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On some properties of the category of cocommutative Hopf Algebras

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Abstract

By a recent work of Gran-Kadjo-Vercruysse, the category of cocommutative Hopf algebras over a field of characteristic zero is semi-abelian. In this paper, we explore some properties of this category, in particular we show that its abelian core is the category of commutative and cocommutative Hopf algebras.

Keywords: Hopf algebra, abelian category, semi-abelian category.

MSC: 16T05, 18B99, 18E10.

Introduction

It is a classical result that the category of commutative and cocommutative Hopf algebras is an abelian category (see for example Corollary 4.16[2] or Theorem 4.3[3]). It is also known that this is no more the case for the category of cocommutative (resp. commutative) Hopf algebras since the coproduct and the product are not equivalent in each of these categories.

In 2002, the more general notion of semi-abelian category emerges in category theory[4]. In a semi-abelian category, classical diagram lemmas (five lemma, snake lemma ...) are valid. Among the examples of semi-abelian category we have the categories of groups, ring without unit, Lie algebras (and more generally algebras over a reduced linear operad) and sheaves or presheaves of these. Abelian categories are also examples of semi-abelian categories. In fact, a category $\mathcal{C}$ is abelian precisely when both $\mathcal{C}$ and $\mathcal{C}^{\mathrm{op}}$ are semi-abelian. Since then, semi-abelian categories become widely-known as the good generalization of the category of groups just as abelian categories is the good generalization of the category of abelian groups.

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[2]Takeuchi, 1972, "A correspondence between Hopf ideals and sub-Hopf algebras".

[3]Newman, 1975, "A correspondence between bi-ideals and sub-Hopf algebras in cocommutative Hopf algebras".

[4]Janelidze, Márki, and Tholen, 2002, "Semi-abelian categories".

A category is semi-abelian if it has a zero object and finite products and is Barr-exact[5] and protomodular in the sense of Bourn[6]. For more details on exact, protomodular and semi-abelian categories, we refer the reader to the excellent book by Borceux and Bourn[7].

In this paper, we follow the characterization of semi-abelian categories given by Hartl and Loiseau[8]. Namely, a category $\mathcal{C}$ is semi-abelian if and only if the following four axioms are satisfied.

(A_1) The category $\mathcal{C}$ is pointed, finitely complete and finitely cocomplete.

(A_2) For any split epimorphism $p\colon X \to Y$ with section $s\colon Y \to X$ and with kernel $\kappa\colon K \hookrightarrow X$, the arrow $<\kappa, s>\colon K \coprod Y \to X$ is a cokernel.

(A_3) The pullback of a cokernel is a cokernel.

(A_4) The image of a kernel by a cokernel is a kernel.

To be specific, we use this characterization to give a new proof of the following theorem, established by Gran, Kadjo and Vercruysse[9].

Theorem 1 – *The category of cocommutative Hopf algebras over a field of characteristic zero is semi-abelian.*

Eventually, we compute the abelian core of this semi-abelian category (*i.e.* the subcategory of abelian objects). We obtain the main result:

Theorem 2 – *The abelian core of the category of cocommutative Hopf algebras over a field of characteristic zero is the category of commutative and cocommutative Hopf algebras.*

Roughly speaking the abelian core of a semi-abelian category $\mathcal{C}$ is the biggest abelian subcategory of $\mathcal{C}$. In particular, we recover as a corollary of this theorem the fact that the category of commutative and cocommutative Hopf algebras is abelian.

The paper is organized as follows. In Section 2, we check that the category of cocommutative Hopf algebras satisfies Axiom (A_1). The verifications of the axioms (A_2) and (A_3) are heavily based on a result of Newman[10] (see also Masuoka 1991) recalled in Section 3. In Section 4, we consider Axiom(A_3) which corresponds to

[5] Barr, 1971, "Exact categories".

[6] Bourn, 1991, "Normalization equivalence, kernel equivalence and affine categories".

[7] Borceux and Bourn, 2004, *Mal'cev, protomodular, homological and semi-abelian categories*.

[8] Hartl and Loiseau, 2011, "A characterization of finite cocomplete homological and of semi-abelian categories".

[9] Gran, Kadjo, and Vercruysse, 2016, "A torsion theory in the category of cocommutative Hopf algebras".

[10] Newman, 1975, "A correspondence between bi-ideals and sub-Hopf algebras in cocommutative Hopf algebras".

Theorem 3.7[11]. Section 5 is devoted to the proof of Axiom (A_2) and, in Section 6, we prove Axiom (A_4). The last section is devoted to the proof of Theorem 2.

1 Conventions and prerequisites.

In the whole article, $\Bbbk$ is a commutative field. By *module* we will understand module over $\Bbbk$. The unadorned symbol $\otimes$ between two $\Bbbk$-modules will stand for $\otimes_{\Bbbk}$. We denote by $\mathcal{H}^{\text{coco}}$ the category of cocommutative Hopf algebras over $\Bbbk$ and by $\mathcal{H}^{\text{co-coco}}$ the category of commutative and cocommutative Hopf algebras over $\Bbbk$.

Let H be a Hopf algebra. Its structure maps will be denoted as follows: multiplication $\mu_H\colon H\otimes H \to H$, comultiplication $\Delta_H\colon H \to H\otimes H$, unit $\eta_H\colon \Bbbk \to H$, counit $\varepsilon_H\colon H \to \Bbbk$ and antipode $S_H\colon H \to H$. Moreover, for any $a, b \in H$, we will denote $\mu_H(a\otimes b)$ by ab. The unit $\eta_H(1)$ will be denoted by 1_A or simply 1. We also adopt the Sweedler-Heyneman notation $\Delta_H(a) = a_1 \otimes a_2$. More generally, a generic element in a tensor product of $\Bbbk$-modules $A\otimes B$, will be denoted by $a\otimes b$, the summation sign being omitted.

We will call *Hopf ideal* of a Hopf algebra H any two-sided ideal I of the algebra H which is also a two-sided coideal of the coalgebra H (*i.e.* $\Delta_H(I) \subset I\otimes H + H\otimes I$ and $\varepsilon_H(I) = 0$) such that, moreover, one has $S_H(I) \subset I$. In particular, the structure on H induces a Hopf algebra structure on the quotient H/I.

A sub-Hopf algebra A of a Hopf algebra H will be called *normal* if, for any $a \in A$ and $y \in H$, one has $y_1 a S(y_2) \in A$. In particular, when H is commutative, one has $y_1 a S(y_2) = y_1 S(y_2) a = \varepsilon(y) a$. Thus, in that case, all sub-algebras of H are normal.

For of any morphism of Hopf algebras φ, we will denote by $\operatorname{im}(\varphi)$ its linear image and by $\ker(\varphi)$ its linear kernel. A morphism of Hopf algebras φ is injective if $\ker(\varphi) = 0$. The kernel $\ker(\varepsilon_H)$ of the counit of a Hopf algebra H will more specifically be denoted by H^+.

2 Completeness and cocompleteness

In this section we prove Axiom (A_1).

Theorem 3 – *The category $\mathcal{H}^{\text{coco}}$ is pointed, finitely complete and finitely cocomplete.*

Proof. First, we remark that the category $\mathcal{H}^{\text{coco}}$ is pointed. Indeed, its zero object is the ground field $\Bbbk$ with initial and terminal morphisms given by the unit $\eta_H\colon H \to \Bbbk$ and the counit $\varepsilon_H\colon \Bbbk \to H$.

Finite (co)completeness is the existence of finite (co)limits which is equivalent to the existence of finite (co)products and (co)equalizer. For the finite (co)products,

[11] Gran, Kadjo, and Vercruysse, 2016, "A torsion theory in the category of cocommutative Hopf algebras".

as the category is pointed, it is in fact sufficient to prove the existence of binary (co)products. Details can be found in Mac Lane (1998, §V.2).

The explicit descriptions of the binary (co)products and (co)equalizer in $\mathcal{H}^{\text{coco}}$ are given below. The reader may check, by straightforward computations, that the given constructions fulfill the definitions. □

The definition of equalizers for morphisms of general Hopf algebras was first given by Andruskiewitsch and Devoto[12] generalizing the notions of kernel given in Sweedler (1969) or Blattner, Cohen, and Montgomery (1986). The same authors give explicit description of coequalizers and cokernels. For finite coproducts of Hopf algebras we refer to Pareigis (2002, §2) and for products to Agore (2011b), Brzezinski and Wisbauer (2003), Agore (2011a). We simply follow the cited authors. It happens that their constructions for Hopf algebras restricts to $\mathcal{H}^{\text{coco}}$.

2.1 Equalizers, kernels and products

First, we give the constructions of equalizers and kernels in $\mathcal{H}^{\text{coco}}$. By Lemma 1.1.3[13], for any two morphisms $A \overset{f}{\underset{g}{\rightrightarrows}} B$ of $\mathcal{H}^{\text{coco}}$ the set

$$\text{Heq}(f,g) = \{x \in A \mid f(x_1)\otimes x_2 = g(x_1)\otimes x_2\} = \{x \in A \mid x_1 \otimes f(x_2) = x_1 \otimes g(x_2)\}$$

is a sub-Hopf algebra of A. The equalizer of $A \overset{f}{\underset{g}{\rightrightarrows}} B$ in $\mathcal{H}^{\text{coco}}$ is the inclusion morphism $\text{heq}(f,g)\colon \text{Heq}(f,g) \to A$.

We denote by $\text{hker}(f)$ the kernel of a morphism $A \xrightarrow{f} B$ in $\mathcal{H}^{\text{coco}}$ which is, by definition, the equalizer $\text{heq}(f, \eta_B \circ \varepsilon_A)$. Explicitly, $\text{hker}(f)$ is the inclusion $\text{Hker}(f) \to A$ with

$$\text{Hker}(f) = \{x \in A \mid x_1 \otimes f(x_2) = x \otimes 1\} = \{x \in A \mid f(x_1)\otimes x_2 = 1 \otimes x\}.$$

It can be easily check by straightforward computation that the kernel $\text{Hker}(f)$ of a morphism $A \xrightarrow{f} B$ is a normal sub-Hopf algebra of A.

The direct product of two objects A and B in $\mathcal{H}^{\text{coco}}$ is given by the tensor product over $\Bbbk$. Indeed the product is $< \pi_A, \pi_B >$ where the projections

$$A \xleftarrow{\pi_A} A\otimes B \xrightarrow{\pi_B} B$$

are given by $\pi_A(a\otimes b) = \varepsilon_B(b)a$ and $\pi_B(a\otimes b) = \varepsilon_A(a)b$ for $a\otimes b \in A\otimes B$. For any two morphisms $f\colon H \to A$ and $g\colon H \to B$ of $\mathcal{H}^{\text{coco}}$, the morphism $\varphi\colon H \to A\otimes B$ fulfilling the universal property of the product is defined by $\varphi(x) = f(x_1)\otimes g(x_2)$.

[12] Andruskiewitsch and Devoto, 1996, "Extensions of Hopf algebras".
[13] Ibid.

This expression of the product is very specific of the cocommutative case. It is a consequence of the fact that the comultiplication Δ_H of a Hopf algebra H is a morphism of coalgebras if and only if H is cocommutative.

2.2 Coequalizers, cokernels and coproducts

We now give explicit description of coequalizers and cokernels in $\mathcal{H}^{\text{coco}}$. Let A and B be two cocommutative Hopf algebras. For any two morphisms $A \overset{f}{\underset{g}{\rightrightarrows}} B$, set

$$J = \{f(x) - g(x) \mid x \in A\} \quad \text{and} \quad \text{Hcoeq}(f,g) = B/BJB$$

The projection $\text{hcoeq}(f,g)\colon B \to \text{Hcoeq}(f,g)$ is the coequalizer of $A \overset{f}{\underset{g}{\rightrightarrows}} B$ in $\mathcal{H}^{\text{coco}}$.

In the sequel, for simplicity of notations, we sometimes denote the Hopf ideal BJB by $\langle J \rangle$. Notice that J is a co-ideal of B as a consequence of the relation

$$\Delta\big(f(x)\big) = f(x_1) \otimes f(x_2) = \big(f(x_1) - g(x_1)\big) \otimes f(x_2) - g(x_1) \otimes \big(f(x_2) - g(x_2)\big)$$

with $x \in A$. It follows that $\langle J \rangle = BJB$ is a Hopf ideal of B and $\text{Hcoeq}(f,g)$ an object of our category $\mathcal{H}^{\text{coco}}$.

The cokernel $\text{hcoker}(f)$ of a morphism $A \xrightarrow{f} B$ in $\mathcal{H}^{\text{coco}}$ is, by definition, the coequalizer $\text{hcoeq}(f, \eta_B \circ \varepsilon_A)$. Notice that, when $g = \eta_B \circ \varepsilon_A$, the set J reduces to $J = f(A^+)$ where $A^+ = \ker(\varepsilon_A)$ is the linear kernel of the counit of A. Indeed, in this particular case, on has $J = \{f(x) - \varepsilon(x)1_B \mid x \in A\}$. The inclusion $f(A^+) \subset J$ is straightforward. On the other hand, for $x \in A$, one has $f(x) - \varepsilon(x)1_B = f(x - \varepsilon(x)1_A)$. As $x - \varepsilon(x)1_A$ belongs to A^+, we get the reverse inclusion.

So the cokernel of f is the projection map

$$\text{hcoker}(f)\colon B \to B/\langle f(A^+) \rangle.$$

We set

$$\text{Hcoker}(f) = B/\langle f(A^+) \rangle.$$

Let A and B be two objects in $\mathcal{H}^{\text{coco}}$. The coproduct object $A \amalg B$ of A and B in $\mathcal{H}^{\text{coco}}$ has the following explicit description (see Pareigis 2002 or Agore 2011a). The coproduct $A \amalg B$ is the module spanned over $\Bbbk$ as an algebra by the elements 1, t_a and t_b with $a \in A$ and $b \in B$ submitted to the relations

$$t_\lambda = \lambda, \quad t_{\lambda a + b} = \lambda t_a + t_b, \quad t_{aa'} = t_a t_{a'}, \quad t_{bb'} = t_b t_{b'}$$

with $\lambda \in \Bbbk$, with $a, a' \in A$, and $b, b' \in B$.

The coproduct $A \amalg B$ is endowed with a Hopf algebra structure given by:

$$\Delta(t_a) = \sum t_{a_1} \otimes t_{a_2}, \quad \varepsilon(t_a) = \varepsilon(a), \quad S(t_a) = t_{S(a)}.$$

with $a \in A$ or $a \in B$. The coproduct diagram is given by

$$A \xrightarrow{\iota_A} A \amalg B \xleftarrow{\iota_B} B$$

with $\iota_A(a) = t_a$ and $\iota_B(b) = t_b$ for $a \in A$, and $b \in B$. This construction satisfies the universal property of the coproduct. Indeed, for any two morphisms $f : A \to H$ and $g : B \to H$ of $\mathcal{H}^{\mathrm{coco}}$, the unique morphism $h : A \amalg B \to H$ such as one has $h \circ \iota_A = f$ and $h \circ \iota_B = g$ is defined on the generators of $A \amalg B$ by $h(t_a) = f(a)$ and $h(t_b) = g(b)$ with $a \in A$ and $b \in B$.

3 Newman correspondence, semi-direct product

In this section, we recall some constructions and results involving kernels and cokernels which we will use in the sequel.

3.1 The Newman correspondence

The following result is crucial for our next proofs.

Theorem 4 ([14]) – *For any cocommutative Hopf algebra over a field, there is a one-to-one correspondence between its sub-Hopf algebras and its left ideals which are also two-sided coideals.*

With a sub-Hopf algebra G of a Hopf algebra H, Newman associates the ideal $\tau(G) = HG^+$. He proves that τ is bijective with inverse map $\sigma(I) = \mathrm{Hker}(H \to H/I)$.

We state three lemmas directly induced by this result.

Lemma 1 – *Let H be a cocommutative Hopf algebra over a field. For any Hopf ideal I of H, there exists a sub-coalgebra G of H such that one has the isomorphism of cocommutative Hopf algebras*

$$H/I \cong H/HG^+H.$$

In other words, the projection map $H \to H/I$ is a cokernel in the category $\mathcal{H}^{\mathrm{coco}}$.

Proof. The two-sided ideal I is in particular a left ideal. So after Corollary 3.4[15], one has $I \cong HG^+$ for $G = \mathrm{Hker}(H \to H/I)$. As I is also a right ideal, we deduce $I = IH \cong HG^+H$. □

[14] Newman, 1975, "A correspondence between bi-ideals and sub-Hopf algebras in cocommutative Hopf algebras".

As a consequence, we have:

Lemma 2 – *Let $f\colon H \to H'$ be a surjective map of cocommutative Hopf algebras over a field. The map f is a cokernel in the category $\mathcal{H}^{\mathrm{coco}}$.*

Proof. Consider the linear ideal $I = \ker(f)$ of $f\colon H \to H'$. It is well know that I is a Hopf ideal. Moreover, one has $H/I \cong H'$. After Lemma 1, the map is a cokernel. □

Lemma 3 – *Let H be a cocommutative Hopf algebra over a field and G one of its normal sub-Hopf algebras. Then, we have $G \cong \mathrm{Hker}(H \to H/HG^+H)$.*

In other words, any inclusion of a normal sub-Hopf algebra into a Hopf algebra is a kernel.

Proof. As G is a normal sub-Hopf algebra, the usual trick $gh = h_1(S(h_2)gh_3)$ for $g \in G$ and $h \in H$ shows that $HG^+H = HG^+$. Moreover, one has the equality $HG^+ = H(\mathrm{Hker}(H \to H/HG^+H))^+$ by Corollary 3.4[16] and, as τ is injective by Corollary 2.5[17], on has the isomorphism $G \cong \mathrm{Hker}(H \to H/HG^+H)$. □

We also point out an important lemma which can be found as a part of the proof of Theorem 4.4[18].

Lemma 4 – *A monomorphism in $\mathcal{H}^{\mathrm{coco}}$ is injective.*

Proof. Let $m\colon X \to Y$ be a monomorphism in $\mathcal{H}^{\mathrm{coco}}$. The linear kernel $\ker(m)$ of m is a Hopf ideal of X by Theorem 4.17[19]. By Lemma 1, it exists a Hopf algebra inclusion $G \xrightarrow{\iota} X$ such as $X\iota(G^+)X = \ker(m)$. This implies

$$(m\circ\iota)(g) = \varepsilon_X(g)1_Y = (m\circ\eta_Y\circ\varepsilon_X\circ\iota)(g)$$

for $g \in G$. So we get $\iota = \eta_Y\circ\varepsilon_X\circ\iota$ and thus we have $G = \mathbb{k}$. Finally, one has $\ker(m) = X\mathbb{k}^+X = \{0\}$. □

3.2 The semi-direct product of Hopf algebras

We will need a notion of semi-direct product of Hopf algebras. The construction we will recall for our purpose is a very special case of the now classical semi-direct product introduced by Blattner, Cohen and Montgomery[20] to whom we refer for

[15] Newman, 1975, "A correspondence between bi-ideals and sub-Hopf algebras in cocommutative Hopf algebras".
[16] Ibid.
[17] Ibid.
[18] Ibid.
[19] Sweedler, 1969, *Hopf algebras*.
[20] Blattner, Cohen, and Montgomery, 1986, "Crossed products and inner actions of Hopf algebras".

proofs. Anyway, all the properties of the product we state here may also be checked through direct calculation.

Let Y and K be two Hopf algebras, we will say that K is a Y-*Hopf algebra* if Y acts on K. In other words, if there exists an action map $- \rightharpoonup -\colon Y \otimes K \to K$ which is a morphism of coalgebras and satisfies the following axioms:

$$y \rightharpoonup (ab) = (y_1 \rightharpoonup a)(y_2 \rightharpoonup b) \qquad 1_Y \rightharpoonup a = a$$
$$(yy') \rightharpoonup a = y \rightharpoonup (y' \rightharpoonup a) \qquad y \rightharpoonup 1_K = \varepsilon_Y(y)1_K$$

with $y, y' \in Y$ and $a, b \in K$. We denote by Y-$\mathcal{H}^{\text{coco}}$ the subcategory of $\mathcal{H}^{\text{coco}}$ whose objects are the Y-Hopf algebras and whose morphisms are those of $\mathcal{H}^{\text{coco}}$ compatible with the action.

Given an object K in Y-$\mathcal{H}^{\text{coco}}$, one may define the semi-direct product $K\#Y$ of K and Y. It is by definition the module $K \otimes Y$ endowed with the Hopf algebra structure defined by

$$(a \otimes y)(b \otimes y') = a(y_1 \rightharpoonup b) \otimes y_2 y' \qquad \Delta(a \otimes y) = (a_1 \otimes y_1) \otimes (a_2 \otimes y_2)$$
$$\eta(1) = 1 \otimes 1 \qquad \varepsilon(a \otimes y) = \varepsilon(a)\varepsilon(y)$$
$$\text{and}$$
$$S(x \otimes y) = (S(y_1) \rightharpoonup S(a)) \otimes S(y_2)$$

given for $a, b \in K$ and $y, y' \in Y$.

This product is nothing else than the product $K\#_\sigma Y$[21] where σ is the cocycle given by $\sigma = \eta_Y \circ (\varepsilon_K \otimes \varepsilon_K)\colon K \otimes K \to Y$. In the sequel, an element $a \otimes y \in K\#Y$ will be denoted by $a\#y$.

Consider the category $\mathcal{P}t_Y$ of pointed objects over an object Y of $\mathcal{H}^{\text{coco}}$. Its objects are the couples of maps $(p, s)\colon\ X \underset{p}{\overset{s}{\rightleftarrows}} Y$ of $\mathcal{H}^{\text{coco}}$ such that s is a section of p (*i.e.* $p \circ s = \mathrm{id}_Y$). The morphisms of $\mathcal{P}t_Y$ between two objects $X \underset{p}{\overset{s}{\rightleftarrows}} Y$ and $X' \underset{p'}{\overset{s'}{\rightleftarrows}} Y$ are the maps $f\colon X \to X'$ satisfying $p' \circ f = p$ and $f \circ s = s'$.

Lemma 5 – *Let Y be an object of $\mathcal{H}^{\text{coco}}$. The categories $\mathcal{P}t_Y$ and Y-$\mathcal{H}^{\text{coco}}$ are equivalent.*

Proof. We will only describe the correspondence between objects. Details may be found in Blattner, Cohen, and Montgomery (1986). With an action $- \rightharpoonup -\colon Y \otimes K \to K$ one associates the maps

$$p = \varepsilon_K \otimes \underset{Y}{\mathrm{id}}\colon K\#Y \to Y \quad \text{and} \quad s = \eta_K \otimes \underset{Y}{\mathrm{id}}\colon Y \to K\#Y.$$

[21] Blattner, Cohen, and Montgomery, 1986, "Crossed products and inner actions of Hopf algebras".

On the other hand, given the data $X \underset{p}{\overset{s}{\rightleftarrows}} Y$, one sets $K = \mathrm{Hker}(p)$. It is easy to check that $y \rightharpoonup k = s(y_1)ks(S_Y(y_2))$ defines an action of Y on K.

The equivalence is based on the isomorphism between $K\#Y$ and X given by the maps

$$\begin{aligned}\mathcal{F}\colon\ & X \to K\#Y \\ & x \mapsto x_1 s(Sp(x_2))\#p(x_3)\end{aligned} \quad \text{and} \quad \begin{aligned}\mathcal{G}\colon\ & K\#Y \to X \\ & k\#y \mapsto ky\end{aligned} \qquad \square$$

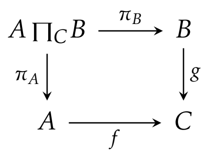
4 Pullbacks of cokernels, Regularity

In this section we survey the construction of pullbacks in the $\mathcal{H}^{\mathrm{coco}}$. A result of Gran, Kadjo and Vercruysse[22] gives the claim of Axiom (A_3). As a consequence, one deduces that the category $\mathcal{H}^{\mathrm{coco}}$ is regular and homological.

From the definitions of products and equalizers, one easily derives the definition of pullbacks. Let A, B, and C be cocommutative Hopf algebras and let $f\colon A \to C$ and $g\colon B \to C$ be morphisms of Hopf algebras. The pullback object of A and B over C is the module

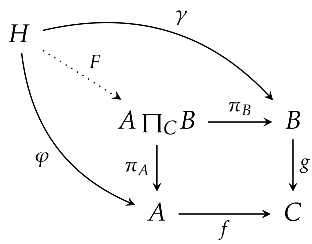
$$A \textstyle\prod_C B = \{a \otimes b \in A \otimes B \mid a_1 \otimes f(a_2) \otimes b = a \otimes g(b_1) \otimes b_2\}.$$

It is a sub-Hopf algebra of the Hopf algebra product $A \otimes B$. One has the commutative diagram



with $\pi_A(a \otimes b) = \varepsilon(b)a$ and $\pi_B(a \otimes b) = \varepsilon(a)b$.

The universal property of pullbacks is given in the following way. For any cocommutative Hopf algebra H and any two morphisms $\gamma\colon H \to B$ and $\varphi\colon H \to A$, there exists a unique morphism $F\colon H \to A \prod_C B$ such that the diagram



is commutative. The morphism F is defined by $F(d) = \varphi(d_1) \otimes \gamma(d_2)$ for any $d \in H$.

[22] Gran, Kadjo, and Vercruysse, 2016, "A torsion theory in the category of cocommutative Hopf algebras".

The following theorem shows that the category of cocommutative Hopf algebras satisfies Axiom (A_3).

Theorem 5 (Theorem 3.7[23]) – *In the category $\mathcal{H}^{\mathrm{coco}}$, the pullback of a cokernel is a cokernel when the ground field has characteristic zero.*

Remark 1 – We do not know if the similar statement for a field of positive characteristic is still true. In fact, the proof of Gran, Kadjo, and Vercruysse (2016) uses in an essential way the Cartier-Milnor-Moore theorem which requires the condition on the characteristic of the ground field.

5 Coproducts and split epimorphisms

The following proposition proves (A_2) for the category $\mathcal{H}^{\mathrm{coco}}$.

Proposition 1 – *Let p be a morphism in $\mathcal{H}^{\mathrm{coco}}$, let s be one of its sections (i.e. $p \circ s = \mathrm{id}_Y$) in $\mathcal{H}^{\mathrm{coco}}$ and $\kappa\colon K \to X$ be the kernel of p in $\mathcal{H}^{\mathrm{coco}}$:*

$$K \xrightarrow{\ \kappa\ } X \underset{p}{\overset{s}{\rightleftarrows}} Y$$

The arrow $<\kappa, s>\colon K \coprod Y \to X$ is a cokernel in $\mathcal{H}^{\mathrm{coco}}$.

Proof. First remark that if, an element $x \in X$, is also an element of $K = \mathrm{Hker}(p)$, then, for any $y \in Y$, one also has $s(y_1)xs(S_Y(y_2)) \in K$. A straightforward computation proves that the formula $y \rightharpoonup x = s(y_1)xs(S_Y(y_2))$ defines an action of Y over K.

Now consider the two linear maps $f, g\colon K \otimes Y \to K \coprod Y$ defined by $f(x \otimes y) = t_{y_1 \rightharpoonup x} t_{y_2}$ and $g(x \otimes y) = t_y t_x$ with $y \in Y$ and $x \in K$. We denote by L the linear image $\mathrm{im}(f-g)$.

Note that both maps preserve the coalgebra structure. So, we have

$$\Delta \circ (f-g) = \Big((f-g) \otimes f + g \otimes (f-g)\Big) \circ \Delta.$$

This later relation implies that L is a two-sided coideal of $K \coprod Y$. Let us set $U = (K \coprod Y) L (K \coprod Y)$ which is both a two-sided ideal a two-sided coideal.

Moreover, for any $x \in K$ and $y \in Y$, one computes

$$\begin{aligned} S(t_{y_1 \rightharpoonup x} t_{y_2} - t_y t_x) &= t_{S(y_1)} t_{y_2 \rightharpoonup S(x)} - t_{S(x)} t_{S(y)} \\ &= t_{S(y_1)} t_{y_2 \rightharpoonup S(x)} t_{y_3} t_{S(y_4)} - t_{S(y_1)} t_{y_2} t_{S(x)} t_{S(y_3)} \\ &= t_{S(y_1)} \big((f-g)(S(x) \otimes y_2)\big) t_{S(y_3)} \end{aligned}$$

[23] Gran, Kadjo, and Vercruysse, 2016, "A torsion theory in the category of cocommutative Hopf algebras".

Notice that, for the first equality, we used cocomutativity and the relation $S(y\rightharpoonup x) = y\rightharpoonup S(x)$ which is a consequence of $S^2 = \mathrm{id}$. Here we remember that the antipode of a cocommutative Hopf algebra is involutive (cf. Proposition 4.0.1[24]). Our computation proves $S(U) \subset U$ and consequently that U is a Hopf ideal.

One clearly has $(K \coprod Y)/U \simeq K\#Y$ which is isomorphic to X after the proof of Lemma 5. Moreover, after Lemma 1, the map $(K \coprod Y) \to (K \coprod Y)/U$ is a cokernel.□

Corollary 1 – *If the ground field has characteristic zero, the category $\mathcal{H}^{\mathrm{coco}}$ is finitely cocomplete homological.*

Proof. The category satisfies the axioms (A_1), (A_2) and (A_3)[25]. □

Corollary 2 – *If the ground field has characteristic zero, the category $\mathcal{H}^{\mathrm{coco}}$ is regular.*

Proof. The result of Proposition 5.1 combined with the existence of finite limits and coequilazers fulfills the axioms defining regular categories. □

6 Images of kernels

It remains to check Axiom (A_4). As the category $\mathcal{H}^{\mathrm{coco}}$ is regular, the image of a morphism is canonically defined as the coequalizer of its kernel pair[26].

Let $f\colon X \to Y$ be a morphism in $\mathcal{H}^{\mathrm{coco}}$. The coequalizer object of the kernel pair of f is $\mathrm{Hcoeq}(\pi_1,\pi_2)$ where π_1 and π_2 are the canonical maps of the pullback diagram

$$\begin{array}{ccc} X \prod_Y X & \xrightarrow{\pi_1} & X \\ {\scriptstyle \pi_2}\downarrow & & \downarrow{\scriptstyle f} \\ X & \xrightarrow[f]{} & Y \end{array}$$

We have $X \prod_Y X = \{x \otimes x' \in X \otimes X \mid x_1 \otimes f(x_2) \otimes x' = x \otimes f(x'_1) \otimes x'_2\}$ and also $\pi_1(x\otimes x') = \varepsilon(x')x$ and $\pi_2(x\otimes x') = \varepsilon(x)x'$. By section 2.2, one has $\mathrm{Hcoeq}(\pi_1,\pi_2) = X/XJX$ where J is the space $\{\varepsilon(x')x - \varepsilon(x)x' \mid x\otimes x' \in X\prod_Y X\}$.

The image object $\mathrm{HIm}(f)$ of $f\colon X \to Y$ in $\mathcal{H}^{\mathrm{coco}}$ is the quotient X/XJX. In fact one has the following lemma which slightly simplify the construction of $\mathrm{HIm}(f)$.

[24] Sweedler, 1969, *Hopf algebras*.

[25] Hartl and Loiseau, 2011, "A characterization of finite cocomplete homological and of semi-abelian categories".

[26] Barr, 1971, "Exact categories".

Lemma 6 – *Let $f : X \to Y$ be a morphism in $\mathcal{H}^{\mathrm{coco}}$. One has* $\mathrm{HIm}(f) = X/J$ *where* $J = \{\varepsilon(x')x - \varepsilon(x)x' \mid x \otimes x' \in X \prod_Y X\}$.

Proof. We shall prove $XJX = J$. First we detail the proof of $XJ = J$.

Note that X acts on $X \prod_Y X$ by $a \cdot (x \otimes x') = a_1 x \otimes a_2 x'$ for $a \in X$ and $x \otimes x' \in X \prod_Y X$. Let us show that $a_1 x \otimes a_2 x'$ is indeed also an element of $X \prod_Y X$. As $x \otimes x'$ is in $X \prod_Y X$, one has

$$x_1 \otimes f(x_2) \otimes x' = x \otimes f(x'_1) \otimes x'_2.$$

By multiplying this relation by $a_1 \otimes f(a_2) \otimes a_3$ on the left side, one gets

$$a_1 x_1 \otimes f(a_2 x_2) \otimes a_3 x' = a_1 x \otimes f(a_2 x'_1) \otimes a_3 x'_2$$

which implies $a_1 x \otimes a_2 x' \in X \prod_Y X$.

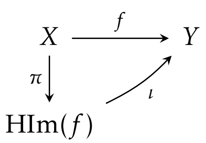
Let now be $a \in X$ and $u = \varepsilon(x')x - \varepsilon(x)x' \in J$. By the property of the counit, we have

$$au = \varepsilon(x')ax - \varepsilon(x)ax' = \varepsilon(a_2 x')a_1 x - \varepsilon(a_1 x)a_2 x'$$

which has the form of the element of J associated with $a \cdot (x \otimes x') \in X \prod_Y X$. We proved $XJ \subset J$. The reverse inclusion being trivially true, one has $J = XJ$

Similarly, one shows $J = JX$. We finally have $J = XJX$. □

After Borceux and Bourn[27], the morphism f factorizes as a product of the regular epimorphism $\pi = \mathrm{hcoeq}(\pi_1, \pi_2)$ and a monomorphism ι in $\mathcal{H}^{\mathrm{coco}}$. One has the diagram



the morphism ι being induced by $f : X \to Y$. The factorization is unique up to an isomorphism.

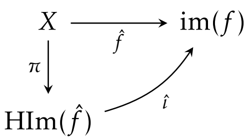
On the other hand, one can consider the linear image $\mathrm{im}(f) = \{f(x) \mid x \in X\}$ of f. As f is a morphism of cocommutative Hopf algebras, $\mathrm{im}(f)$ is a sub-cocommutative Hopf algebra of Y.

In our case, in fact, the two notions of image coincide.

Lemma 7 – *If the ground field has characteristic zero, in the category $\mathcal{H}^{\mathrm{coco}}$, for any morphism f, one has* $\mathrm{HIm}(f) \cong \mathrm{im}(f)$.

[27] Borceux and Bourn, 2004, *Mal'cev, protomodular, homological and semi-abelian categories*.

Proof. Consider the morphism $f\colon X \to Y$. We denote by $\hat{f}\colon X \to \mathrm{im}(f)$ the morphism obtained considering the linear image of f. It is still a morphism of $\mathcal{H}^{\mathrm{coco}}$, so one has the factorization diagram



□

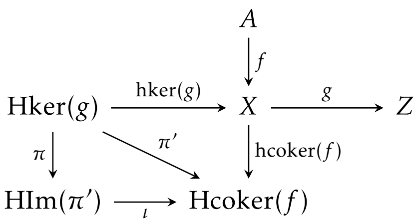
where $\hat{\iota}$ is a monomorphism. By Lemma 4, the morphism $\hat{\iota}$ is injective. On the other hand, $\hat{f}$ is surjective and consequently $\hat{\iota}$ is. This implies the isomorphism $\mathrm{HIm}(\hat{f}) \cong \mathrm{im}(f)$.

Finally, using the construction of the categorical image, one checks the equality $\mathrm{HIm}(f) = \mathrm{HIm}(\hat{f})$.

We can now prove that Axiom (A_4) is fulfilled.

Proposition 2 – *If the ground field has characteristic zero, in $\mathcal{H}^{\mathrm{coco}}$ the image of a kernel is a kernel.*

Proof. Consider the following commutative diagram in $\mathcal{H}^{\mathrm{coco}}$:



As $\mathrm{Hker}(g)$ is a normal sub-Hopf algebra of X, its linear image, through the projection $X \to \mathrm{Hcoker}(f)$ is a normal sub-Hopf algebra of $\mathrm{Hcoker}(f)$. The later linear image is nothing else than $\mathrm{HIm}(\pi')$. After Lemma 3, it is a kernel object under our assumptions. □

At this point, we proved that all the axioms (A_1), (A_2), (A_3) and (A_4) are fulfilled for $\mathcal{H}^{\mathrm{coco}}$ and so one recovers Theorem 1.

7 Abelian core, categorical semi-abelian product

This section is widely inspired by Borceux[28]. In a first time, we determine the abelian core of $\mathcal{H}^{\text{coco}}$. In a second time, we prove that the categorical semi-direct product in $\mathcal{H}^{\text{coco}}$ is nothing else than the semi-direct product defined in section 3. In all this section, we assume that the ground field has characteristic zero.

Lemma 8 – *Let A be a sub-algebra of a cocommutative Hopf algebra H. The sub-Hopf algebra is normal if and only if the inclusion $A \to H$ is normal in $\mathcal{H}^{\text{coco}}$.*

Proof. Consider a normal map $A \to H$ in $\mathcal{H}^{\text{coco}}$. The sub-object A is a sub-Hopf algebra of H such as it exists a morphism $\varphi\colon H \to H'$ and $A = \text{Hker}(\varphi)$. We already noticed that kernel objects are normal sub-Hopf algebras. The converse assertion is Lemma 3. □

The following proposition is Theorem 2.

Proposition 3 – *The full sub-category of abelian objects of $\mathcal{H}^{\text{coco}}$ is $\mathcal{H}^{\text{co-coco}}$.*

Proof. We use the characterization of Theorem 6.9[29] which states that an object C in a semi-abelian category is abelian if and only if its diagonal $C \to C\otimes C$ is normal. In our case, if C is an object of $\mathcal{H}^{\text{coco}}$, the diagonal map is nothing else than the comutiplication Δ_C. So, after Lemma 7.1, it suffices to prove that C is abelian if and only if $\text{im}(\Delta_C)$ is a normal sub-Hopf algebra of $C\otimes C$.

If C is commutative, so is $C\otimes C$ and as sub-Hopf algebras of a commutative algebra are normal, it follows that $\text{im}(\Delta_C)$ is.

On the other hand, suppose that $\text{im}(\Delta_C)$ is a normal sub-Hopf algebra of $C\otimes C$. For any two elements $a, c \in C$ it exists $d \in C$ such that we have

$$\Delta(d) = \big(c_1\otimes 1\big)\big(a_1\otimes a_2\big)S\big(c_2\otimes 1\big) = c_1 a_1 S(c_2)\otimes a_2.$$

By applying ε_C to the first and second tensor factors of the above identity we respectively get $d = \varepsilon_C(c)a$ and $d = c_1 a S(c_2)$. Thus, we have

$$d = \varepsilon_C(c)a = c_1 a S(c_2).$$

As the identity is true for any $a\otimes c \in C$, we may apply it to the first tensor factor of $a\otimes c_1\otimes c_2$ and get

$$\varepsilon_C(c_1)a\otimes c_2 = c_1 a S(c_2)\otimes c_3 \Longrightarrow \varepsilon_C(c_1)ac_2 = c_1 a S(c_2)c_3 \Longrightarrow ac = ca.$$

Thus, C is commutative. □

[28] Borceux, 2004, "A survey of semi-abelian categories".

We retrieve the known result: the category $\mathcal{H}^{\text{co-coco}}$ is abelian.

To end the article, we prove that the semi-direct product defined in Section 3 is the semi-direct product in $\mathcal{H}^{\text{coco}}$ in the categorical sense defined in Borceux (2004). We will follow the latter reference.

For any object Y in $\mathcal{H}^{\text{coco}}$, we have the pair of adjoint functors:

$$\begin{array}{lll} \text{Hker}\colon \quad \mathcal{P}t_Y \to \mathcal{H}^{\text{coco}} & & \mathcal{H}^{\text{coco}} \to \mathcal{P}t_Y \\ X \underset{p}{\overset{s}{\rightleftarrows}} Y \;\mapsto \text{Hker}(p) & \text{and} & K \mapsto \; K \coprod Y \underset{(\varepsilon_K, \text{id}_Y)}{\overset{\iota_Y}{\rightleftarrows}} Y \end{array}$$

where the functor Hker is monadic. Then, one can consider the monad $\mathbb{T}_Y$ associated to Hker. By definition, the semi-direct product of an algebra (K,ξ) for the monad $\mathbb{T}_Y$ and the object Y is the domain of the pointed object (p,s): $X \underset{p}{\overset{s}{\rightleftarrows}} Y$ corresponding to (K,ξ) via the equivalence $\mathcal{P}t_Y \cong (\mathcal{H}^{\text{coco}})^{\mathbb{T}_Y}$.

Theorem 6 – *Let Y be an object in $\mathcal{H}^{\text{coco}}$. Let (K,ξ) be an algebra for the monad $\mathbb{T}_Y$. The semi-direct product of an algebra (K,ξ) and Y is $K\#Y$.*

Proof. The proof given for the category of groups[30] is still valid in our case if one replaces Proposition 5.7[31] by Lemma 5. □

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[30] Ibid., Section 5.

[31] Ibid.

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