



On the functional equation $\alpha \mathbf{u} + \mathcal{C} \star (\chi \mathbf{u}) = \mathbf{f}$

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Abstract

We study in this paper the functional equation

$$\alpha \mathbf{u}(t) + \mathcal{C} \star (\chi \mathbf{u})(t) = \mathbf{f}(t)$$

where $\alpha \in \mathbb{C}^{d \times d}$, $\mathbf{u}, \mathbf{f}: \mathbb{R} \rightarrow \mathbb{C}^d$, \mathbf{u} being unknown. The term $\mathcal{C} \star (\chi \mathbf{u})(t)$ denotes the discrete convolution of an almost zero matricial mapping \mathcal{C} with discrete support together with the product of \mathbf{u} and the characteristic function χ of a fixed segment.

This equation combines some aspects of recurrence equations and/or delayed functional equations, so that we may construct a matricial based framework to solve it. We investigate existence, unicity and determination of the solution to this equation. In order to do this, we use some new results about linear independence of monomial words in matrix algebras.

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msc: 39A06, 39A70, 39A12, 68R15.

1 Introduction

Let $[t_0, t_f]$ be some interval of time and χ its characteristic function, and $\varepsilon > 0$ a fixed time delay. Let $(c_k)_{k \in \mathbb{Z}}$ be an almost zero sequence of matrices in $\mathbb{C}^{d \times d}$ where d denotes the “physical” dimension.

We define next $\mathcal{C}: \mathbb{R} \rightarrow \mathbb{C}^{d \times d}$ by

$$\mathcal{C}(t) = \begin{cases} c_{-t/\varepsilon} & \text{if } t \in \varepsilon \mathbb{Z} \\ 0 & \text{otherwise} \end{cases}.$$

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We denote by I_d the identity matrix of size d and we fix throughout a matrix $\alpha \in \mathbb{C}^{d \times d}$. We have the discrete convolution product over $\varepsilon \mathbb{Z}$:

$$(\mathbf{u} \star \mathbf{v})(t) = \sum_{s \in \varepsilon \mathbb{Z}} \mathbf{u}(s) \mathbf{v}(t-s), \quad t \in \mathbb{R}$$

where \mathbf{u} and \mathbf{v} are two mappings defined in \mathbb{R} with matrix values of format $(d \times d, d \times 1)$ or $(1 \times d, d \times 1)$.

This paper is devoted to the study of the functional equation

$$\alpha \mathbf{u} + \mathcal{C} \star (\chi \mathbf{u}) = \mathbf{f} \tag{1}$$

which may be rewritten equivalently as

$$\alpha \mathbf{u}(t) + \sum_{k \in \mathbb{Z}} c_k \mathbf{u}(t+k\varepsilon) \chi(t+k\varepsilon) = \mathbf{f}(t), \tag{2}$$

where $\mathbf{u}, \mathbf{f}: \mathbb{R} \rightarrow \mathbb{C}^d$, \mathbf{u} being unknown. The sum in the left-hand side of (2) will be denoted throughout the paper by $\square \mathbf{u} = \mathcal{C} \star (\chi \mathbf{u})$, where the operator \square may occur in the context of discrete calculus of variations². Equation (1) may be thought as a discrete version of the differential equation $\alpha \mathbf{u}(t) + \mathbf{u}'(t) = \mathbf{f}(t)$ where \mathbf{u}' has been replaced by \square . An analogous functional equation appears in the field of time-scale calculus and has led to many works, see for instance Bohner and Peterson (2001) and the literature therein. However, we do not require here that the operator \square behaves as a derivative.

We define N as the largest integer such that $c_N \neq 0$ or $c_{-N} \neq 0$. For example, when $N = 1$, equation (2) rewrites as

$$\mathbf{f}(t) - \alpha \mathbf{u}(t) = \begin{cases} 0 & \text{if } t < t_0 - \varepsilon \\ c_1 \mathbf{u}(t + \varepsilon) & \text{if } t_0 - \varepsilon \leq t < t_0 \\ c_1 \mathbf{u}(t + \varepsilon) + c_0 \mathbf{u}(t) & \text{if } t_0 \leq t < t_0 + \varepsilon \\ c_1 \mathbf{u}(t + \varepsilon) + c_0 \mathbf{u}(t) + c_{-1} \mathbf{u}(t - \varepsilon) & \text{if } t_0 + \varepsilon \leq t \leq t_f - \varepsilon \\ c_0 \mathbf{u}(t) + c_{-1} \mathbf{u}(t - \varepsilon) & \text{if } t_f - \varepsilon < t \leq t_f \\ c_{-1} \mathbf{u}(t - \varepsilon) & \text{if } t_f < t \leq t_f + \varepsilon \\ 0 & \text{if } t_f + \varepsilon < t \end{cases} \tag{3}$$

In general, the equations (2) may be thought as a mixture between recurrence equations and delayed functional equations.

²Ryckelynck and Smoch, 2013, "Discrete Calculus of Variations for quadratic Lagrangians"; Ryckelynck and Smoch, 2014, "Quadratic choreographies".

2. Analytical properties of the functions $\square \mathbf{u}$

We focus in this paper on the following two problems. What are the analytical properties of functions of the shape $\square \mathbf{u}$ and is the functional equation (1) well-posed?

The paper is organized as follows. In Section 2 we provide some analytical properties of the functions $\square \mathbf{u}$. Section 3 presents some technical lemmas for matricial linear recurrences. Section 4 is concerned with the non-vanishing property for entries and determinants of matricial polynomial functions. It ensures that the linear compatibility equations extracted when solving (1) are Cramer for the specific case $N = 1$. We prove in Sections 5 and 6 that (1) is well-posed if α is invertible, with proper accuracies, according to the respective cases $N = 1$ and arbitrary N . The case $\alpha = 0$ is also investigated, and stands for the characterization of the range of the operator \square . At last, Section 7 gives some concluding remarks and perspectives.

2 Analytical properties of the functions $\square \mathbf{u}$

In order to comprehend equation (1) and present some robust methods for solving it, we provide first some features of the operator \square .

Theorem 1 – *Let $v \in \mathbb{N} \cup \{\infty\}$.*

1. *The operator \square maps the space of the functions $\mathbf{u}: \mathbb{R} \rightarrow \mathbb{C}^d$ of piecewise- \mathcal{C}^v regularity, to the space of compactly supported functions $\mathbf{v}: \mathbb{R} \rightarrow \mathbb{C}^d$ of piecewise- \mathcal{C}^v regularity, such that $\text{Supp}(\mathbf{v}) \subset [t_0 - N\varepsilon, t_f + N\varepsilon]$.*
2. *If $\mathbf{u}: \mathbb{R} \rightarrow \mathbb{C}^d$ has p points of discontinuity on \mathbb{R} , then $\square \mathbf{u}$ has at most $(4N + 2) + (2N + 1)p$ points of discontinuity on \mathbb{R} .*
3. *If c_N or c_{-N} is invertible, then the kernel of \square consists of the functions which vanish on $[t_0, t_f]$.*
4. *If c_N or c_{-N} is invertible, then for all $\mathbf{u}: \mathbb{R} \rightarrow \mathbb{C}^d$, $\square \mathbf{u}$ is measurable, (respectively integrable, of piecewise- \mathcal{C}^v regularity) on \mathbb{R} if and only if \mathbf{u} is measurable (respectively integrable, of piecewise- \mathcal{C}^v regularity) on $[t_0, t_f]$.*

Proof. The main ingredient of the proof is the explicit formula for $\square \mathbf{u}(t)$ provided in Ryckelynck and Smoch (2013):

$$\left\{ \begin{array}{ll}
 0 & \text{if } t < t_0 - N\varepsilon \\
 c_N \mathbf{u}(t + N\varepsilon) & \text{if } t_0 - N\varepsilon \leq t < t_0 - N\varepsilon + \varepsilon \\
 c_{N-1} \mathbf{u}(t + N\varepsilon - \varepsilon) + c_N \mathbf{u}(t + N\varepsilon) & \text{if } t_0 - N\varepsilon + \varepsilon \leq t < t_0 - N\varepsilon + 2\varepsilon \\
 \vdots & \vdots \\
 c_{N-p} \mathbf{u}(t + N\varepsilon - p\varepsilon) + \dots + c_N \mathbf{u}(t + N\varepsilon) & \text{if } p \in \{0, \dots, 2N-1\} \text{ and} \\
 & t_0 + (p-N)\varepsilon \leq t < t_0 + (p+1-N)\varepsilon \\
 \vdots & \vdots \\
 c_{-N} \mathbf{u}(t - N\varepsilon) + \dots + c_N \mathbf{u}(t + N\varepsilon) & \text{if } t_0 + N\varepsilon \leq t \leq t_f - N\varepsilon \\
 \vdots & \vdots \\
 c_{-N} \mathbf{u}(t - N\varepsilon) + \dots + c_{p-N} \mathbf{u}(t - N\varepsilon + p\varepsilon) & \text{if } p \in \{2N-1, \dots, 0\} \text{ and} \\
 & t_f + (N-p-1)\varepsilon < t \leq t_f + (N-p)\varepsilon \\
 \vdots & \vdots \\
 c_{-N} \mathbf{u}(t - N\varepsilon) + c_{1-N} \mathbf{u}(t - N\varepsilon + \varepsilon) & \text{if } t_f + N\varepsilon - 2\varepsilon < t \leq t_f + N\varepsilon - \varepsilon \\
 c_{-N} \mathbf{u}(t - N\varepsilon) & \text{if } t_f + N\varepsilon - \varepsilon < t \leq t_f + N\varepsilon \\
 0 & \text{if } t_f + N\varepsilon < t
 \end{array} \right. \quad (4)$$

Assertion 1. is an obvious consequence of formula (4). Next, let $\mathbf{u}: \mathbb{R} \rightarrow \mathbb{C}^d$, then formula (4) shows that the $(4N+2)$ points $t_0 + k\varepsilon$ and $t_f + k\varepsilon$, $|k| \leq N$ may be points of discontinuity of $\square \mathbf{u}$. Moreover, if t_δ is any point of discontinuity of \mathbf{u} , the values $t_\delta + k\varepsilon$, $|k| \leq N$, give rise in general to points of discontinuity of $\square \mathbf{u}$. Lastly, since there does not exist any other point of discontinuity of $\square \mathbf{u}$, assertion 2. is proved. Let us prove assertion 3., i.e., the conditional injectivity of \square . We first notice due to (4) that if \mathbf{u} vanishes in $[t_0, t_f]$, then all the terms occurring in (4) are equal to 0 so that $\square \mathbf{u}$ is zero everywhere. Conversely, suppose that $\det(c_N) \neq 0$. If $\mathbf{u} \in \ker(\square)$, second row of formula (4) shows that $\mathbf{u} = 0$ in $[t_0, t_0 + \varepsilon[$. Next, by using this result, the third row of (4) shows that $\mathbf{u} = 0$ in $[t_0 + \varepsilon, t_0 + 2\varepsilon[$ and so on. Then we may prove easily that $\mathbf{u} = 0$ in $[t_0, t_0 + \varepsilon[\cup [t_0 + \varepsilon, t_0 + 2\varepsilon[\cup \dots \cup]t_f - \varepsilon, t_f]$. We may proceed similarly when $\det(c_{-N}) \neq 0$ by starting from the last but one row of (4). As a consequence, if $\mathbf{u}, \mathbf{v}: \mathbb{R} \rightarrow \mathbb{C}^d$, we get

$$\square \mathbf{u} = \square \mathbf{v} \quad \Leftrightarrow \quad \mathbf{u} = \mathbf{v} \text{ in } [t_0, t_f]$$

which proves the third assertion. We proceed in the very same way to prove that measurability or integrability of \mathbf{u} on $[t_0, t_f]$ is equivalent to the same property for $\square \mathbf{u}$ on \mathbb{R} . \square

3 Preliminary lemmas for matricial linear recurrences

As we shall see in Sections 5 and 6, solving (1) queries to deal with matricial nonstationary recurrences of the shape

$$\mathbf{w}_{n+1} = \mathbf{M}_n \mathbf{w}_n + \mathbf{g}_n, \quad n \in \mathbb{N}, \quad (5)$$

where \mathbf{M}_n is an $s \times s$ matrix and $\mathbf{w}_n, \mathbf{g}_n$ are vector sequences in \mathbb{C}^s , s being a fixed integer. Since non-commutative products occur in the explicit formula, let us denote by

$$\prod_{k=0}^m \mathbf{M}_k = \mathbf{M}_m \mathbf{M}_{m-1} \dots \mathbf{M}_0$$

this left-side product.

Lemma 1 – *The solution to the recurrence (5) is given by*

$$\mathbf{w}_n = \left(\prod_{k=0}^{n-1} \mathbf{M}_k \right) \mathbf{w}_0 + \sum_{k=0}^{n-1} \left(\prod_{\ell=k+1}^{n-1} \mathbf{M}_\ell \right) \mathbf{g}_k.$$

Proof. The cases $n = 0$ and $n = 1$ are true and use the conventions of the empty sum and empty (left-)product. The proof for $n > 1$ is straightforward by using induction. \square

The particular case when \mathbf{M} is independent on n is of some importance and gives rise to the solution to stationary matricial recurrence of the shape (5) as follows:

$$\mathbf{w}_n = \mathbf{M}^n \mathbf{w}_0 + \sum_{k=0}^{n-1} \mathbf{M}^{n-k-1} \mathbf{g}_k. \quad (6)$$

In the following two lemmas, we deal with sequences of non-commutative matricial polynomials.

Lemma 2 – *Let $d \geq 2$, $\beta, \gamma \in \mathcal{M}_{d,d}(\mathbb{C})$ and (δ'_n) and (δ''_n) be the sequences of matrices defined by the recurrences*

$$\delta'_{n+1} = \beta \delta'_n + \gamma \delta'_{n-1}, \quad \delta''_{n+1} = \delta''_n \beta + \delta''_{n-1} \gamma, \quad n \geq 1 \quad (7)$$

with $\delta'_0 = \delta''_0 = 0$ and $\delta'_1 = \delta''_1 = I_d$. Then, for all $n \geq 2$, we have

$$\delta'_n = \delta''_n = \sum \beta^{m_1} \gamma^{m_2} \beta^{m_3} \gamma^{m_4} \dots \quad (8)$$

where the sum is extended over all the multiplets $(m_1, m_2, \dots, m_n) \in \mathbb{N}^n$ with

$$m_j \geq 1 \quad \text{if} \quad 2 \leq j \leq n-1 \quad \text{and} \quad \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} (m_{2i-1} + 2m_{2i}) = n-1.$$

The number of monomials in the formula (8) is the Fibonacci number \mathcal{F}_n .

Proof. First, let us mention that formula (8) is true for $n = 0$ and $n = 1$ by using the empty sum convention and the initial conditions. Let us prove now the formula (8) for the sequence (δ'_n) . Suppose that the result (8) holds for n and $n-1$ and that δ'_{n-1} and δ'_n are of lengths \mathcal{F}_{n-1} and \mathcal{F}_n respectively. Then, due to (8) and (7) we see that δ'_{n+1} is the sum of the words excerpted from the sum δ'_n multiplied on the left by β and those from the sum δ'_{n-1} multiplied on the left by γ . So we have proved by induction that δ'_n has the shape

$$\delta'_n = \sum e_{(n, m_1, m_2, \dots)} \beta^{m_1} \gamma^{m_2} \beta^{m_3} \gamma^{m_4} \dots$$

where the coefficients $e_{(n, m_1, m_2, \dots)}$ are convenient positive integers independent of β and γ . Let us note that if $\beta = \gamma = I_d$, then obviously, $\delta'_n = \mathcal{F}_n \times I_d$ so, the number of monomial words in the sum (8) is \mathcal{F}_n . Let us prove now that all the coefficients $e_{(n, m_1, m_2, \dots)}$ are in fact equal to 1. Indeed, we see that all the \mathcal{F}_{n-1} words coming from δ'_{n-1} are distinct from the \mathcal{F}_n words arising from δ'_n since the first ones begin with β while the second ones start with γ . All the words δ'_{n+1} occurring in (8) being distinct, their cardinality is $\mathcal{F}_{n-1} + \mathcal{F}_n = \mathcal{F}_{n+1}$. So far, we have proved the result (8) for δ'_n and the proof for δ''_n is the same. \square

From now on, we set $\delta_n = \delta'_n = \delta''_n, \forall n \in \mathbb{N}$.

Lemma 3 – Let $d \geq 2$, $\beta, \gamma \in \mathcal{M}_{d,d}(\mathbb{C})$, $(\mathbf{g}_n)_n \in (\mathbb{C}^d)^{\mathbb{N}}$, and let $(\mathbf{w}_n)_n \in (\mathbb{C}^d) \in \mathbb{N}$ the vector sequence satisfying for all $n \geq 1$ the second-order matricial recurrence equation:

$$\mathbf{w}_{n+1} = \beta \mathbf{w}_n + \gamma \mathbf{w}_{n-1} + \mathbf{g}_n. \quad (9)$$

Then for all $n \in \mathbb{N}^*$ we have

$$\mathbf{w}_n = \delta_n \mathbf{w}_1 + \delta_{n-1} \gamma \mathbf{w}_0 + \sum_{k=1}^{n-1} \delta_{n-k} \mathbf{g}_k. \quad (10)$$

Proof. Let us define

$$\mathbf{M} = \begin{pmatrix} \beta & \gamma \\ I_d & 0 \end{pmatrix}.$$

3. Preliminary lemmas for matricial linear recurrences

We may easily prove by induction that

$$\mathbf{M}^n = \begin{pmatrix} \delta_{n+1} & \delta_n \gamma' \\ \delta_n & \delta_{n-1} \gamma' \end{pmatrix},$$

for all $n \in \mathbb{N}^*$, by using the recurrence (7) for the sequence (δ_n) . Next, (9) may be rewritten as

$$\begin{pmatrix} \mathbf{w}_{n+1} \\ \mathbf{w}_n \end{pmatrix} = \mathbf{M} \begin{pmatrix} \mathbf{w}_n \\ \mathbf{w}_{n-1} \end{pmatrix} + \begin{pmatrix} \mathbf{g}_n \\ 0 \end{pmatrix}.$$

By setting

$$\mathbf{w}'_n = \begin{pmatrix} \mathbf{w}_n \\ \mathbf{w}_{n-1} \end{pmatrix}, \quad \mathbf{g}'_n = \begin{pmatrix} \mathbf{g}_n \\ 0 \end{pmatrix}$$

and by using a slightly adapted version of formula (6), we get

$$\mathbf{w}_n = \mathbf{M}^{n-1} \mathbf{w}_1 + \sum_{k=1}^{n-1} \mathbf{M}^{n-1-k} \mathbf{g}_k.$$

Now, the formula (10) is obtained by reading the first row of the previous formula. \square

Remark 1 – Blanchard and Watson³ have introduced, in mathematical finance, the model of rational expectations (RE) bubbles, based on the dynamical equation $X_{t+1} = a_t X_t + b_t$, where $(a_t), (b_t)$ are two given discrete stochastic processes, which may depend on the process X_t . The bidimensional generalization of this model⁴ is based on a dynamical system of the shape $X_{t+1} = a_t X_t + b_t$, where $(a_t), (b_t)$ are two given matricial discrete stochastic processes. In the study of the stability of this dynamics, the role of the ordered products $a_t \cdot a_{t-1} \cdot a_{t-2} \dots a_0$ is crucial.

Remark 2 – The study of Lyapunov stability of non-autonomous dynamical systems has given rise to a lot of papers. Here is a situation where stochastic processes play a central role. Let us consider a linear discrete-time evolution of an N -dimensional system described by d complex degrees of freedom. The state of the system at time t is given by a d -dimensional vector \mathbf{x}_t and the dynamics of the system is simply $\mathbf{x}_{t+1} = Y_{t+1} \mathbf{x}_t$, where the evolution operator Y_{t+1} is represented by a $d \times d$ matrix. In that situation, the total evolution from the initial state $\mathbf{x}_t = Z(t) \mathbf{x}_0$ is effectively driven by the product matrix $Z(t) = Y_t Y_{t-1} \dots Y_1$. When the matrices Y_j are i.i.d. complex non-Hermitian random matrices, interesting problems related to stability

³Blanchard and Watson, 1982, *Bubbles, Rational Expectations and Speculative Markets*, in: P. Wachtel, ed., *Crisis in Economic and Financial Structure: Bubbles, Bursts, and Shocks*.

⁴Sornette and Malevergne, 2001, "From Rational Bubbles to Crashes".

of the system and distribution of eigenvalues of $Y_t Y_{t-1} \cdots Y_1$ have been studied by G. Akemann, Z. Burda, M. Kieburg⁵, who give also a large bibliography. For previous works on products of random matrices, we may also cite the book of Bougerol and Lacroix⁶.

4 Linear independence of words in matrix algebras

We shall obtain a result which ensures that the foregoing linear compatibility equations which arise when solving the functional equation (1) are Cramer for almost all choices of matrices $\alpha, c_{-1}, c_0, c_1 \in \mathbb{C}^{d \times d}$ if $N = 1$.

Let us call a *matricial polynomial function* of two matricial indeterminates (ζ, ξ) an expression of the shape

$$\psi(\zeta, \xi) = \sum \kappa_{m_1, m_2, \dots} \zeta^{m_1} \xi^{m_2} \zeta^{m_3} \xi^{m_4} \dots \quad (11)$$

with coefficients in \mathbb{C} . As an example, we may cite the sequence of matrices occurring in Lemma 2. Given such a function, for all $(p, q) \in \mathbb{N}^2$, we denote by K_{pq} the sum of all the coefficients $\kappa_{m_1, m_2, \dots}$ such that $m_1 + m_3 + \dots = p$ and $m_2 + m_4 + \dots = q$. Similarly, for all $(p, q, r, s) \in \mathbb{N}^4$, we introduce the sum $\tilde{K}_{p,q,r,s}$ of all the coefficients $m_1 \times m_2 \times \dots \times \kappa_{m_1, m_2, \dots}$ such that $m_1 + m_3 + \dots = p$, $m_2 + m_4 + \dots = q$, and where r and s count the occurrences of ζ and ξ respectively in the monomial of (11) associated to the multiplerts (m_1, m_2, \dots) . We must have necessarily $|r - s| \leq 1$. We shall say that the function (11) is *generic* when at least one sum K_{pq} and one sum $\tilde{K}_{p,q,r,s}$ are nonzero. For instance, the matrices δ_n occurring in (8) are the specializations at (β, γ) of generic polynomials, since the coefficients are either 1 or 0.

Let us give now the main result of this section.

Theorem 2 – *Suppose we are given l generic matricial polynomial functions $\psi^{(k)}(\zeta, \xi)$, and let us set $\psi_{i,j}^{(k)}(\zeta, \xi) = (\psi^{(k)}(\zeta, \xi))_{ij}$. Then, the set of couples (ζ, ξ) such that*

$$\begin{aligned} \forall i, j, k, \quad & \psi_{i,j}^{(k)}(\zeta, \xi) \neq 0, \quad \text{with } 1 \leq i, j \leq d, 1 \leq k \leq l, \quad \text{and} \\ \forall k, \quad & \det(\psi^{(k)}(\zeta, \xi)) \neq 0, \quad \text{with } 1 \leq k \leq l \end{aligned}$$

consists in an open dense subset of $(\mathbb{C}^{d \times d})^2$.

Proof. In order to prove this result, we first show that if $\psi(\zeta, \xi)$ is the generic matricial polynomial function defined by (11), then all its entries $\psi_{i,j}(\zeta, \xi)$ are

⁵Akemann, Burda, and Kieburg, 2014, “Universal distribution of Lyapunov exponents for products of Ginibre matrices”.

⁶Bougerol and Lacroix, 1985, *Products of Random Matrices with Applications to Schrödinger Operators*.

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nonzero polynomials w.r.t. their $2d^2$ indeterminates. The proof goes as follows. Let us specialize ψ at matrices of the shape

$$\zeta = \begin{pmatrix} x & z & 0_{1,d-2} \\ 0 & x & 0_{1,d-2} \\ 0_{d-2,1} & 0_{d-2,1} & I_{d-2} \end{pmatrix} \quad \text{and} \quad \xi = \begin{pmatrix} y & t & 0_{1,d-2} \\ 0 & y & 0_{1,d-2} \\ 0_{d-2,1} & 0_{d-2,1} & I_{d-2} \end{pmatrix}$$

for convenient $x, y, z, t \in \mathbb{C}$. We easily get

$$\psi(\zeta, \xi) = \begin{pmatrix} \psi(x, y) & \tilde{\psi}(x, y, z, t) & 0_{1,d-2} \\ 0 & \psi(x, y) & 0_{1,d-2} \\ 0_{d-2,1} & 0_{d-2,1} & \psi(1, 1)I_{d-2} \end{pmatrix} \quad (12)$$

where ψ is the function (11) evaluated at complex numbers and is equal to $\sum K_{p,q} x^p y^q$ while $\tilde{\psi}$ is the polynomial $\sum \tilde{K}_{p,q,r,s} x^{p-r} y^{q-s} z^r t^s$. Due to the hypotheses, the sets of couples (x, y) or quadruplets (x, y, z, t) such that $\psi(x, y) \neq 0$ or $\tilde{\psi}(x, y, z, t) \neq 0$ respectively, are open everywhere dense subsets in \mathbb{C}^2 and in \mathbb{C}^4 respectively. Now, in order to deal with the zero-like entries of (12), we use the following property of ψ . We have

$$\psi(g^{-1}\zeta g, g^{-1}\xi g) = g^{-1}\psi(\zeta, \xi)g,$$

where g is any invertible matrix in $\mathbb{C}^{d \times d}$. Especially, if we choose g as a permutation matrix μ_σ , we obtain

$$\psi_{ij}(\mu_\sigma^{-1}\zeta\mu_\sigma, \mu_\sigma^{-1}\xi\mu_\sigma) = \psi_{\sigma(i),\sigma(j)}(\zeta, \xi).$$

The symmetric group \mathfrak{S}_d acts transitively on the set of couples (i, i) with $1 \leq i \leq d$ and transitively on the set of couples (i, j) with $1 \leq i, j \leq d$ and $i \neq j$. Thus, since $\psi_{1,1}$ and $\psi_{1,2}$ are nonzero polynomials, we may claim by using suitable specializations of (ζ, ξ) that every other entry $\psi_{i,j}$ of $\psi(\zeta, \xi)$ gives rise to a nonzero polynomial.

Let us prove now that if ψ is generic, then $\det(\psi(\zeta_{11}, \dots, \xi_{dd}))$ is a nonzero polynomial. Indeed, if we particularize ζ and ξ to be two multiples of the identity matrix, i.e., $(\zeta, \xi) = (\rho I_d, \sigma I_d)$ with ρ and σ in \mathbb{C} , we get the polynomial

$$\det(\psi(\rho I_d, \sigma I_d)) = \left(\sum_{p,q} K_{p,q} \rho^p \sigma^q \right)^d = \psi(\rho, \sigma)^d$$

which is nonzero in $\mathbb{C}[\rho, \sigma]$ since there exists at least one coefficient $K_{pq} \neq 0$.

Collecting all these informations, we may work with open dense sets in \mathbb{C}^{2d^2} instead of dealing with nonzero polynomials. Let us consider for all $k \in \{1, \dots, l\}$ and $i, j \in \{1, \dots, d\}$ the closed algebraic varieties of codimension 1

$$\mathcal{F}_{ij}^{(k)} = \{(\zeta, \xi) \in (\mathbb{C}^{d \times d})^2 / \psi_{ij}^{(k)}(\zeta_{11}, \dots, \xi_{dd}) = 0\}$$

and

$$\mathcal{F}_{\det}^{(k)} = \left\{ (\zeta, \xi) \in (\mathbb{C}^{d \times d})^2 / \det(\psi^{(k)}(\zeta_{11}, \dots, \xi_{dd})) = 0 \right\}.$$

Since each hypersurface is meager, the union $(\bigcup_{ijk} \mathcal{F}_{ij}^{(k)}) \cup (\bigcup_k \mathcal{F}_{\det}^{(k)})$ is a closed meager subset of $(\mathbb{C}^{d \times d})^2$. Thus, its complementary is an everywhere open dense subset of $(\mathbb{C}^{d \times d})^2$. This ends the proof of the result. \square

Remark 3 – As an example, for all integer m , the set of couples $(\beta, \gamma) \in (\mathbb{C}^{d \times d})^2$ such that

$$\det(\beta) \neq 0, \quad \det(\gamma) \neq 0, \quad \beta\gamma \neq \gamma\beta \quad \text{and} \quad \det(\delta_n) \neq 0, \quad \forall n \in \{0, \dots, m\},$$

is an open dense subset of $(\mathbb{C}^{d \times d})^2$. In the very special case when the matrices β, γ are multiples of I_d , i.e., $\beta = \tilde{\beta}I_d$ and $\gamma = \tilde{\gamma}I_d$ with $\tilde{\beta}$ and $\tilde{\gamma}$ in \mathbb{C} , we easily find due to (7) that

$$\delta_n = \frac{1}{\sqrt{\tilde{\beta}^2 + 4\tilde{\gamma}}} \left(\left(\frac{\tilde{\beta} + \sqrt{\tilde{\beta}^2 + 4\tilde{\gamma}}}{2} \right)^n - \left(\frac{\tilde{\beta} - \sqrt{\tilde{\beta}^2 + 4\tilde{\gamma}}}{2} \right)^n \right) I_d.$$

The set of couples $(\tilde{\beta}, \tilde{\gamma})$ such that

$$\frac{\tilde{\beta} + \sqrt{\tilde{\beta}^2 + 4\tilde{\gamma}}}{\tilde{\beta} - \sqrt{\tilde{\beta}^2 + 4\tilde{\gamma}}}$$

is not of the shape $\exp(\frac{2ik\pi}{n})$ is open dense in \mathbb{C}^2 and, for those couples, we have $\det(\delta_n) \neq 0$.

Remark 4 – The set of generic polynomials is an open dense subset in the vector space of matricial polynomial functions of given degree.

Remark 5 – Let us mention how to proceed if we deal with non-generic polynomials. We may use in this case the so-called defect Theorem of P.M. Cohn⁷. If \mathcal{K} is a commutative field and Ω is an alphabet of non commuting variables, we may denote by $\mathcal{K}\langle\langle\Omega\rangle\rangle$ the algebra of non commutative formal series in these variables. Then the defect theorem states that *if φ_1, φ_2 are elements of $\mathcal{K}\langle\langle\Omega\rangle\rangle$ without constant term, satisfying a non-trivial relation $\Phi(\varphi_1, \varphi_2) = 0$ for some non commutative series Φ in two variables, then φ_1, φ_2 commute*. This theorem allows to exhibit at least one nonzero polynomial entry to $\psi(\zeta, \xi)$, but the conclusion is not as so accurate than in Theorem 2.

⁷Lothaire, 2002, *Algebraic Combinatorics On Words*, Theorem 9.6.1, p. 283.

5. Solving explicitly the functional equation when $N = 1$

5 Solving explicitly the functional equation when $N = 1$

Let us show now how to solve the functional equation (1) when $N = 1$ by using the preceding results. We restrict ourselves to the quite opposite cases α invertible or $\alpha = 0$. Let us define $M \geq 1$ as the integer part of $\frac{t_f - t_0}{\varepsilon}$.

Theorem 3 – *If $N = 1$, for all quadruplets $(c_{-1}, c_0, c_1, \alpha)$ in an open dense subset of $(\mathbb{C}^{d \times d})^4$, the equation*

$$\alpha \mathbf{u} + \mathcal{C} \star (\chi \mathbf{u}) = \mathbf{f}$$

admits one and only one solution.

Proof. We may assume by density that c_1 is invertible. If $M = 1$, the result follows easily from the inspection of (3).

The main part of the proof consists in solving (3) and is valid for all α . Outside $[t_0 - \varepsilon, t_f + \varepsilon]$, one has $\mathbf{f}(t) = \alpha \mathbf{u}(t)$ which allows to determine \mathbf{u} if α is invertible. Next, we focus on the interval $[t_0 + \varepsilon, t_f - \varepsilon]$. By using (2), we get the recurrence equation:

$$\mathbf{u}(\tau + (n+1)\varepsilon) = c_1^{-1}(-(\alpha + c_0)\mathbf{u}(\tau + n\varepsilon) - c_{-1}\mathbf{u}(\tau + (n-1)\varepsilon) + \mathbf{f}(\tau + n\varepsilon)), \quad (13)$$

$\forall \tau \in [t_0, t_0 + \varepsilon]$ and $n \geq 1$. So we see that (13) is a recurrence of the shape (9). Using Lemma 3 with

$$\beta = -c_1^{-1}(\alpha + c_0), \quad \gamma = -c_1^{-1}c_{-1}, \quad (14)$$

and $\mathbf{w}_n = \mathbf{u}(\tau + n\varepsilon)$, $\mathbf{g}_n = c_1^{-1}\mathbf{f}(\tau + n\varepsilon)$ for τ fixed, we get

$$\mathbf{u}(\tau + n\varepsilon) = \delta_n \mathbf{u}(\tau + \varepsilon) + \delta_{n-1} \gamma \mathbf{u}(\tau) + \sum_{k=1}^{n-1} \delta_{n-k} c_1^{-1} \mathbf{f}(\tau + k\varepsilon), \quad (15)$$

for all instant $\tau + n\varepsilon \leq t_f$. Since $\tau \in [t_0, t_0 + \varepsilon[$, we have in any case $n \leq M - 1$. More explicitly, we shall use (15) in the following ranges of indices:

$$\tau \in [t_0, t_f - M\varepsilon[\quad \Rightarrow \quad \max\{n/\tau + n\varepsilon \leq t_f\} = M, \quad (16)$$

$$\tau \in [t_f - M\varepsilon, t_0 + \varepsilon[\quad \Rightarrow \quad \max\{n/\tau + n\varepsilon \leq t_f\} = M - 1. \quad (17)$$

We find it convenient to denote by $\ell = \ell(\tau)$ the maximal value of n for which we may use (15), given by (16) or (17). Although the formula (15) is true for $n = 1$, this result is tautological in this case. For $n \geq 2$, (15) expresses the solution to the functional equation (1) in the interval $[t_0 + 2\varepsilon, t_f]$, as a linear combination of the

restrictions $\mathbf{u}(\tau)$ and $\mathbf{u}(\tau + \varepsilon)$ of \mathbf{u} to the two intervals $[t_0, t_0 + \varepsilon[$ and $[t_0 + \varepsilon, t_0 + 2\varepsilon[$. The coefficients of this linear relationship are independent on time and constitute the sequence of matrices (δ_n) which depend only on β and γ .

Now, let us determine the two additional unknown functions $\mathbf{u}(\tau)$ and $\mathbf{u}(\tau + \varepsilon)$ by solving (2) near the boundaries t_0 and t_f . In order to do this, let us write explicitly the four remaining equations:

$$\text{if } t \in [t_0 - \varepsilon, t_0[, \quad \mathbf{f}(t) = \alpha \mathbf{u}(t) + c_1 \mathbf{u}(t + \varepsilon), \quad (18)$$

$$\text{if } t \in [t_0, t_0 + \varepsilon[, \quad \mathbf{f}(t) = (\alpha + c_0) \mathbf{u}(t) + c_1 \mathbf{u}(t + \varepsilon), \quad (19)$$

$$\text{if } t \in]t_f - \varepsilon, t_f], \quad \mathbf{f}(t) = (\alpha + c_0) \mathbf{u}(t) + c_{-1} \mathbf{u}(t - \varepsilon), \quad (20)$$

$$\text{if } t \in]t_f, t_f + \varepsilon], \quad \mathbf{f}(t) = \alpha \mathbf{u}(t) + c_{-1} \mathbf{u}(t - \varepsilon). \quad (21)$$

We note that the restriction of \mathbf{u} to $[t_0 - \varepsilon, t_0[$ is undetermined. Let us call this function φ . Next we solve (18) and (19) as follows, by adapting slightly the range of the time variable. If $t \in [t_0, t_0 + \varepsilon[$, (18) provides

$$\mathbf{u}(t) = c_1^{-1} \mathbf{f}(t - \varepsilon) - c_1^{-1} \alpha \varphi(t - \varepsilon). \quad (22)$$

If $t \in [t_0 + \varepsilon, t_0 + 2\varepsilon[$, (19) and (22) give

$$\mathbf{u}(t) = c_1^{-1} \mathbf{f}(t - \varepsilon) - c_1^{-1} (\alpha + c_0) c_1^{-1} \mathbf{f}(t - 2\varepsilon) + c_1^{-1} (\alpha + c_0) c_1^{-1} \alpha \varphi(t - 2\varepsilon). \quad (23)$$

The formulas (22) and (23) give the two additional restrictions of \mathbf{u} needed in the expansion (15). We note that these restrictions, at the time being, depend on the auxiliary restriction φ of \mathbf{u} .

Now it remains to solve (20) and (21). First, the two functions occurring in the r.h.s. of the equation (20) are already known and may be expressed through (15). The equation (20) may be thought as a constraint either on φ or on the l.h.s. \mathbf{f} of (1). We may observe that $\varphi(\tau - \varepsilon)$ occurs twice in $\mathbf{u}(t)$ and two times in $\mathbf{u}(t - \varepsilon)$. So, the coefficient we are looking for is equal to

$$\begin{aligned} \theta &= ((\alpha + c_0) \delta_\ell + c_{-1} \delta_{\ell-1}) [\text{coeff of } \varphi \text{ in } \mathbf{u}(\tau + \varepsilon)] \\ &\quad + ((\alpha + c_0) \delta_{\ell-1} + c_{-1} \delta_{\ell-2}) \gamma [\text{coeff of } \varphi \text{ in } \mathbf{u}(\tau)], \end{aligned}$$

where $\ell = M$ or $\ell = M - 1$, depending on the location of τ w.r.t. $t_f - M\varepsilon$. Direct inspection of these coefficients from (22) and (23) yields the explicit expression of the coefficient of φ in (20):

$$\begin{aligned} \theta &= ((\alpha + c_0) \delta_\ell + c_{-1} \delta_{\ell-1}) (c_1^{-1} (\alpha + c_0) c_1^{-1} \alpha) \\ &\quad + ((\alpha + c_0) \delta_{\ell-1} + c_{-1} \delta_{\ell-2}) \gamma (-c_1^{-1} \alpha) \\ &= [((\alpha + c_0) \delta_\ell + c_{-1} \delta_{\ell-1}) c_1^{-1} (\alpha + c_0) - ((\alpha + c_0) \delta_{\ell-1} + c_{-1} \delta_{\ell-2}) \gamma] c_1^{-1} \alpha. \end{aligned}$$

5. Solving explicitly the functional equation when $N = 1$

Now, in order to simplify the matrix θ , we use the formulas (14) to eliminate $\alpha + c_0$ and c_{-1} and we get

$$\theta = c_1[(\beta\delta_\ell + \gamma\delta_{\ell-1})\beta + (\beta\delta_{\ell-1} + \gamma\delta_{\ell-2})\gamma]c_1^{-1}\alpha.$$

By using three times the recurrence (7), we obtain

$$\theta = c_1\delta_{\ell+2}c_1^{-1}\alpha.$$

At the time being, no assumption has been made about α . From now on, we consider the case when α is invertible and chosen in such a way that $\det(\delta_{\ell+2}) \neq 0$ (see (14) and Remark 3). Therefore, $\det(\theta) \neq 0$. Hence, (20) may be solved w.r.t. $\varphi(\tau - \varepsilon)$. As a by-product of the previous results, $\varphi(\tau - \varepsilon)$ is a linear combination of the values of \mathbf{f} on the set $\{\tau + k\varepsilon \mid k \leq N\}$. It remains to solve equation (21) on $]t_f, t_f + \varepsilon]$. We get simply $\mathbf{u}(t) = \alpha^{-1}(\mathbf{f}(t) - c_{-1}\mathbf{u}(t - \varepsilon))$ for all $t \in]t_f, t_f + \varepsilon]$, where the r.h.s. has already been determined. Then we have proved that, for a generic quadruplet $(c_{-1}, c_0, c_1, \alpha)$ and for all $\mathbf{f}: \mathbb{R} \rightarrow \mathbb{C}^d$, there exists one and only one solution to the functional equation (1). \square

Next, let us consider the case when α is zero, which stands for the characterization of $\text{Im}(\square)$.

Theorem 4 – *For all triplets (c_{-1}, c_0, c_1) in an open dense subset of $(\mathbb{C}^{d \times d})^3$ and for all $\mathbf{g}: [t_0, t_f] \rightarrow \mathbb{C}^d$, there exists one and only one extension \mathbf{f} of \mathbf{g} to \mathbb{R} such that $\text{Supp}(\mathbf{f}) \subset [t_0 - \varepsilon, t_f + \varepsilon]$ and the functional equation $\mathbf{f} = \square\mathbf{u}$ admits one and only one solution in $[t_0, t_f]$.*

Proof. We may assume by density that c_1 and c_{-1} are invertible. Let be given a function $\mathbf{g}: [t_0, t_f] \rightarrow \mathbb{C}^d$ and let us construct an appropriate extension $\mathbf{f}: \mathbb{R} \rightarrow \mathbb{C}^d$ of \mathbf{g} as follows. We set

$$\mathbf{f}(t) = \begin{cases} 0 & \text{if } t < t_0 - \varepsilon \\ \mathbf{p}(t) & \text{if } t_0 - \varepsilon \leq t < t_0 \\ \mathbf{g}(t) & \text{if } t_0 \leq t \leq t_f \\ \mathbf{q}(t) & \text{if } t_f < t \leq t_f + \varepsilon \\ 0 & \text{if } t > t_f + \varepsilon \end{cases}$$

for some auxiliary functions \mathbf{p} and \mathbf{q} with values in \mathbb{C}^d . The proof consists in showing that \mathbf{f} lies in the range of \square if and only if \mathbf{p} and \mathbf{q} are uniquely determined by \mathbf{g} . We keep the notations of the previous proof. When $\alpha = 0$, the formulas (13) to (23) still hold, we have $\theta = 0$ and the auxiliary function φ does not occur anymore. Let us solve the two equations (20) and (21) w.r.t. the two restrictions \mathbf{p} and \mathbf{q} of \mathbf{f} . To write these equations, we introduce the operators

$$\Delta_\ell \mathbf{f}(\tau) = \sum_{k=1}^{\ell-1} \delta_{\ell-k} c_1^{-1} \mathbf{f}(\tau + k\varepsilon).$$

Let us notice that these operators may be thought as discrete integral operators. Plugging formula (15), with the maximal value ℓ allowed for n , in the equation (20), we get for all $t \in]t_f - \varepsilon, t_f]$

$$\begin{aligned} \mathbf{f}(t) = & c_0(\delta_\ell \mathbf{u}(\tau + \varepsilon) + \delta_{\ell-1} \gamma \mathbf{u}(\tau) + \Delta_\ell \mathbf{f}(\tau)) + c_{-1}(\delta_{\ell-1} \mathbf{u}(\tau + \varepsilon) + \delta_{\ell-2} \gamma \mathbf{u}(\tau) \\ & + \Delta_{\ell-1} \mathbf{f}(\tau)), \end{aligned}$$

t and τ being connected as previously by the requirements that $\tau \in [t_0, t_0 + \varepsilon[$ and $\frac{t-\tau}{\varepsilon} \in \mathbb{N}$. Remembering the values of β, γ given by (14) and the recurrence (7), we get:

$$\mathbf{f}(t) = -c_1(\delta_{\ell+1} \mathbf{u}(\tau + \varepsilon) + \delta_\ell \gamma \mathbf{u}(\tau)) + c_0 \Delta_\ell \mathbf{f}(\tau) + c_{-1} \Delta_{\ell-1} \mathbf{f}(\tau). \quad (24)$$

Since we may respectively rewrite (22) and (23) as

$$\mathbf{u}(\tau) = c_1^{-1} \mathbf{p}(\tau) \quad \text{and} \quad \mathbf{u}(\tau + \varepsilon) = c_1^{-1} \mathbf{f}(\tau) - c_1^{-1} c_0 c_1^{-1} \mathbf{p}(\tau),$$

we see that the coefficient of $\mathbf{p} = \mathbf{f}(\tau - \varepsilon)$ in (24) is given by:

$$(\delta_{\ell+1} c_1^{-1} c_0 - \delta_\ell \gamma) c_1^{-1} = -\delta_{\ell+2} c_1^{-1}.$$

We assume the triplet (c_{-1}, c_0, c_1) lies in the open dense subset where $\det(\delta_{\ell+2}) \neq 0$ and thus, we have shown that the restriction \mathbf{p} exists and is unique. This being done, the equation (20) yields

$$\mathbf{f}(t + \varepsilon) = \mathbf{q}(t) = c_{-1}(\delta_\ell \mathbf{u}(\tau + \varepsilon) + \delta_{\ell-1} \gamma \mathbf{u}(\tau) + \Delta_\ell \mathbf{f}(\tau)),$$

t and τ being connected as before, which is well-determined. As a consequence, the formula (15) gives rise to one unique function \mathbf{u} on $[t_0, t_f]$ such that $\square \mathbf{u} = \mathbf{f}$ and this ends the proof. \square

6 A matricial-based framework when N is arbitrary

Let us use a matricial-based framework for solving the functional equation for arbitrary N . In this section we assume throughout that c_N and α are invertible, mainly for the ease of exposition. We shall say that the $(2N + 2)$ -tuple $(c_{-N}, \dots, c_0, \dots, c_N, \alpha)$ is *generic* in $(\mathbb{C}^{d \times d})^{2N+2}$ provided the foregoing system (35) is Cramer. This system (35) consists in N equations w.r.t. N unknowns, generalizing the auxiliary function φ occurring in Section 5. Its coefficients are matricial polynomial functions w.r.t. $c_N^{-1}(c_0 + \alpha)$ and the $2N - 1$ matricial indeterminates $c_N^{-1} c_k$, $k \in \{-N, \dots, N - 1\}$ and $k \neq 0$, which in a sense generalize (11).

Theorem 5 – *If the $(2N + 2)$ -tuple $(c_{-N}, \dots, c_0, \dots, c_N, \alpha)$ is generic, the functional equation (1) admits one and only one solution.*

6. A matricial-based framework when N is arbitrary

Similarly, we shall say that the $(2N + 1)$ -tuple $(c_{-N}, \dots, c_0, \dots, c_N)$ is *generic* in $(\mathbb{C}^{d \times d})^{2N+1}$ provided the foregoing matrix Θ defined by (36) is invertible. This matrix is associated to a linear system of N equations w.r.t. N unknowns $\mathbf{p}_1, \dots, \mathbf{p}_N$, which are the restrictions of \mathbf{f} to parts of $[t_0 - N\varepsilon, t_0[$ in order that \mathbf{f} lies in the range of the operator \square .

Theorem 6 – *If the $(2N + 1)$ -tuple $(c_{-N}, \dots, c_0, \dots, c_N)$ is generic, for all $\mathbf{g}: [t_0, t_f] \rightarrow \mathbb{C}^d$ there exists one and only one extension $\mathbf{f}: \mathbb{R} \rightarrow \mathbb{C}^d$ such that $\text{Supp}(\mathbf{f}) \subset [t_0 - N\varepsilon, t_f + N\varepsilon]$ and the functional equation $\mathbf{f} = \square \mathbf{u}$ admits one and only one solution in $[t_0, t_f]$.*

Proof (Of both theorems). Let $\mathbf{u}: \mathbb{R} \rightarrow \mathbb{C}^d$ be a solution to equation (1), related to \mathbf{A}, \mathbf{B} and \mathbf{U} defined over \mathbb{R} as follows:

$$\mathbf{B}(t) = \begin{pmatrix} c_N^{-1} \mathbf{f}(t) \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{C}^{2dN}, \quad \mathbf{U}(t) = \begin{pmatrix} \mathbf{u}(t + (N-1)\varepsilon) \\ \vdots \\ \mathbf{u}(t) \\ \vdots \\ \mathbf{u}(t - N\varepsilon) \end{pmatrix} \in \mathbb{C}^{2dN}, \quad (25)$$

$$\mathbf{A}(t) = \begin{pmatrix} -c_N^{-1} c_{N-1} \chi(t + (N-1)\varepsilon) & \dots & -c_N^{-1} (\alpha + c_0 \chi(t)) & \dots & -c_N^{-1} c_{-N} \chi(t - N\varepsilon) \\ I_d & 0 & \dots & \dots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & I_d & 0 \end{pmatrix}. \quad (26)$$

The matrix $\mathbf{A}(t) \in \mathbb{C}^{2dN \times 2dN}$ is locally constant w.r.t. t . Especially, when $t \in [t_0 + N\varepsilon, t_f - N\varepsilon]$, all the characteristic functions of the entries of $\mathbf{A}(t)$ are equal to 1 and we shall agree to denote by \mathbf{A} the constant value of $\mathbf{A}(t)$ in this segment. Let us mention that the matrix \mathbf{A} depends only on the $2N + 2$ matrices c_i and α and is nothing but the block companion matrix of the functional equation (1).

When $t < t_0 - N\varepsilon$ or $t > t_f + N\varepsilon$, we have already seen that the solution to the equation (1) is given by $\mathbf{u}(t) = \alpha^{-1} \mathbf{f}(t)$. Now, if $t \in [t_0 - N\varepsilon, t_f - N\varepsilon]$, (2) may be written as

$$\mathbf{u}(t + N\varepsilon) = c_N^{-1} [\mathbf{f}(t) - (c_{N-1} \chi(t + (N-1)\varepsilon) \mathbf{u}(t + (N-1)\varepsilon) + \dots + (\alpha + c_0 \chi(t)) \mathbf{u}(t) + \dots + c_{-N} \chi(t - N\varepsilon) \mathbf{u}(t - N\varepsilon))]. \quad (27)$$

When $t \in [t_0 - N\varepsilon, t_0[$, this equation expresses $\mathbf{u}(t + N\varepsilon)$ explicitly as a combination of the N auxiliary functions $\varphi_k(\tau) = \mathbf{u}(\tau - k\varepsilon)$, $1 \leq k \leq N$, which are at this level unknown. Obviously, the functions $\varphi_k(\tau)$, $1 \leq k \leq N$, stand for the last N components of $\mathbf{U}(\tau)$. By using definitions (25) and (26), recurrence (27) may be rewritten as

$$\mathbf{U}(t + \varepsilon) = \mathbf{A}(t) \mathbf{U}(t) + \mathbf{B}(t), \quad \forall t \in [t_0 - N\varepsilon, t_f - N\varepsilon].$$

The preceding recurrence and Lemma 1 yield the formula

$$\mathbf{U}(t + n\varepsilon) = \mathbf{C}_n(t)\mathbf{U}(t) + \mathbf{F}_n(t) \quad (28)$$

for $t_0 - N\varepsilon \leq t \leq t + (n-1)\varepsilon \leq t_f - N\varepsilon$, where

$$\mathbf{C}_n(t) = \begin{cases} \prod_{k=0}^{n-1} \mathbf{A}(t+k\varepsilon) & \text{if } t + (n-1)\varepsilon < t_0 + N\varepsilon, \\ \mathbf{A}^{n-N} \left(\prod_{k=0}^{N-1} \mathbf{A}(t+k\varepsilon) \right) & \text{otherwise} \end{cases} \quad (29)$$

and

$$\mathbf{F}_n(t) = \sum_{k=0}^{n-1} \left(\prod_{\ell=k+1}^{n-1} \mathbf{A}(t+\ell\varepsilon) \right) \mathbf{B}(t+k\varepsilon). \quad (30)$$

Focusing on the first row of (28), i.e., the first component $\mathbf{u}(t + (n+N-1)\varepsilon)$ of $\mathbf{U}(t+n\varepsilon)$, we emphasize the fact that the unknown function $\mathbf{u}(t)$ may be expressed as a linear combination of the delayed functions $\mathbf{f}(t+k\varepsilon)$, $k \in \mathbb{Z}$ through $\mathbf{F}_n(t)$, as well as the auxiliary functions $\varphi_k(\tau)$, $1 \leq k \leq N$, through $\mathbf{U}(t)$. So, as a first conclusion of these calculations, $\mathbf{u}(t)$ is well-determined in the whole interval $] -\infty, t_f]$, as a combination of the auxiliary functions φ_k .

When $t \in [t_f - N\varepsilon, t_f + N\varepsilon]$, we may deduce from (1) and (4) the two systems of N functional equations which are analogous to (20) and (21), i.e.,

$$\begin{cases} \mathbf{f}(t) = \alpha \mathbf{u}(t) + c_{-N} \mathbf{u}(t-N\varepsilon) + \dots + c_{N-1} \mathbf{u}(t+(N-1)\varepsilon) & \text{in }]t_f - N\varepsilon, t_f - (N-1)\varepsilon] \\ \mathbf{f}(t) = \alpha \mathbf{u}(t) + c_{-N} \mathbf{u}(t-N\varepsilon) + \dots + c_{N-2} \mathbf{u}(t+(N-2)\varepsilon) & \text{in }]t_f - (N-1)\varepsilon, t_f - (N-2)\varepsilon] \\ \vdots \\ \mathbf{f}(t) = \alpha \mathbf{u}(t) + c_{-N} \mathbf{u}(t-N\varepsilon) + \dots + c_0 \mathbf{u}(t) & \text{in }]t_f - \varepsilon, t_f] \end{cases} \quad (31)$$

and

$$\begin{cases} \mathbf{f}(t) = \alpha \mathbf{u}(t) + c_{-N} \mathbf{u}(t-N\varepsilon) + \dots + c_{-1} \mathbf{u}(t-\varepsilon) & \text{in }]t_f, t_f + \varepsilon] \\ \mathbf{f}(t) = \alpha \mathbf{u}(t) + c_{-N} \mathbf{u}(t-N\varepsilon) + \dots + c_{-2} \mathbf{u}(t-2\varepsilon) & \text{in }]t_f + \varepsilon, t_f + 2\varepsilon] \\ \vdots \\ \mathbf{f}(t) = \alpha \mathbf{u}(t) + c_{-N} \mathbf{u}(t-N\varepsilon) & \text{in }]t_f + (N-1)\varepsilon, t_f + N\varepsilon] \end{cases} \quad (32)$$

Let us consider the case when α is invertible. We convert (31) into a linear system for the N auxiliary unknown functions φ_k . To do this, we note first that (28), with $n = M$ and $t = \tau - N\varepsilon$, writes as

$$\mathbf{U}(\tau + (M-N)\varepsilon) = \mathbf{C}_M(\tau - N\varepsilon)\mathbf{U}(\tau - N\varepsilon) + \mathbf{F}_M(\tau - N\varepsilon). \quad (33)$$

6. A matricial-based framework when N is arbitrary

The first N components of $\mathbf{U}(\tau - N\varepsilon)$ which occur in the right-hand side of the previous equality are the auxiliary functions while the last N components are values of the function $\alpha^{-1}\mathbf{f}$ at instants $\tau - (N + k)\varepsilon$, $1 \leq k \leq N$. Next, by shifting adequately t in the system of equations (31), we get for all $t \in]t_f - N\varepsilon, t_f - (N - 1)\varepsilon]$ the equality

$$\begin{pmatrix} \mathbf{f}(t) \\ \mathbf{f}(t + \varepsilon) \\ \vdots \\ \mathbf{f}(t + (N - 1)\varepsilon) \end{pmatrix} = \mathbf{D} \begin{pmatrix} \mathbf{u}(t + (N - 1)\varepsilon) \\ \vdots \\ \mathbf{u}(t - N\varepsilon) \end{pmatrix} \quad (34)$$

where

$$\mathbf{D} = \begin{pmatrix} c_{N-1} & \dots & \dots & c_1 & (\alpha + c_0) & c_{-1} & \dots & c_{-N} \\ c_{N-2} & \dots & c_1 & (\alpha + c_0) & c_{-1} & \dots & c_{-N} & 0 \\ \vdots & \ddots & \ddots & \ddots & & \ddots & \ddots & \vdots \\ c_1 & \ddots & \ddots & & \ddots & \ddots & & \vdots \\ (\alpha + c_0) & c_{-1} & \dots & c_{-N} & 0 & \dots & \dots & 0 \end{pmatrix} \in \mathbb{C}^{Nd \times (2Nd)}$$

is a rectangular block Hankel matrix. By setting $t = \tau + (M - N)\varepsilon$ in (34) and using (33), we easily get

$$\mathbf{DC}_M(\tau - N\varepsilon)\mathbf{U}(\tau - N\varepsilon) = \begin{pmatrix} \mathbf{f}(\tau + (M - N)\varepsilon) \\ \vdots \\ \mathbf{f}(\tau + (M - 1)\varepsilon) \end{pmatrix} - \mathbf{DF}_M(\tau - N\varepsilon). \quad (35)$$

By partitioning the $(Nd) \times (2Nd)$ matrix $\mathbf{DC}_M(\tau - N\varepsilon)$ as $(\Theta_1 \ \Theta_2)$, where $\Theta_1, \Theta_2 \in \mathbb{C}^{Nd \times Nd}$, we see that the previous linear system consists in N vectorial equations in \mathbb{C}^d depending on the N auxiliary functions φ_k and is Cramer provided that Θ_1 is invertible. Under the assumption of genericity, the previous condition is satisfied and thus, there exists one and only one solution to (35). The auxiliary functions being determined, as a rule, the function \mathbf{u} is well defined and unique on $]-\infty, t_f]$. Since α is also invertible by assumption, we may compute the various extensions of \mathbf{u} on $[t_f, t_f + N\varepsilon]$ using the system (32). This ends the proof of Theorem 5. The proof of the second theorem proceeds in the same way. Let \mathbf{p}_k be the unknown restriction of \mathbf{f} to $[t_0 - k\varepsilon, t_0 - (k - 1)\varepsilon]$, for all $k \in \{1, \dots, N\}$. Similarly let us define \mathbf{q}_k as the unknown restriction of \mathbf{f} to $]t_f + (k - 1)\varepsilon, t_f + k\varepsilon]$, for all $k \in \{1, \dots, N\}$. When α is zero and $\det(c_N) \neq 0$, the system (4) shows that $\mathbf{u}(t)$ may be arbitrarily chosen outside $[t_0, t_f]$. Indeed, the auxiliary functions φ_k do not occur in the calculation of \mathbf{u} . However, \mathbf{u} is well-determined in the whole interval $[t_0, t_f]$ provided \mathbf{g} is given in this interval and the \mathbf{p}_k , $1 \leq k \leq N$, are already determined. In order that \mathbf{u} satisfies the systems (31) and (32), the restrictions \mathbf{p}_k and \mathbf{q}_k , $1 \leq k \leq N$, must be chosen adequately. Let us focus on (31) or, equivalently, on (35). Indeed, both systems still hold when

$\alpha = 0$. However, our approach is different from previously since the unknowns which are the \mathbf{p}_k 's are located exclusively in the term $-\mathbf{D}\mathbf{F}_M(\tau - N\varepsilon)$. Indeed, the above discussion shows that $\mathbf{U}(\tau - N\varepsilon)$ does not depend on the restrictions \mathbf{p}_k nor the matrix $(\mathbf{f}(\tau + (M - k)\varepsilon))_{1 \leq k \leq n}$. Let us determine the coefficient $[\Theta]_{ij} \in \mathbb{C}^{d \times d}$ of $\mathbf{p}_j(\tau)$ in the i -th equation in the system (35). To do this, we use the convention that for some matrix $\Theta \in \mathbb{C}^{(Nd) \times (Nd)}$, the symbol $[\Theta]_{ij}$ denotes the $(d \times d)$ -block located at the (i, j) -entry of Θ . Then, due to formulas (25) and (30), we see that

$$[\Theta]_{ij} = - \left[D \prod_{k=N-j+1}^{M-1} \mathbf{A}(\tau + (k - N)\varepsilon) \right]_{i1} c_N^{-1}. \quad (36)$$

We assume that the multiplet (c_{-N}, \dots, c_N) is generic, which amounts to say that Θ is invertible. Therefore, (35) is Cramer and admits one and only one solution $(\mathbf{p}_1, \dots, \mathbf{p}_N)$. Next, by using the same transformation which led to (34), the system (32) may be rewritten as

$$\begin{pmatrix} \mathbf{f}(t) \\ \vdots \\ \mathbf{f}(t + (N - 1)\varepsilon) \end{pmatrix} = \begin{pmatrix} \mathbf{q}_1(t) \\ \vdots \\ \mathbf{q}_N(t) \end{pmatrix} = \begin{pmatrix} c_{-1} & c_{-2} & \dots & c_{-N} \\ c_{-2} & \dots & c_{-N} & 0 \\ \vdots & \ddots & \ddots & \vdots \\ c_{-N} & 0 & \dots & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u}(t - \varepsilon) \\ \vdots \\ \mathbf{u}(t - N\varepsilon) \end{pmatrix},$$

for all $t \in]t_f, t_f + \varepsilon]$. Since the matrix $(\mathbf{u}(t - k\varepsilon))_{1 \leq k \leq N}$ is known, we obtain as a conclusion that the restrictions $(\mathbf{p}_k, \mathbf{q}_k)$, $1 \leq k \leq N$, are uniquely determined by \mathbf{g} , which ends the proof of Theorem 6. \square

7 Final comments

Before concluding this paper, it is natural to ask if there exist other ways to handle the functional equation $\alpha \mathbf{u} + \mathcal{C} \star (\chi \mathbf{u}) = \mathbf{f}$. Since (1) is a convolution equation, one might want to use the discrete Laplace (or Z -) transform to solve it. Indeed, if the Z -transform is defined by $\mathcal{L}\mathbf{u}(p) = \sum_{t \in \varepsilon\mathbb{Z}} \exp(-pt)\mathbf{u}(t)$, then one has the formula $\mathcal{L}(\mathbf{u} \star \mathbf{v})(p) = (\mathcal{L}\mathbf{u})(\mathcal{L}\mathbf{v})$ for all functions \mathbf{u} and \mathbf{v} on the real line, with compact support and values in suitable algebras. So, when $\alpha = 0$, the image of (1) is $\mathcal{L}(\mathcal{C})\mathcal{L}(\chi \mathbf{u}) = \mathcal{L}(\mathbf{f})$ and, thus, if $\mathcal{L}(\mathcal{C})$ is invertible, we get

$$\mathcal{L}(\chi \mathbf{u})(p) = \left(\sum_{k \in \mathbb{Z}} c_k \exp(pk\varepsilon) \right)^{-1} \left(\sum_{k \in \mathbb{Z}} \exp(-pk\varepsilon) \mathbf{f}(k\varepsilon) \right).$$

Unfortunately, the determination of the original \mathbf{u} has been transferred to a difficult problem of moments. Moreover, this approach does not apply to the general case $\alpha \neq 0$.

Acknowledgments

Another way to proceed would be to apply Picard's method. Indeed, one sees that (1) is equivalent to

$$\begin{aligned}\mathbf{u} &= \alpha^{-1}\mathbf{f} - \alpha^{-1}\square\mathbf{u} = \alpha^{-1}\mathbf{f} - \alpha^{-1}\square(\alpha^{-1}\mathbf{f} - \alpha^{-1}\square\mathbf{u}) = \dots \\ &= \alpha^{-1}\mathbf{f} + \sum_{k=1}^{\infty} (-1)^k \alpha^{-k-1} \square^k \mathbf{f}.\end{aligned}$$

Thus if $\|\alpha^{-1}\|$ is small enough, one may hope that \square is a contraction of a suitable Banach or Fréchet function space \mathcal{F} . The main difficulty in this approach is the requirement that the vector space \mathcal{F} is complete, and it is related to the increasing number of points of discontinuity of $\square^k \mathbf{f}$, as k increases.

By comparison, the approach developed in this paper is less complicated and more robust in the sense that the solution is given explicitly. Indeed, the central parts of the proofs of the four previous theorems highlight not only the well-posedness of (1) but also an effective computational way to construct interval by interval the solution \mathbf{u} .

In scope of future work, we suggest the three following problems. First, the formulas (28) and (29) imply that, in the very special case when \mathbf{f} is constant, $\mathbf{u}(t)$ may be expressed through exponential-monomial functions inside $[t_0 + N\varepsilon, t_f - N\varepsilon]$. So, in this case, the search for pseudo-periodic solutions of (1) is linked to the difficult elimination problem of requiring that the spectrum $Sp(\mathbf{A})$ of \mathbf{A} is included in the unit circle of \mathbb{C} . Second, we conjecture the following variant of Theorem 2. If the coefficients $\kappa_{m_1, m_2, \dots}$ occurring in formula (11) are rational or even algebraic over \mathbb{Q} and, in contrast, the entries $\zeta_{i,j}$ and $\xi_{i,j}$ generate a subfield of \mathbb{C} of transcendence degree equal to $2d^2$, then all the quantities $\psi_{i,j}^{(k)}(\zeta, \xi)$ and $\det(\psi^{(k)})(\zeta, \xi)$ do not vanish. Lastly, we conjecture also that for all $N \geq 2$, the genericity assumption on the multiplets in Theorems 5 and 6 hold on open dense subsets of appropriate products of matricial algebras. This conjecture seems natural in light of the two theorems presented in Section 5.

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