**} \leq \frac{\left(\int_\Omega |u_0|\,dx\right)^{2-q}}{(q-2)\lambda|\Omega|^{2-q}} =: T_1.$$

Proof. Assume by contradiction that $T_{\max}(u) = \infty$. First, note that by the aforementioned hypothesis on λ and u_0, Theorem 2 implies that $u \geq 0$ for all $t \geq 0$ and a.e. in Ω. Integrating the partial differential equation and using Hölder's inequality yield

$$\int_\Omega \partial_t u\,dx = \lambda \int_\Omega u^{q-1}\,dx \geq \lambda|\Omega|^{2-q}\left(\int_\Omega u\;dx\right)^{q-1}.$$

Set $N(t) = \|u(t,\cdot)\|_{1,\Omega}$ for $t > 0$, we are led to

$$\frac{dN(t)}{dt} \geq \lambda|\Omega|^{2-q} N(t)^{q-1} \tag{45}$$

and, integrating between 0 and $t > 0$, we obtain

$$N(t)^{2-q} \leq \left(N(0)^{2-q} - (q-2)\lambda|\Omega|^{2-q} t\right),$$

which implies that N becomes infinite at T_1. □

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