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Elementary Evaluation of Convolution Sums Involving the Divisor Function for a Class of Levels

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Abstract

We discuss the evaluation of convolution sums involving the divisor function, $\sum_{\substack{(l,m)\in\mathbb{N}^2\\ \alpha l+\beta m=n}} \sigma(l)\sigma(m)$, for the class of levels $\alpha\beta$ belonging to all natural numbers. The evaluation of convolution sums belonging to this class is achieved by applying modular forms and primitive Dirichlet characters. We illustrate our method with the explicit examples for the levels $\alpha\beta = 33$, 40, 45, 50, 54, and 56. As a corollary, the known convolution sums for the levels $\alpha\beta = 10$, 11, 12, 15, 16, 18, 24, 25, 27, 32 and 36 are improved when we revisit their evaluations. If the level $\alpha\beta \equiv 0 \pmod 4$, we determine natural numbers a, b and use the evaluated convolution sums together with other known convolution sums to carry out the number of representations of n by the octonary quadratic forms $a\sum_{i=1}^{4} x_i^2 + b\sum_{i=5}^{8} x_i^2$. Similarly, if the level $\alpha\beta \equiv 0 \pmod 3$, we compute natural numbers c, d and make use of the evaluated convolution sums together with other known convolution sums to determine the number of representations of n by the octonary quadratic forms $c\sum_{i=1}^{2}(x_{2i-1}^2 + x_{2i-1}x_{2i} + x_{2i}^2) + d\sum_{i=3}^{4}(x_{2i-1}^2 + x_{2i-1}x_{2i} + x_{2i}^2)$. In addition, we determine formulae for the number of representations of a positive integer n when $(a,b) = (1,1), (1,3), (1,6), (2,3)$.

Keywords: Sums of Divisors; Dedekind eta function, Convolution Sums, Modular Forms, Dirichlet Characters, Eisenstein forms, Cusp Forms, Octonary quadratic Forms, Number of Representations.

MSC: 11A25, 11F11, 11F20, 11E20, 11E25, 11F27.

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1 Introduction

In this work, we denote by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ the sets of natural numbers, integers, rational numbers, real numbers and complex numbers, respectively. Let in addition $\mathbb{N}_0$ denote the set of natural numbers without zero, i.e., $\mathbb{N}_0 = \mathbb{N} \setminus \{0\}$. Let $k \in \mathbb{N}$ and let $n \in \mathbb{N}_0$. The sum $\sigma_k(n)$ of the k^{th} powers of the positive divisors of n is defined by

$$\sigma_k(n) = \sum_{0<d|n} d^k. \tag{1}$$

We let $\sigma(n)$ stand for $\sigma_1(n)$. For $m \notin \mathbb{N}$ we set $\sigma_k(m) = 0$ and for all $k \in \mathbb{N}_0$ we set $\sigma_k(0) = 0$.

Let $\alpha, \beta \in \mathbb{N}_0$ be such that $\alpha \leq \beta$. We define the convolution sum, $W_{(\alpha,\beta)}(n)$, as follows:

$$W_{(\alpha,\beta)}(n) = \sum_{\substack{(l,m)\in\mathbb{N}^2 \\ \alpha\, l+\beta\, m=n}} \sigma(l)\sigma(m). \tag{2}$$

We write $W_\beta(n)$ as a shorthand for $W_{(1,\beta)}(n)$. If for all $(l,m) \in \mathbb{N}^2$ it holds that $\alpha\, l + \beta\, m \neq n$, then we set $W_{(\alpha,\beta)}(n) = 0$

So far known convolution sums are displayed in Table 1.

Level $\alpha\beta$	**References**
1	Besge (1885), Glaisher (1862), and Ramanujan (1916)
2, 3, 4	Huard et al. (2002)
5, 7	Cooper and Toh (2009) and Lemire and Williams (2006)
6	Ş. Alaca and Williams (2007)
8, 9	Williams (2005, 2006)
10, 11, 13, 14	Royer (2007)
12, 16, 18, 24	A. Alaca, Ş. Alaca, and Williams (2006, 2007a,b, 2008)
15	Ramakrishnan and Sahu (2013)
10, 20	Cooper and Ye (2014)
23	Chan and Cooper (2008)
25	Xia, Tian, and Yao (2014)
27, 32	Ş. Alaca and Kesicioğlu (2016)
36	Ye (2015)
14, 26, 28, 30, 22, 44, 52, 48, 64	Ntienjem (2015, 2017a,b)

Table 1 – Known convolution sums $W_{(\alpha,\beta)}(n)$ of level $\alpha\beta$

Let $\mathfrak{N}$ be a subset of $\mathbb{N}_0$ which is defined as follows:

$$\mathfrak{N} = \{\, 2^\nu \mho \mid \nu \in \{0,1,2,3\} \text{ and } \mho \text{ is a finite product of distinct odd primes} \}.$$

We evaluate the convolution sum $W_{(\alpha,\beta)}(n)$ for the class of levels $\alpha\beta$ such that

(1) $\alpha\beta \in \mathfrak{N}$ and

(2) $\alpha\beta \in \mathbb{N}_0 \setminus \mathfrak{N}$.

Therefore, we evaluate the convolution sum $W_{(\alpha,\beta)}(n)$ for the class of levels $\alpha\beta$ belonging to all natural numbers. We use in particular Dirichlet characters and modular forms to evaluate these convolution sums. The evaluation of the convolution sum for these classes of levels is new.

We observe that the levels $\alpha\beta = 9$, 16, 18, 25, 27, 32, 36, 48, 64 from Table 1 belong to the class of levels mentioned in item (2) while the other levels from Table 1 are handled in item (1).

As an immediate consequence of the generalization of the evaluation of convolution sums, we revisit the evaluation of the convolution sums for the levels $\alpha\beta = 9$, 10, 11, 12, 15, 16, 18, 24, 25, 27, 32 and 36, which leads to the improvement of the result of the evaluation of the convolution sums for $\alpha\beta = 10$, 11, 12, 15, 16, 18, 24, 25, 27, 32 and 36.

We illustrate our general approach in case

1. $\alpha\beta \in \mathfrak{N}$ by evaluating the convolution sum for $\alpha\beta = 33$, 40 and 56.

2. $\alpha\beta \in \mathbb{N}_0 \setminus \mathfrak{N}$ by evaluating the convolution sum for $\alpha\beta = 45$, 50 and 54.

Again, these convolution sums have not been evaluated as yet.

As an application, convolution sums are used to determine explicit formulae for the number of representations of a positive integer n by the octonary quadratic forms

$$a\sum_{i=1}^{4} x_i^2 + b\sum_{i=5}^{8} x_i^2 \tag{3}$$

and

$$c\sum_{i=1}^{2} (x_{2i-1}^2 + x_{2i-1}x_{2i} + x_{2i}^2) + d\sum_{i=3}^{4} (x_{2i-1}^2 + x_{2i-1}x_{2i} + x_{2i}^2), \tag{4}$$

respectively, where $a, b, c, d \in \mathbb{N}_0$.

So far known explicit formulae for the number of representations of n by the octonary form (3) are displayed in Table 2.

(a,b)	**References**
$(1,2)$	Williams (2006)
$(1,4)$	A. Alaca, Ş. Alaca, and Williams (2007a)
$(1,5)$	Cooper and Ye (2014)
$(1,6)$	Ramakrishnan and Sahu (2013)
$(1,8)$	Ş. Alaca and Kesicioğlu (2016)
$(1,7)$, $(1,11)$, $(1,13)$, $(1,12)$, $(1,16)$, $(3,4)$	Ntienjem (2015, 2017a,b)

Table 2 – Known representations of n by the form (3)

Similarly, so far known explicit formulae for the number of representations of n by the octonary form (4) are referenced in Table 3.

(c,d)	**References**
$(1,1)$	Lomadze (1989)
$(1,2)$	Ş. Alaca and Williams (2007)
$(1,3)$	Williams (2005)
$(1,4)$, $(1,6)$, $(1,8)$, $(2,3)$	A. Alaca, Ş. Alaca, and Williams (2006, 2007a,b)
$(1,5)$	Ramakrishnan and Sahu (2013)
$(1,9)$	Ş. Alaca and Kesicioğlu (2016)
$(1,10)$, $(2,5)$, $(1,16)$	Ntienjem (2015, 2017b)
$(1,12)$, $(3,4)$	Ye (2015)

Table 3 – Known representations of n by the form (4)

We first discuss a method to determine all pairs $(a,b)\in\mathbb{N}_0^2$ and $(c,d)\in\mathbb{N}_0^2$ that are necessary for the determination of the formulae for the number of representations of a positive integer by the octonary quadratic forms (3) and (4) when the level $\alpha\beta$ is contained in the above-mentioned class of levels. We then determine explicit formulae for the number of representations of a positive integer n by octonary quadratic forms (3) and (4) whenever $\alpha\beta\equiv 0 \pmod 3$ and $\alpha\beta\equiv 0 \pmod 4$, respectively.

We next use the convolution sums, $W_{(\alpha,\beta)}(n)$, where

- $\alpha\beta = 33$, 45 and 54 to give examples of explicit formulae for the number of representations of a positive integer n by the octonary quadratic forms (3).
- $\alpha\beta = 40$ and 56, to provide examples of explicit formulae for the number of representations of a positive integer n by the octonary quadratic forms (4).

Software for symbolic scientific computation is used to obtain the results of this paper. This software comprises the open source software packages *GiNaC*, *Maxima*, *Reduce*, *SAGE* and the commercial software package *MAPLE*.

2 Essentials to the Understanding of the Problem

2.1 Modular Forms

Let $\mathbb{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$, be the upper half-plane and let $\Gamma = G = \mathrm{SL}_2(\mathbb{R}) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mid a, b, c, d \in \mathbb{R}$ and $ad - bc = 1\}$ be the group of 2×2-matrices. Let $N \in \mathbb{N}_0$. Then

$$\Gamma(N) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) \mid \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \pmod{N} \right\}$$

is a subgroup of G and is called the *principal congruence subgroup of level N*. A subgroup H of G is called a *congruence subgroup of level N* if it contains $\Gamma(N)$.

For our purposes the following congruence subgroup is relevant:

$$\Gamma_0(N) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) \mid c \equiv 0 \pmod{N} \right\}.$$

Let $k, N \in \mathbb{N}$ and let $\Gamma' \subseteq \Gamma$ be a congruence subgroup of level N. Let $k \in \mathbb{Z}$, $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ and $f : \mathbb{H} \cup \mathbb{Q} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$. We denote by $f^{[\gamma]_k}$ the function whose value at z is $(cz+d)^{-k} f(\gamma(z))$, i.e., $f^{[\gamma]_k}(z) = (cz+d)^{-k} f(\gamma(z))$. The following definition is according to N. Koblitz[2].

Definition 1 – Let $N \in \mathbb{N}_0$, $k \in \mathbb{Z}$, f be a meromorphic function on $\mathbb{H}$ and $\Gamma' \subset \Gamma$ a congruence subgroup of level N.

(a) f is called a *modular function of weight k* for Γ' if

  (a1) for all $\gamma \in \Gamma'$ it holds that $f^{[\gamma]_k} = f$.

  (a2) for any $\delta \in \Gamma$ it holds that $f^{[\delta]_k}(z)$ has the form $\sum_{n \in \mathbb{Z}} a_n e^{\frac{2\pi i z n}{N}}$ and $a_n \neq 0$ for finitely many $n \in \mathbb{Z} \setminus \mathbb{N}$.

(b) f is called a *modular form of weight k* for Γ' if

  (b1) f is a modular function of weight k for Γ',

  (b2) f is holomorphic on $\mathbb{H}$,

  (b3) for all $\delta \in \Gamma$ and for all $n \in \mathbb{Z} \setminus \mathbb{N}$ it holds that $a_n = 0$.

(c) f is called a *cusp form of weight k for* Γ' if

  (c1) f is a modular form of weight k for Γ',

  (c2) for all $\delta \in \Gamma$ it holds that $a_0 = 0$.

[2] Koblitz, 1993, *Introduction to Elliptic Curves and Modular Forms*, p. 108.

Let us denote by $M_k(\Gamma')$ the set of modular forms of weight k for Γ', by $S_k(\Gamma')$ the set of cusp forms of weight k for Γ' and by $E_k(\Gamma')$ the set of Eisenstein forms. The sets $M_k(\Gamma')$, $S_k(\Gamma')$ and $E_k(\Gamma')$ are vector spaces over $\mathbb{C}$. Therefore, $M_k(\Gamma_0(N))$ is the space of modular forms of weight k for $\Gamma_0(N)$, $S_k(\Gamma_0(N))$ is the space of cusp forms of weight k for $\Gamma_0(N)$, and $E_k(\Gamma_0(N))$ is the space of Eisenstein forms. The decomposition of the space of modular forms as a direct sum of the space generated by the Eisenstein series and the space of cusp forms, i.e., $M_k(\Gamma_0(N)) = E_k(\Gamma_0(N)) \oplus S_k(\Gamma_0(N))$, is well-known; see for example the online version of the book[3].

We assume in this paper that $2 \leq k$ is even and that χ and ψ are primitive Dirichlet characters with conductors L and R, respectively. It has been noted by W. A. Stein[4] that

$$E_{k,\chi,\psi}(q) = C_0 + \sum_{n=1}^{\infty}\Big(\sum_{d|n}\psi(d)\chi\Big(\frac{n}{d}\Big)\, d^{k-1}\Big)q^n, \tag{5}$$

where

$$C_0 = \begin{cases} 0 & \text{if } L > 1 \\ -\frac{B_{k,\chi}}{2k} & \text{if } L = 1 \end{cases}$$

and $B_{k,\chi}$ are the generalized Bernoulli numbers. Theorems 5.8 and 5.9 in Section 5.3 in Stein (2011, p. 86) are then applicable.

If the primitive Dirichlet characters χ and ψ are trivial, then their conductors L and R are one, respectively. Therefore, (5) can be normalized and then given as follows: $E_k(q) = 1 - \frac{2k}{B_k}\sum\limits_{n=1}^{\infty}\sigma_{k-1}(n)\, q^n$. This will be the case whenever the level $\alpha\beta$ belongs to $\mathfrak{N}$.

2.2 Eta Quotients

On the upper half-plane $\mathbb{H}$, the Dedekind η-function, $\eta(z)$, is defined by $\eta(z) = e^{\frac{2\pi i z}{24}}\prod\limits_{n=1}^{\infty}(1 - e^{2\pi i n z})$. Let us set $q = e^{2\pi i z}$. Then it follows that

$$\eta(z) = q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n) = q^{\frac{1}{24}}F(q), \quad \text{where } F(q) = \prod_{n=1}^{\infty}(1-q^n).$$

Let $j, \kappa \in \mathbb{N}$ and $e_j \in \mathbb{Z}$. According to G. Köhler[5] an *eta product* or *eta quotient*,

[3]Stein, 2011, *Modular Forms, A Computational Approach*, p. 81.
[4]Ibid., p. 86.
[5]Köhler, 2011, *Eta Products and Theta Series Identities*, p. 31.

$f(z)$, is a finite product of eta functions of the form

$$\prod_{j=1}^{\kappa} \eta(jz)^{e_j}. \tag{6}$$

Based on this definition of an eta quotient there exists $N \in \mathbb{N}$ such that $N = \operatorname{lcm}\{j \mid 1 \leq j \leq \kappa\}$. We call such an N the *level* of an eta product. Therefore, an eta quotient will simply be understood as

$$\prod_{j \mid N} \eta(jz)^{e_j}.$$

If $k = \frac{1}{2} \sum_{j=1}^{\kappa} e_j$, then an eta quotient $f(z)$ behaves like a modular form of weight k on $\Gamma_0(N)$ with some multiplier system.

We will use eta function, eta quotient and eta product interchangeably as synonyms.

The Dedekind η-function is applied by M. Newman[6] to systematically construct modular forms for $\Gamma_0(N)$. Newman then establishes conditions (i)-(iv) in the following theorem. G. Ligozat[7] determined the order of vanishing of an η-function at all cusps of $\Gamma_0(N)$, which is condition (v) or (v′) in the following theorem.

The following theorem is formulated by L. J. P. Kilford[8] and G. Köhler[9]; it will be used to exhaustively determine η-quotients, $f(z)$, which belong to $M_k(\Gamma_0(N))$, and especially those η-quotients which are in $S_k(\Gamma_0(N))$.

Theorem 1 (M. Newman and G. Ligozat) – *Let $N \in \mathbb{N}_0$, $D(N)$ be the set of all positive divisors of N, $\delta \in D(N)$ and $r_\delta \in \mathbb{Z}$. Let furthermore $f(z) = \prod_{\delta \in D(N)} \eta^{r_\delta}(\delta z)$ be an eta quotient. If the following four conditions are satisfied*

(i) $\sum_{\delta \in D(N)} \delta r_\delta \equiv 0 \pmod{24}$,

(ii) $\sum_{\delta \in D(N)} \frac{N}{\delta} r_\delta \equiv 0 \pmod{24}$,

(iii) $\prod_{\delta \in D(N)} \delta^{r_\delta}$ *is a square in* $\mathbb{Q}$,

[6] Newman, 1957, "Construction and Application of a Class of Modular Functions"; Newman, 1959, "Construction and Application of a Class of Modular Functions II".

[7] Ligozat, 1975, "Courbes Modulaires de Genre 1".

[8] Kilford, 2008, *Modular forms: A classical and computational introduction*, p. 99.

[9] Köhler, 2011, *Eta Products and Theta Series Identities*, Cor. 2.3, p. 37.

(iv) $0 < \sum_{\delta\in D(N)} r_\delta \equiv 0 \pmod 4$,

(v) $\forall d \in D(N)$ *it holds* $\sum_{\delta\in D(N)} \frac{\gcd(\delta,d)^2}{\delta} r_\delta \geq 0$,

then $f(z) \in M_k(\Gamma_0(N))$, *where* $k = \frac{1}{2}\sum_{\delta\in D(N)} r_\delta$.

Moreover, the η*-quotient* $f(z)$ *is an element of* $S_k(\Gamma_0(N))$ *if* ***(v)*** *is replaced by*

(v′) $\forall d \in D(N)$ *it holds* $\sum_{\delta\in D(N)} \frac{\gcd(\delta,d)^2}{\delta} r_\delta > 0$.

2.3 Convolution Sums $W_{(\alpha,\beta)}(n)$

Given $\alpha,\beta \in \mathbb{N}_0$ such that $\alpha \leq \beta$, let the convolution sum be defined by (2).

Suppose in addition that $\gcd(\alpha,\beta) = \delta > 1$ for some $\delta \in \mathbb{N}_0$. Then there exist $\alpha_1,\beta_1 \in \mathbb{N}_0$ such that $\gcd(\alpha_1,\beta_1) = 1$, $\alpha = \delta\,\alpha_1$ and $\beta = \delta\,\beta_1$. Hence,

$$W_{(\alpha,\beta)}(n) = \sum_{\substack{(l,k)\in\mathbb{N}_0^2\\ \alpha\, l+\beta\, k=n}} \sigma(l)\sigma(k) = \sum_{\substack{(l,k)\in\mathbb{N}_0^2\\ \delta\,\alpha_1\, l+\delta\,\beta_1\, k=n}} \sigma(l)\sigma(k) = W_{(\alpha_1,\beta_1)}\left(\frac{n}{\delta}\right). \tag{7}$$

Therefore, we may simply assume that $\gcd(\alpha,\beta) = 1$ as does A. Alaca et al.[10].

The formula proved by M. Besge[11], J. W. L. Glaisher[12] and S. Ramanujan[13] is applied to (7) to deduce that

$$\forall\,\alpha \in \mathbb{N}_0 \quad W_{(\alpha,\alpha)}(n) = W_{(1,1)}\left(\frac{n}{\alpha}\right) = \frac{5}{12}\sigma_3\left(\frac{n}{\alpha}\right) + \left(\frac{1}{12} - \frac{1}{2\,\alpha}n\right)\sigma\left(\frac{n}{\alpha}\right). \tag{8}$$

Let $q \in \mathbb{C}$ be such that $|q| < 1$. Let furthermore χ and ψ be primitive Dirichlet characters with conductors L and R, respectively.

We assume that the primitive Dirichlet characters χ and ψ

1. are trivial whenever $\alpha\beta \in \mathfrak{N}$ holds.
2. are such that $\chi = \psi$ and that χ is a Legendre-Jacobi-Kronecker symbol otherwise.

[10] A. Alaca, Ş. Alaca, and Williams, 2006, “Evaluation of the convolution sums $\sum_{l+12m=n}\sigma(l)\sigma(m)$ and $\sum_{3l+4m=n}\sigma(l)\sigma(m)$”.

[11] Besge, 1885, “Extrait d’une lettre de M Besge à M Liouville”.

[12] Glaisher, 1862, “On the square of the series in which the coefficients are the sums of the divisors of the exponents”.

[13] Ramanujan, 1916, “On certain arithmetical functions”.

Then the following Eisenstein series hold:

$$L(q) = E_2(q) = 1 - 24\sum_{n=1}^{\infty}\sigma(n)q^n, \tag{9}$$

$$M(q) = E_4(q) = 1 + 240\sum_{n=1}^{\infty}\sigma_3(n)q^n, \tag{10}$$

$$\begin{aligned} M_\chi(q^\lambda) &= E_4(q^\lambda)\otimes\chi \\ &= \chi(\lambda)\left(C_0 + \sum_{n=1}^{\infty}\chi(n)\,\sigma_3(n)\,q^{\lambda n}\right) \\ &= \chi(\lambda)\,C_0 + \sum_{n=1}^{\infty}\chi(\lambda\,n)\,\sigma_3(n)\,q^{\lambda n} \\ &= \chi(\lambda)\,C_0 + \sum_{n=1}^{\infty}\chi(n)\,\sigma_3\left(\frac{n}{\lambda}\right)q^n, \end{aligned} \tag{11}$$

where $\lambda \in \mathbb{N}_0$,

$$C_0 = \begin{cases} 0 & \text{if } L > 1 \\ -\frac{B_{4,\chi}}{8} & \text{if } L = 1 \end{cases}$$

and $B_{4,\chi}$ are the specially generalized Bernoulli numbers.

Note that $M(q)$ is a special case of (5) or (11) and hold if $\alpha\beta \in \mathfrak{N}$. We state two relevant results for the sequel of this work.

Lemma 1 – *Let $\alpha, \beta \in \mathbb{N}_0$. Then*

$$(\alpha\,L(q^\alpha) - \beta\,L(q^\beta))^2 \in M_4(\Gamma_0(\alpha\beta)).$$

Proof. If $\alpha = \beta$, then trivially $0 = (\alpha\,L(q^\alpha) - \alpha\,L(q^\alpha))^2 \in M_4(\Gamma_0(\alpha))$ and there is nothing to prove. Therefore, we may suppose that $\alpha \neq \beta$ in the sequel. We apply the result proved by W. A. Stein[14] to deduce $L(q) - \alpha\,L(q^\alpha) \in M_2(\Gamma_0(\alpha)) \subseteq M_2(\Gamma_0(\alpha\beta))$ and $L(q) - \beta\,L(q^\beta) \in M_2(\Gamma_0(\beta)) \subseteq M_2(\Gamma_0(\alpha\beta))$. Therefore,

$$\alpha\,L(q^\alpha) - \beta\,L(q^\beta) = (L(q) - \beta\,L(q^\beta)) - (L(q) - \alpha\,L(q^\alpha)) \in M_2(\Gamma_0(\alpha\beta))$$

and so $(\alpha\,L(q^\alpha) - \beta\,L(q^\beta))^2 \in M_4(\Gamma_0(\alpha\beta))$. □

[14] Stein, 2011, *Modular Forms, A Computational Approach*, Thrms 5.8, 5.9, p. 86.

Theorem 2 – *Let $\alpha, \beta, N \in \mathbb{N}_0$ be such that $N = \alpha\beta, \alpha < \beta$, and α and β are relatively prime. The case $\alpha = \beta$ is discussed above. Then*

$$(\alpha\, L(q^{\alpha}) - \beta\, L(q^{\beta}))^2 = (\alpha - \beta)^2 + \sum_{n=1}^{\infty}\Bigg(240\,\alpha^2\,\sigma_3\left(\frac{n}{\alpha}\right) + 240\,\beta^2\,\sigma_3\left(\frac{n}{\beta}\right)$$
$$+ 48\,\alpha\,(\beta - 6n)\,\sigma\left(\frac{n}{\alpha}\right) + 48\,\beta\,(\alpha - 6n)\,\sigma\left(\frac{n}{\beta}\right) - 1152\,\alpha\beta\, W_{(\alpha,\beta)}(n)\Bigg)q^n. \tag{12}$$

Proof. We first observe that

$$(\alpha\, L(q^{\alpha}) - \beta\, L(q^{\beta}))^2 = \alpha^2\, L^2(q^{\alpha}) + \beta^2\, L^2(q^{\beta}) - 2\,\alpha\beta\, L(q^{\alpha})L(q^{\beta}). \tag{13}$$

J. W. L. Glaisher[15] has proved the following identity

$$L^2(q) = 1 + \sum_{n=1}^{\infty}\Big(240\,\sigma_3(n) - 288\, n\,\sigma(n)\Big)q^n \tag{14}$$

which we apply to deduce

$$L^2(q^{\alpha}) = 1 + \sum_{n=1}^{\infty}\left(240\,\sigma_3\left(\frac{n}{\alpha}\right) - 288\,\frac{n}{\alpha}\,\sigma\left(\frac{n}{\alpha}\right)\right)q^n \tag{15}$$

and

$$L^2(q^{\beta}) = 1 + \sum_{n=1}^{\infty}\left(240\,\sigma_3\left(\frac{n}{\beta}\right) - 288\,\frac{n}{\beta}\,\sigma\left(\frac{n}{\beta}\right)\right)q^n. \tag{16}$$

Since

$$\left(\sum_{n=1}^{\infty}\sigma\left(\frac{n}{\alpha}\right)q^n\right)\left(\sum_{n=1}^{\infty}\sigma\left(\frac{n}{\beta}\right)q^n\right) = \sum_{n=1}^{\infty}\Bigg(\sum_{\alpha k + \beta l = n}\sigma(k)\sigma(l)\Bigg)q^n = \sum_{n=1}^{\infty} W_{(\alpha,\beta)}(n)q^n,$$

we conclude, when using an accordingly modified version of (9), that

$$L(q^{\alpha})L(q^{\beta}) = 1 - 24\sum_{n=1}^{\infty}\sigma\left(\frac{n}{\alpha}\right)q^n - 24\sum_{n=1}^{\infty}\sigma\left(\frac{n}{\beta}\right)q^n + 576\sum_{n=1}^{\infty} W_{(\alpha,\beta)}(n)q^n. \tag{17}$$

Therefore,

$$\Big(\alpha\, L(q^{\alpha}) - \beta\, L(q^{\beta})\Big)^2 = (\alpha - \beta)^2 + \sum_{n=1}^{\infty}\Bigg(240\,\alpha^2\,\sigma_3\left(\frac{n}{\alpha}\right) + 240\,\beta^2\,\sigma_3\left(\frac{n}{\beta}\right)$$
$$+ 48\,\alpha\,(\beta - 6n)\,\sigma\left(\frac{n}{\alpha}\right) + 48\,\beta\,(\alpha - 6n)\,\sigma\left(\frac{n}{\beta}\right) - 1152\,\alpha\beta\, W_{(\alpha,\beta)}(n)\Bigg)q^n$$

as asserted. $\square$

3 Evaluating $W_{(\alpha,\beta)}(n)$, where $\alpha\beta \in \mathbb{N}_0$

We carry out an explicit formula for the convolution sum $W_{(\alpha,\beta)}(n)$.

3.1 Bases of $E_4(\Gamma_0(\alpha\beta))$ and $S_4(\Gamma_0(\alpha\beta))$

Let $\mathcal{D}(\alpha\beta)$ denote the set of all positive divisors of $\alpha\beta$.

We apply the dimension formulae as discussed in T. Miyake[16] or W. A. Stein[17] to conclude that

- for the space of Eisenstein forms,

$$\dim(E_4(\Gamma_0(\alpha\beta))) = \sum_{d|\alpha\beta} \varphi\Big(\gcd\Big(d, \frac{\alpha\beta}{d}\Big)\Big) = m_E, \tag{18}$$

  where $m_E \in \mathbb{N}_0$ and φ is the Euler's totient function.

  We observe that, if $\alpha\beta \in \mathfrak{N}$, then

$$\dim(E_4(\Gamma_0(\alpha\beta))) = \sum_{\delta|\alpha\beta} \varphi\Big(\gcd\Big(\delta, \frac{\alpha\beta}{\delta}\Big)\Big) = \sum_{\delta|\alpha\beta} 1 = \sigma_0(\alpha\beta) = d(\alpha\beta). \tag{19}$$

- for the space of cusp forms, $\dim(S_4(\Gamma_0(\alpha\beta))) = m_S$, where $m_S \in \mathbb{N}$.

We use Theorem 1 (i)–(v′) to exhaustively determine as many elements of the space $S_4(\Gamma_0(\alpha\beta))$ as possible. From these elements of the space $S_4(\Gamma_0(\alpha\beta))$ we select relevant ones for the purpose of the determination of a basis of this space. The proof of the following theorem provides a method to effectively determine such a basis.

The so-determined basis of the vector space of cusp forms is in general not unique. However, due to the change of basis which is an automorphism, it is sufficient to only consider this basis for our purpose.

Let $\mathcal{C}$ denote a set of primitive Dirichlet characters $\chi(n) = (\frac{m}{n})$, where $m, n \in \mathbb{Z}$ with m fixed and $(\frac{m}{n})$ is the Legendre-Jacobi-Kronecker symbol. Let furthermore $D_\chi(\alpha\beta) \subseteq \mathcal{D}(\alpha\beta)$ denote the subset of $\mathcal{D}(\alpha\beta)$ associated with the Dirichlet character χ. Then it is obvious that not every primitive Dirichlet character is a good candidate which, applied in (11), will make it constitute a basis element of the space of Eisenstein series for a given level that belongs to $\mathbb{N}_0 \setminus \mathfrak{N}$. For example, if the levels are 25 and 32, the primitive Dirichlet characters $(\frac{5}{n})$ and $(\frac{-4}{n})$ will not permit one to build a basis of $E_4(\Gamma_0(25))$ and $E_4(\Gamma_0(32))$, respectively.

[15] Glaisher, 1862, "On the square of the series in which the coefficients are the sums of the divisors of the exponents".

[16] Miyake, 1989, *Modular Forms*, Thrm 2.5.2, p. 60.

[17] Stein, 2011, *Modular Forms, A Computational Approach*, Prop. 6.1, p. 91.

Let i, κ be natural numbers. It is standard to express a natural number in the for $\prod_{i=1}^{\kappa} p_i^{e_i}$, where p_i is a prime number and e_i is in $\mathbb{N}_0$, modulo a permutation of the primes p_i. Therefore, we will also in the following use this form to express a level $\alpha\beta \in \mathbb{N}_0 \setminus \mathfrak{N}$.

Definition 2 – Let $i, \kappa \in \mathbb{N}_0$ and $n \in \mathbb{N}$. Let $C \in \mathbb{Z}$ be fixed. Suppose that the level $\alpha\beta \in \mathbb{N}_0 \setminus \mathfrak{N}$ is fixed and of the form $\prod_{i=1}^{\kappa} p_i^{e_i}$, where p_i is a prime number and e_i is in $\mathbb{N}_0$. We say that the primitive Dirichlet character $\chi(n) = (\frac{C}{n})$ *annihilates* $E_4(\Gamma_0(\alpha\beta))$ or is an *annihilator* of $E_4(\Gamma_0(\alpha\beta))$ if for some $1 \leq j \leq \kappa$ we have $1 < p_j^{e_j} \in \mathbb{N}_0 \setminus \mathfrak{N}$ and $M_\chi(q^\delta)$ vanishes for all $1 < \delta$ positive divisor of $p_j^{e_j}$.

A set $\mathcal{C}$ of primitive Dirichlet characters *annihilates* $E_4(\Gamma_0(\alpha\beta))$ or is an *annihilator* of $E_4(\Gamma_0(\alpha\beta))$ if each $\chi(n) \in \mathcal{C}$ is an annihilator of $E_4(\Gamma_0(\alpha\beta))$.

To illustrate the above definition, suppose that $\alpha\beta = 2 \times 3^2$ and the primitive Dirichlet characters is $\chi(n) = (\frac{-3}{n})$. Then $C = -3$ so that |C|is a positive divisor of $9 = 3^2$. For all $1 < \delta \in \mathcal{D}(9) = \{1, 3, 9\}$ one easily verify when applying (11) that

$$M_\chi(q^\delta) = \sum_{n=1}^{\infty} \chi(n)\,\sigma_3\left(\frac{n}{\delta}\right) q^n = 0,$$

since it holds that

$$\chi(n) = \left(\frac{-3}{n}\right) = \begin{cases} -1 & \text{if } n \equiv 2 \pmod 3, \\ 0 & \text{if } \gcd(3, n) \neq 1, \\ 1 & \text{if } n \equiv 1 \pmod 3 \end{cases}$$

and

$$\sigma_3\left(\frac{n}{\delta}\right) = \begin{cases} 0 & \text{if } \frac{n}{\delta} \notin \mathbb{N}_0, \\ \text{nonzero} & \gcd(3, n) \neq 1. \end{cases}$$

Hence, the primitive Dirichlet character $\chi(n) = (\frac{-3}{n})$ is an annihilator of $E_4(\Gamma_0(2 \times 3^2))$.

The following theorem provides a strong criterion for the selection of a primitive Dirichlet character for a given level $\alpha\beta \in \mathbb{N}_0 \setminus \mathfrak{N}$.

Theorem 3 – *Let i, κ be in $\mathbb{N}_0$. Let $C \in \mathbb{Z}$ be fixed. Let $\chi(n) = (\frac{C}{n})$ be a primitive Dirichlet character with conductor $|C| > 1$ and let the level $\alpha\beta \in \mathbb{N}_0 \setminus \mathfrak{N}$ be fixed and of the form $\prod_{i=1}^{\kappa} p_i^{e_i}$, where p_i is a prime number and e_i is in $\mathbb{N}_0$. Suppose furthermore that $p_j^{e_j} \in \mathbb{N}_0 \setminus \mathfrak{N}$ is a positive divisor of $\alpha\beta$ for some $1 \leq j \leq \kappa$. If the conductor of $\chi(n)$ is a positive divisor of p_j and hence of the level $\alpha\beta$, then $\chi(n) = (\frac{C}{n})$ is an annihilator of $E_4(\Gamma_0(\alpha\beta))$.*

Proof. Suppose that $\alpha\beta \in \mathbb{N}_0 \setminus \mathfrak{N}$ is fixed and of the form $\prod_{i=1}^{\kappa} p_i^{e_i}$, where p_i is a prime number and e_i is in $\mathbb{N}_0$. As an immediate consequence of the structure of $\alpha\beta$ there exists $1 \le j \le \kappa$ such that $p_j^{e_j} \in \mathbb{N}_0 \setminus \mathfrak{N}$ is a positive divisor of $\alpha\beta$. Now, it is well-known that for each $1 \le f \le e_j$ it holds that p_j^f is a positive divisor of $p_j^{e_j}$. Since the conductor of $\chi(n)$ is a positive divisor of the level $p_j^{e_j}$, there exists $1 \le f \le e_j$ such that $p_j^f = |C|$. On the other hand, it holds that

$$\chi(n) = \left(\frac{C}{n}\right) = \begin{cases} 0 & \text{if } \gcd(|C|, n) \neq 1, \\ \text{nonzero} & \text{otherwise.} \end{cases} \tag{20}$$

For each $1 < \delta \in \mathcal{D}(p_j^{e_j})$ it holds that $\gcd(|C|, \delta) \neq 1$; hence, $(\frac{C}{n}) = 0$ for each $1 < \delta \in \mathcal{D}(p_j^{e_j})$ such that $\gcd(|C|, \delta) \neq 1$. Since the conductor of $\chi(n)$ is greater than one, it follows that $C_0 = 0$. Then we apply (11) we obtain

$$M_\chi(q^\delta) = \sum_{n=1}^{\infty} \chi(n)\sigma_3\left(\frac{n}{\delta}\right) q^n.$$

Since it also holds that

$$\sigma_3\left(\frac{n}{\delta}\right) = \begin{cases} 0 & \text{if } \frac{n}{\delta} \notin \mathbb{N}_0, \\ \text{nonzero} & \gcd(|C|, n) \neq 1, \end{cases} \tag{21}$$

we obtain the stated result by simply putting altogether; that is $M_\chi(q^\delta) = 0$ for all $1 < \delta \in \mathcal{D}(p_j^{e_j})$. □

If $\alpha\beta \in \mathfrak{N}$ holds, then the primitive Dirichlet characters are trivial. Therefore, the set $\mathcal{C}$ is empty. Hence, the case where $\alpha\beta \in \mathfrak{N}$ holds is a special case of the following theorem.

Theorem 4 – ***(a)*** *Let $\mathcal{C}$ be a set of primitive Dirichlet characters such that for each $\chi(n) \in \mathcal{C}$ it holds that $\chi(n)$ is not an annihilator of $E_4(\Gamma_0(\alpha\beta))$. Then the set $\mathcal{B}_E = \{M(q^t) \mid t \in \mathcal{D}(\alpha\beta)\} \cup \bigcup_{\chi \in \mathcal{C}} \{M_\chi(q^t) \mid t \in D_\chi(\alpha\beta)\}$ is a basis of $E_4(\Gamma_0(\alpha\beta))$.*

(b) *Let $1 \le i \le m_S$ be positive integers, $\delta \in D(\alpha\beta)$ and $(r(i,\delta))_{i,\delta}$ be a table of the powers of $\eta(\delta z)$. Let furthermore $\mathfrak{B}_{\alpha\beta,i}(q) = \prod_{\delta|\alpha\beta} \eta^{r(i,\delta)}(\delta z)$ be selected elements of $S_4(\Gamma_0(\alpha\beta))$. Then the set $\mathcal{B}_S = \{\mathfrak{B}_{\alpha\beta,i}(q) \mid 1 \le i \le m_S\}$ is a basis of $S_4(\Gamma_0(\alpha\beta))$.*

(c) *The set $\mathcal{B}_M = \mathcal{B}_E \cup \mathcal{B}_S$ constitutes a basis of $M_4(\Gamma_0(\alpha\beta))$.*

Remark 1 – (r1) Each $\mathfrak{B}_{\alpha\beta,i}(q)$ can be expressed in the form $\sum\limits_{n=1}^{\infty}\mathfrak{b}_{\alpha\beta,i}(n)q^n$, where $1 \leq i \leq m_S$ and for each $n \geq 1$ the coefficient $\mathfrak{b}_{\alpha\beta,i}(n)$ is an integer.

(r2) If we divide the sum that results from Theorem 1 (v′), when $d = N$, by 24, then we obtain the smallest positive degree of q in $\mathfrak{B}_{\alpha\beta,i}(q)$.

The existence of a basis of the space of cusp forms in terms of theta series has been proved by A. Pizer[18] for all square-free levels. M. Newman and G. Ligozat, see Theorem 1 (i)–(v′), have determined a method to find as many theta series belonging to a space of cusp forms as possible. Therefore, the proof of this theorem is essentially restricted to show that the selected elements of the space of modular forms of the given level are linearly independent.

Proof. We only consider the case where $\alpha\beta$ is in $\mathbb{N}_0 \setminus \mathfrak{N}$ since the case $\alpha\beta \in \mathfrak{N}$ is proved similarly using the fact that $\mathcal{C} = \varnothing$.

(a) W. A. Stein[19] has shown that for each t positive divisor of $\alpha\beta$ it holds that $M(q^t)$ is in $M_4(\Gamma_0(t))$. Since $M_4(\Gamma_0(t))$ is a vector space and the set $\mathcal{C}$ of primitive Dirichlet characters does not annihilates $E_4(\Gamma_0(\alpha\beta))$, it also holds for each primitive Dirichlet character $\chi(n) = (\frac{C}{n}) \in \mathcal{C}$ and $t \in D_\chi(\alpha\beta)$ that $0 \neq M_\chi(q^t)$ is in $M_4(\Gamma_0(t))$. Since the dimension of $E_4(\Gamma_0(\alpha\beta))$ is finite, it suffices to show that $\mathcal{B}_E$ is linearly independent. Suppose that for each $\chi \in \mathcal{C}, s \in D_\chi(\alpha\beta)$ we have $z(\chi)_s \in \mathbb{C}$ and that for each $t \in D(\alpha\beta)$ we have $x_t \in \mathbb{C}$. Then

$$\sum_{t|\alpha\beta} x_t\, M(q^t) + \sum_{\chi\in\mathcal{C}}\Bigg(\sum_{s\in D_\chi(\alpha\beta)} z(\chi)_s M_\chi(q^s)\Bigg) = 0.$$

We recall that χ is a Legendre-Jacobi-Kronecker symbol; therefore, for all $0 \neq a \in \mathbb{Z}$ it holds that $(\frac{a}{0}) = 0$. Since the primitive Dirichlet characters χ and ψ are not trivial and have the conductors L and R which we may assume greater than one, we can deduce that $C_0 = 0$ in (11). Then we equate the coefficients of q^n for $n \in D(\alpha\beta) \cup \bigcup\limits_{\chi\in\mathcal{C}} \{s|s \in D_\chi(\alpha\beta)\}$ to obtain the homogeneous system of linear equations in m_E unknowns:

$$\sum_{u|\alpha\beta} \sigma_3\Big(\frac{t}{u}\Big)x_u + \sum_{\chi\in\mathcal{C}}\sum_{v\in D_\chi(\alpha\beta)} \chi(t)\sigma_3\Big(\frac{t}{v}\Big)Z(\chi)_v = 0, \qquad t \in D(\alpha\beta).$$

The determinant of the matrix of this homogeneous system of linear equations is not zero. Hence, the unique solution is $x_t = z(\chi)_s = 0$ for all $t \in D(\alpha\beta)$ and for all $\chi \in \mathcal{C}, s \in D_\chi(\alpha\beta)$. So, the set $\mathcal{B}_E$ is linearly independent and hence is a basis of $E_4(\Gamma_0(\alpha\beta))$.

[18] Pizer, 1976, “The representability of modular forms by theta series”.

(b) We show that each $\mathfrak{B}_{\alpha\beta,i}(q)$, where $1 \leq i \leq m_S$, is in the space $S_4(\Gamma_0(\alpha\beta))$. This is obviously the case since $\mathfrak{B}_{\alpha\beta,i}(q), 1 \leq i \leq m_S$, are obtained using an exhaustive search which applies items (i)–(v′) in Theorem 1.

Since the dimension of $S_4(\Gamma_0(\alpha\beta))$ is finite, it suffices to show that the set $\mathcal{B}_S$ is linearly independent. Suppose that $x_i \in \mathbb{C}$ and $\sum_{i=1}^{m_S} x_i\,\mathfrak{B}_{\alpha\beta,i}(q) = 0$. Then $\sum_{i=1}^{m_S} x_i\,\mathfrak{B}_{\alpha\beta,i}(q) = \sum_{n=1}^{\infty}(\sum_{i=1}^{m_S} x_i\,\mathfrak{b}_{\alpha\beta,i}(n))q^n = 0$ which gives the homogeneous system of m_S linear equations in m_S unknowns:

$$\sum_{i=1}^{m_S} \mathfrak{b}_{\alpha\beta,i}(n)\,x_i = 0, \qquad 1 \leq n \leq m_S. \tag{22}$$

Two cases arise:

The smallest degree of $\mathfrak{B}_{\alpha\beta,i}(q)$ **is** i **for each** $1 \leq i \leq m_S$ Then the square matrix which corresponds to this homogeneous system of m_S linear equations is triangular with 1's on the diagonal. Hence, the determinant of that matrix is 1 and so the unique solution is $x_i = 0$ for all $1 \leq i \leq m_S$.

The smallest degree of $\mathfrak{B}_{\alpha\beta,i}(q)$ **is** i **for** $1 \leq i < m_S$ Let n' be the largest positive integer such that $1 \leq i \leq n' < m_S$. Let $\mathcal{B}_S' = \{\mathfrak{B}_{\alpha\beta,i}(q) \mid 1 \leq i \leq n'\}$ and $\mathcal{B}_S'' = \{\mathfrak{B}_{\alpha\beta,i}(q) \mid n' < i \leq m_S\}$. Then $\mathcal{B}_S = \mathcal{B}_S' \cup \mathcal{B}_S''$ and we may consider $\mathcal{B}_S$ as an ordered set. By the case above, the set $\mathcal{B}_S'$ is linearly independent. Hence, the linear independence of the set $\mathcal{B}_S$ depends on that of the set $\mathcal{B}_S''$. Let $A = (\mathfrak{b}_{\alpha\beta,i}(n))$ be the $m_S \times m_S$ matrix in (22). If $\det(A) \neq 0$, then $x_i = 0$ for all $1 \leq i \leq m_S$ and we are done. Suppose that $\det(A) = 0$. Then for some $n' < k \leq m_S$ there exists $\mathfrak{B}_{\alpha\beta,k}(q)$ which is causing the system of equations to be inconsistent. We substitute $\mathfrak{B}_{\alpha\beta,k}(q)$ with, say $\mathfrak{B}'_{\alpha\beta,k}(q)$, which does not occur in $\mathcal{B}_S$ and compute the determinant of the new matrix A. Since there are finitely many $\mathfrak{B}_{\alpha\beta,k}(q)$ with $n' < k \leq m_S$ that may cause the system of linear equations to be inconsistent and finitely many elements of $S_4(\Gamma_0(\alpha\beta)) \setminus \mathcal{B}_S$, the procedure will terminate with a consistent system of linear equations. Hence, we will find a set of linearly independent elements of $S_4(\Gamma_0(\alpha\beta))$.

Therefore, the set $\{\mathfrak{B}_{\alpha\beta,i}(q) \mid 1 \leq i \leq m_S\}$ is linearly independent and hence is a basis of $S_4(\Gamma_0(\alpha\beta))$.

(c) Since $M_4(\Gamma_0(\alpha\beta)) = E_4(\Gamma_0(\alpha\beta)) \oplus S_4(\Gamma_0(\alpha\beta))$, the result follows from (a) and (b). □

[19] Stein, 2011, *Modular Forms, A Computational Approach*, Thrms 5.8 and 5.9, p. 86.

If $\mathcal{C} = \emptyset$, that means that the primitive Dirichlet character is trivial, then the formulation and the proof of Theorem 4 is straightforward.

If the level $\alpha\beta$ belongs to the class $\mathfrak{N}$, then Theorem 4 (a) is provable by induction on the set of positive divisors of $\alpha\beta$; see for example E. Ntienjem[20]. Note that each positive divisor of $\alpha\beta$ is in the class $\mathfrak{N}$ whenever the level $\alpha\beta$ belongs to the class $\mathfrak{N}$. This nice property does not hold in general if the level $\alpha\beta$ belongs to the class $\mathbb{N} \setminus \mathfrak{N}$. For example 45 is an element of the class $\mathbb{N} \setminus \mathfrak{N}$; however, 15 which is a positive divisor of 45 does not belong to $\mathbb{N} \setminus \mathfrak{N}$.

The proof of Theorem 4 (b) provides us with an effective method to determine a basis of the space of cusp forms of level $\alpha\beta$ whenever $\alpha\beta$ belongs to $\mathbb{N}_0$.

3.2 Evaluating the Convolution Sum $W_{(\alpha,\beta)}(n)$

We recall that it is sufficient to assume that the primitive Dirichlet character χ is not trivial since the case χ trivial can be concluded as an immediate corollary.

Lemma 2 – *Let $\alpha, \beta \in \mathbb{N}$ be such that $\gcd(\alpha,\beta) = 1$. Let furthermore $\mathcal{B}_M = \mathcal{B}_E \cup \mathcal{B}_S$ be a basis of $M_4(\Gamma_0(\alpha\beta))$. Then there exist $X_\delta, Z(\chi)_s, Y_j \in \mathbb{C}$, where $1 \le j \le m_S, \chi \in \mathcal{C}, s \in D_\chi(\alpha\beta)$ and $\delta|\alpha\beta$, such that*

$$\begin{aligned}(\alpha L(q^\alpha) - \beta L(q^\beta))^2 &= \sum_{\delta|\alpha\beta} X_\delta + \sum_{\chi\in\mathcal{C}} \sum_{s\in D_\chi(\alpha\beta)} C_0 Z(\chi)_s \\ &+ \sum_{n=1}^{\infty} \Bigg(240 \sum_{\delta|\alpha\beta} \sigma_3\Big(\frac{n}{\delta}\Big) X_\delta + \sum_{\chi\in\mathcal{C}} \sum_{s\in D_\chi(\alpha\beta)} \sigma_3\Big(\frac{n}{s}\Big)\chi(n) Z(\chi)_s + \sum_{j=1}^{m_S} \mathfrak{b}_j(n) Y_j\Bigg) q^n.\end{aligned} \tag{23}$$

Proof. That $(\alpha L(q^\alpha) - \beta L(q^\beta))^2 \in M_4(\Gamma_0(\alpha\beta))$ follows from Lemma 1. Hence, by Theorem 4 (c), there exist $X_\delta, Z(\chi)_s, Y_j \in \mathbb{C}, 1 \le j \le m_S, \chi \in \mathcal{C}, s \in D_\chi(\alpha\beta)$ and δ is a divisor of $\alpha\beta$, such that

$$\begin{aligned}(\alpha L(q^\alpha) - \beta L(q^\beta))^2 &= \sum_{\delta|\alpha\beta} X_\delta M(q^\delta) + \sum_{\chi\in\mathcal{C}} \sum_{s\in D_\chi(\alpha\beta)} Z(\chi)_s M_\chi(q^s) + \sum_{j=1}^{m_S} Y_j \mathfrak{B}_j(q) \\ &= \sum_{\delta|\alpha\beta} X_\delta + \sum_{\chi\in\mathcal{C}} \sum_{s\in D_\chi(\alpha\beta)} C_0 Z(\chi)_s + \sum_{n=1}^{\infty} \Bigg(240 \sum_{\delta|\alpha\beta} \sigma_3\Big(\frac{n}{\delta}\Big) X_\delta \\ &\quad + \sum_{\chi\in\mathcal{C}} \sum_{s\in D_\chi(\alpha\beta)} \chi(n)\sigma_3\Big(\frac{n}{s}\Big) Z(\chi)_s + \sum_{j=1}^{m_S} \mathfrak{b}_j(n) Y_j\Bigg) q^n.\end{aligned}$$

[20] Ntienjem, 2017a, "Evaluation of the Convolution Sum involving the Sum of Divisors Function for 22, 44 and 52".

3. Evaluating $W_{(\alpha,\beta)}(n)$, where $\alpha\beta\in\mathbb{N}_0$

We equate the right hand side of (23) with that of (12) to obtain

$$\sum_{n=1}^{\infty}\Bigg(240\sum_{\delta|\alpha\beta}X_\delta\,\sigma_3\Big(\frac{n}{\delta}\Big)+\sum_{\chi\in\mathcal{C}}\Big(\sum_{s\in D_\chi(\alpha\beta)}\chi(n)\,\sigma_3\Big(\frac{n}{s}\Big)Z(\chi)_s\Big)+\sum_{j=1}^{m_S}Y_j\,\mathfrak{b}_j(n)\Bigg)q^n$$
$$=\sum_{n=1}^{\infty}\Bigg(240\,\alpha^2\,\sigma_3\Big(\frac{n}{\alpha}\Big)+240\,\beta^2\,\sigma_3\Big(\frac{n}{\beta}\Big)+48\,\alpha\,(\beta-6n)\,\sigma\Big(\frac{n}{\alpha}\Big)$$
$$+\,48\,\beta\,(\alpha-6n)\,\sigma\Big(\frac{n}{\beta}\Big)-1152\,\alpha\beta\,W_{(\alpha,\beta)}(n)\Bigg)q^n.$$

We then take the coefficients of q^n such that n is in $D(\alpha\beta)$ and $1\le n\le m_S$, but as many as the unknown, $X_1,\ldots,X_{\alpha\beta}$, $Z(\chi)_s$ for all $\chi\in\mathcal{C}, s\in D_\chi(\alpha\beta)$, and $Y_1,\ldots,Y_{m_S}$, to obtain a system of m_E+m_S linear equations whose unique solution determines the values of the unknowns. Hence, we obtain the result. $\square$

For the following theorem, let for the sake of simplicity $X_\delta, Z(\chi)_s$ and Y_j stand for their values obtained in the previous lemma.

Theorem 5 – *Let n be a positive integer. Then*

$$W_{(\alpha,\beta)}(n)=-\frac{5}{24\,\alpha\beta}\sum_{\substack{\delta|\alpha\beta\\ \delta\neq\alpha,\beta}}\sigma_3\Big(\frac{n}{\delta}\Big)X_\delta-\frac{1}{1152\,\alpha\beta}\sum_{\chi\in\mathcal{C}}\sum_{s\in D_\chi(\alpha\beta)}Z(\chi)_s\,\sigma_3\Big(\frac{n}{s}\Big)$$
$$+\frac{5}{24\,\alpha\,\beta}(\alpha^2-X_\alpha)\,\sigma_3\Big(\frac{n}{\alpha}\Big)+\frac{5}{24\,\alpha\,\beta}(\beta^2-X_\beta)\,\sigma_3\Big(\frac{n}{\beta}\Big)$$
$$-\sum_{j=1}^{m_S}\frac{1}{1152\,\alpha\beta}\,\mathfrak{b}_j(n)\,Y_j+\Big(\frac{1}{24}-\frac{1}{4\beta}n\Big)\sigma\Big(\frac{n}{\alpha}\Big)+\Big(\frac{1}{24}-\frac{1}{4\alpha}n\Big)\sigma\Big(\frac{n}{\beta}\Big).$$

Proof. We equate the right hand side of (23) with that of (12) to yield

$$1152\,\alpha\beta\,W_{(\alpha,\beta)}(n)=-240\sum_{\delta|\alpha\beta}\sigma_3\Big(\frac{n}{\delta}\Big)X_\delta-\sum_{\chi\in\mathcal{C}}\sum_{s\in D_\chi(\alpha\beta)}Z(\chi)_s\,\sigma_3\Big(\frac{n}{s}\Big)$$
$$-\sum_{j=1}^{m_S}\mathfrak{b}_j(n)\,Y_j+240\,\alpha^2\,\sigma_3\Big(\frac{n}{\alpha}\Big)+240\,\beta^2\,\sigma_3\Big(\frac{n}{\beta}\Big)$$
$$+\,48\,\alpha\,(\beta-6n)\,\sigma\Big(\frac{n}{\alpha}\Big)+48\,\beta\,(\alpha-6n)\,\sigma\Big(\frac{n}{\beta}\Big).$$

We then solve for $W_{(\alpha,\beta)}(n)$ to obtain the stated result. $\square$

Remark 2 – **(a)** We observe that the following part of Theorem 5 depends only on n, α and β but not on the basis of the modular space $M_4(\Gamma_0(\alpha\beta))$:

$$\left(\frac{1}{24}-\frac{1}{4\beta}n\right)\sigma\left(\frac{n}{\alpha}\right)+\left(\frac{1}{24}-\frac{1}{4\alpha}n\right)\sigma\left(\frac{n}{\beta}\right).$$

(b) For all $\chi \in \mathcal{C}$ and for all $s \in D_{\chi}(\alpha\beta)$ the value of $Z(\chi)_s$ appears to be zero in all explicit examples evaluated as yet. Will the value of $Z(\chi)_s$ always vanish for all $\alpha\beta$ belonging to $\mathbb{N}_0 \setminus \mathfrak{N}$?

We now have the prerequisite to determine a formula for the number of representations of a positive integer n by an octonary quadratic form.

4 Number of Representations of a Positive Integer for this Class of Levels

We discuss in this section the determination of formulae for the number of representations of a positive integer by the octonary quadratic forms (3) and (4), respectively.

4.1 Representations of a Positive Integer by the Octonary Quadratic Form (3)

We determine formulae for the number of representations of a positive integer by the octonary quadratic forms (3).

Formulae for the Number of Representations by (3)

Let $n \in \mathbb{N}$ and let the number of representations of n by the quaternary quadratic form $\sum\limits_{i=1}^{4} x_i^2$ be defined by $r_4(n) = \text{card}(\{(x_1, x_2, x_3, x_4) \in \mathbb{Z}^4 \mid n = \sum\limits_{i=1}^{4} x_i^2\})$. It follows from the definition that $r_4(0) = 1$. Jacobi's identity $r_4(n)$, $n \in \mathbb{N}_0$, is

$$r_4(n) = 8\sigma(n) - 32\sigma\left(\frac{n}{4}\right), \tag{24}$$

a proof of which is given in K. S. Williams[21].

Now, let the number of representations of n by the octonary quadratic form (3) be

$$N_{(a,b)}(n) = \text{card}\left(\left\{(x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8) \in \mathbb{Z}^8 \;\middle|\; n = a\sum_{i=1}^{4} x_i^2 + b\sum_{i=5}^{8} x_i^2\right\}\right),$$

where $a, b \in \mathbb{N}_0$.

[21] Williams, 2011, *Number Theory in the Spirit of Liouville*, Thrm 9.5, p. 83.

It immediately follows from the definition of $N_{(a,b)}(n)$ that if $a,b \in \mathbb{N}_0$ are such that $\gcd(a,b) = d > 1$ for some $d \in \mathbb{N}_0$, then $N_{(a,b)}(n) = N_{(\frac{a}{d},\frac{b}{d})}\left(\frac{n}{d}\right)$. Therefore, one can simply assume that $a,b \in \mathbb{N}_0$ are relatively prime.

We then derive the following result:

Theorem 6 – *Let $n \in \mathbb{N}$ and let $a,b \in \mathbb{N}_0$ be relatively prime. Then*

$$N_{(a,b)}(n) = 8\sigma\left(\frac{n}{a}\right) - 32\sigma\left(\frac{n}{4a}\right) + 8\sigma\left(\frac{n}{b}\right) - 32\sigma\left(\frac{n}{4b}\right) \\ + 64\, W_{(a,b)}(n) + 1024\, W_{(a,b)}\left(\frac{n}{4}\right) - 256\,(W_{(4a,b)}(n) + W_{(a,4b)}(n)).$$

Proof. We have

$$N_{(a,b)}(n) = \sum_{\substack{(l,m)\in\mathbb{N}^2\\ a\,l+b\,m=n}} r_4(l)r_4(m) = r_4\left(\frac{n}{a}\right)r_4(0) + r_4(0)r_4\left(\frac{n}{b}\right) + \sum_{\substack{(l,m)\in\mathbb{N}_0^2\\ a\,l+b\,m=n}} r_4(l)r_4(m).$$

We make use of (24) to obtain

$$N_{(a,b)}(n) = 8\sigma\left(\frac{n}{a}\right) - 32\sigma\left(\frac{n}{4a}\right) + 8\sigma\left(\frac{n}{b}\right) - 32\sigma\left(\frac{n}{4b}\right) \\ + \sum_{\substack{(l,m)\in\mathbb{N}_0^2\\ al+bm=n}} \left(8\sigma(l) - 32\sigma\left(\frac{l}{4}\right)\right)(8\sigma(m) - 32\sigma\left(\frac{m}{4}\right)).$$

We know that

$$\left(8\sigma(l) - 32\sigma\left(\frac{l}{4}\right)\right)\left(8\sigma(m) - 32\sigma\left(\frac{m}{4}\right)\right) \\ = 64\sigma(l)\sigma(m) - 256\sigma\left(\frac{l}{4}\right)\sigma(m) - 256\sigma(l)\sigma\left(\frac{m}{4}\right) + 1024\sigma\left(\frac{l}{4}\right)\sigma\left(\frac{m}{4}\right).$$

In the sequel of this proof, we assume that the evaluation of

$$W_{(a,b)}(n) = \sum_{\substack{(l,m)\in\mathbb{N}_0^2\\ al+bm=n}} \sigma(l)\sigma(m),$$

$W_{(4a,b)}(n)$ and $W_{(a,4b)}(n)$ are known.

Let $1 < \lambda \in \mathbb{N}$ and $\tau : \mathbb{N} \mapsto \mathbb{N}$ be an injective function such that $\tau(n) = \lambda \cdot n$ for each $n \in \mathbb{N}$.

We set $\lambda = 4$ in the sequel. When we use the function τ with l as argument we derive

$$W_{(4a,b)}(n) = \sum_{\substack{(l,m)\in\mathbb{N}_0^2\\ al+bm=n}} \sigma\left(\frac{l}{4}\right)\sigma(m) = \sum_{\substack{(l,m)\in\mathbb{N}_0^2\\ 4a\,l+bm=n}} \sigma(l)\sigma(m).$$

When we apply the function τ with m as argument we infer

$$W_{(a,4b)}(n) = \sum_{\substack{(l,m)\in\mathbb{N}_0^2 \\ al+bm=n}} \sigma(l)\sigma\Big(\frac{m}{4}\Big) = \sum_{\substack{(l,m)\in\mathbb{N}_0^2 \\ al+4b\,m=n}} \sigma(l)\sigma(m).$$

We simultaneously apply the function τ with l and m as arguments, respectively, to conclude

$$\sum_{\substack{(l,m)\in\mathbb{N}_0^2 \\ al+bm=n}} \sigma\Big(\frac{l}{4}\Big)\sigma\Big(\frac{m}{4}\Big) = \sum_{\substack{(l,m)\in\mathbb{N}_0^2 \\ al+bm=\frac{n}{4}}} \sigma(l)\sigma(m) = W_{(a,b)}\Big(\frac{n}{4}\Big).$$

We finally put all these evaluations together to obtain the stated result for $N_{(a,b)}(n)$. □

From this proof, one immediately observe that a formula for the number of representations of a positive integer n by the octonary quadratic form (3) depends on the evaluated convolution Sums for some given levels ab and $4ab$ with $a,b\in\mathbb{N}_0$.

Based on this observation, we only take into consideration those levels $\alpha\beta$ which are multiple of 4; that is $\alpha\beta\equiv 0 \pmod 4$.

Determination of All Relevant $(a,b)\in\mathbb{N}_0^2$ for $N_{(a,b)}(n)$ for a Given $\alpha\beta\in\mathbb{N}_0$

We carry out a method to determine all pairs $(a,b)\in\mathbb{N}_0^2$ which are necessary for the determination of $N_{(a,b)}(n)$ for a given level $\alpha\beta\in\mathbb{N}_0$ such that $\alpha\beta\equiv 0 \pmod 4$ holds.

Let $\Lambda = \frac{\alpha\beta}{4} = 2^{\nu-2}\mho$, $P_4 = \{p_0 = 2^{\nu-2}\}\cup\bigcup_{j>1}\{p_j \mid p_j \text{ is a prime divisor of } \mho\}$ and $\mathcal{P}(P_4)$ be the power set of P_4. Then for each $Q\in\mathcal{P}(P_4)$ we define $\mu(Q) = \prod_{p\in Q} p$. We set $\mu(Q)=1$ if Q is an empty set. Let now

$$\Omega_4 = \Big\{(\mu(Q_1),\mu(Q_2)) \,\Big|\, \text{there exist } Q_1,Q_2\in\mathcal{P}(P_4) \text{ such that } \gcd(\mu(Q_1),\mu(Q_2)) = 1 \text{ and } \mu(Q_1)\,\mu(Q_2) = \Lambda\Big\}.$$

Observe that $\Omega_4\neq\varnothing$ since $(1,\Lambda)\in\Omega_4$.

To illustrate our method, suppose that $\alpha\beta = 2^3\cdot 3\cdot 5$. Then $\Lambda = 2\cdot 3\cdot 5$, $P_4=\{2,3,5\}$ and $\Omega_4 = \{(1,30),(2,15),(3,10),(5,6)\}$.

Proposition 1 – *Suppose that the level $\alpha\beta\in\mathbb{N}_0$ and $\alpha\beta\equiv 0 \pmod 4$. Furthermore, suppose that Ω_4 is defined as above. Then for all $n\in\mathbb{N}$ the set Ω_4 contains all pairs $(a,b)\in\mathbb{N}_0^2$ such that $N_{(a,b)}(n)$ can be obtained by applying $W_{(\alpha,\beta)}(n)$ and some other evaluated convolution sums.*

Proof. We prove this by induction on the structure of the level $\alpha\beta$.

Suppose that $\alpha\beta = 2^\nu p_2$, where $\nu \in \{2,3\}$ and p_2 is an odd prime. Then by the above definitions we have $\Lambda = 2^{\nu-2}p_2$, $P_4 = \{2^{\nu-2}, p_2\}$,

$$\mathcal{P}(P_4) = \{\varnothing, \{2^{\nu-2}\}, \{p_2\}, \{2^{\nu-2}, p_2\}\},$$

and $\Omega_4 = \{(1, 2^{\nu-2}p_2), (2^{\nu-2}, p_2)\}$.

Following the observation made at the end of the proof of Theorem 6, we note that $\alpha\beta = 4ab = 2^\nu p_2$. Hence, $ab = 2^{\nu-2}p_2$ which leads immediately to $N_{(a,b)}(n)$.

We show that Ω_4 is the largest such set. Assume now that there exist another set, say Ω_4', which results from the above definitions. Then there are two cases.

Case $\Omega_4' \subseteq \Omega_4$ There is nothing to show. So, we are done.

Case $\Omega_4 \subset \Omega_4'$ Let $(e,f) \in \Omega_4' \setminus \Omega_4$. Since $ef = 2^{\nu-2}p_2$ and $\gcd(e,f) = 1$, we must have either $(e,f) = (1, 2^{\nu-2}p_2)$ or $(e,f) = (2^{\nu-2}, p_2)$. So, $(e,f) \in \Omega_4$. Hence, $\Omega_4 = \Omega_4'$.

Suppose now that $\alpha\beta = 2^\nu p_2 p_3$, where $\nu \in \{2,3\}$ and p_2, p_3 are distinct odd primes. Then by the induction hypothesis and by the above definitions we have essentially

$$\Omega_4 = \{(1, 2^{\nu-2}p_2p_3), (2^{\nu-2}, p_2p_3), (2^{\nu-2}p_2, p_3), (2^{\nu-2}p_3, p_2)\}.$$

One notes that $\alpha\beta = 4ab = 2^\nu p_2p_3$. Hence, $ab = 2^{\nu-2}p_2p_3$ which immediately gives $N_{(a,b)}(n)$.

Again, we show that Ω_4 is the largest such set. Suppose that there exist another set, say Ω_4', which results from the above definitions. Two cases arise.

Case $\Omega_4' \subseteq \Omega_4$ There is nothing to prove. So, we are done.

Case $\Omega_4 \subset \Omega_4'$ Let $(e,f) \in \Omega_4' \setminus \Omega_4$. Since $ef = 2^{\nu-2}p_2p_3$ and $\gcd(e,f) = 1$, we must have $(e,f) = (1, 2^{\nu-2}p_2p_3)$ or $(e,f) = (2^{\nu-2}, p_2p_3)$ or $(e,f) = (2^{\nu-2}p_2, p_3)$ or $(e,f) = (2^{\nu-2}p_3, p_2)$. So, $(e,f) \in \Omega_4$. Hence, $\Omega_4 = \Omega_4'$. □

We then deduce the following:

Corollary 1 – *Let $n \in \mathbb{N}$, $\alpha\beta \in \mathbb{N}_0$ with $\alpha\beta \equiv 0 \pmod 4$ and Ω_4 be determined as above. Then for each $(a,b) \in \Omega_4$ we have*

$$\begin{aligned} N_{(a,b)}(n) = {} & 8\sigma\left(\frac{n}{a}\right) - 32\sigma\left(\frac{n}{4a}\right) + 8\sigma\left(\frac{n}{b}\right) - 32\sigma\left(\frac{n}{4b}\right) \\ & + 64\, W_{(a,b)}(n) + 1024\, W_{(a,b)}\left(\frac{n}{4}\right) - 256\,(W_{(4a,b)}(n) + W_{(a,4b)}(n)). \end{aligned}$$

4.2 Representations of a Positive Integer by the Octonary Quadratic Form (4)

We now determine formulae for the number of representations of a positive integer by the octonary quadratic forms (4).

Formulae for the Number of Representations by (4)

Let $n \in \mathbb{N}$ and let $s_4(n)$ denote the number of representations of n by the quaternary quadratic form $\sum_{i=1}^{2} (x_{2i-1}^2 + x_{2i-1}x_{2i} + x_{2i}^2)$, that is,

$$s_4(n) = \operatorname{card}\left(\left\{(x_1, x_2, x_3, x_4) \in \mathbb{Z}^4 \;\middle|\; n = \sum_{i=1}^{2} (x_{2i-1}^2 + x_{2i-1}x_{2i} + x_{2i}^2)\right\}\right).$$

It is obvious that $s_4(0) = 1$. For all $n \in \mathbb{N}_0$, J. G. Huard et al.[22], G. A. Lomadze[23] and K. S. Williams[24] have proved that

$$s_4(n) = 12\sigma(n) - 36\sigma\left(\frac{n}{3}\right). \tag{25}$$

Now, let the number of representations of n by the octonary quadratic form (4) be

$$\begin{aligned} R_{(c,d)}(n) = \operatorname{card}\Bigg(\Bigg\{&(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) \in \mathbb{Z}^8 \;\Bigg| \\ &n = c\sum_{i=1}^{2} (x_{2i-1}^2 + x_{2i-1}x_{2i} + x_{2i}^2) + d\sum_{i=3}^{4} (x_{2i-1}^2 + x_{2i-1}x_{2i} + x_{2i}^2)\Bigg\}\Bigg), \end{aligned}$$

where $c, d \in \mathbb{N}_0$.

From this definition of $c, d \in \mathbb{N}_0$ suppose that $c, d \in \mathbb{N}_0$ are such that $\gcd(c, d) = e > 1$ for some $e \in \mathbb{N}_0$. Then $R_{(c,d)}(n) = R_{(\frac{c}{e}, \frac{d}{e})}\left(\frac{n}{e}\right)$. Hence, one can simply assume that $c, d \in \mathbb{N}_0$ are relatively prime.

We infer the following

Theorem 7 – *Let $n \in \mathbb{N}$ and $c, d \in \mathbb{N}_0$ be relatively prime. Then*

$$\begin{aligned} R_{(c,d)}(n) &= 12\sigma\left(\frac{n}{c}\right) - 36\sigma\left(\frac{n}{3c}\right) + 12\sigma\left(\frac{n}{d}\right) - 36\sigma\left(\frac{n}{3d}\right) \\ &\quad + 144\, W_{(c,d)}(n) + 1296\, W_{(c,d)}\left(\frac{n}{3}\right) - 432\,(W_{(3c,d)}(n) + W_{(c,3d)}(n)). \end{aligned}$$

Proof. It holds that

$$R_{(c,d)}(n) = \sum_{\substack{(l,m)\in\mathbb{N}^2 \\ cl+dm=n}} s_4(l)s_4(m) = s_4\left(\frac{n}{c}\right)s_4(0) + s_4(0)s_4\left(\frac{n}{d}\right) + \sum_{\substack{(l,m)\in\mathbb{N}_0^2 \\ cl+dm=n}} s_4(l)s_4(m).$$

[22]Huard et al., 2002, “Elementary evaluation of certain convolution sums involving divisor functions”.
[23]Lomadze, 1989, “Representation of numbers by sums of the quadratic forms $x_1^2 + x_1x_2 + x_2^2$”.
[24]Williams, 2011, *Number Theory in the Spirit of Liouville*, Thrm 17.3, p. 225.

We apply (25) to derive

$$
\begin{aligned}
R_{(c,d)}(n) &= 12\sigma\left(\frac{n}{c}\right) - 36\sigma\left(\frac{n}{3c}\right) + 12\sigma\left(\frac{n}{d}\right) - 36\sigma\left(\frac{n}{3d}\right) \\
&\quad + \sum_{\substack{(l,m)\in\mathbb{N}_0^2 \\ cl+dm=n}} \left(12\sigma(l) - 36\sigma\left(\frac{l}{3}\right)\right)\left(12\sigma(m) - 36\sigma\left(\frac{m}{3}\right)\right).
\end{aligned}
$$

We know that

$$
\begin{aligned}
&\left(12\sigma(l) - 36\sigma\left(\frac{l}{3}\right)\right)\left(12\sigma(m) - 36\sigma\left(\frac{m}{3}\right)\right) \\
&\quad = 144\sigma(l)\sigma(m) - 432\sigma\left(\frac{l}{3}\right)\sigma(m) - 432\sigma(l)\sigma\left(\frac{m}{3}\right) + 1296\sigma\left(\frac{l}{3}\right)\sigma\left(\frac{m}{3}\right).
\end{aligned}
$$

We assume that the evaluation of

$$
W_{(c,d)}(n) = \sum_{\substack{(l,m)\in\mathbb{N}_0^2 \\ cl+dm=n}} \sigma(l)\sigma(m),
$$

$W_{(c,3d)}(n)$ and $W_{(3c,d)}(n)$ are known. We set $\lambda = 3$ in the sequel. We apply the function τ to m to derive

$$
\sum_{\substack{(l,m)\in\mathbb{N}_0^2 \\ cl+dm=n}} \sigma(l)\sigma\left(\frac{m}{3}\right) = \sum_{\substack{(l,m)\in\mathbb{N}_0^2 \\ cl+3d\,m=n}} \sigma(l)\sigma(m) = W_{(c,3d)}(n).
$$

Let λ and τ be defined as in Subsection 4.1. Let us set λ to 3 in the sequel. We make use of the function τ with l as argument to conclude

$$
\sum_{\substack{(l,m)\in\mathbb{N}_0^2 \\ cl+dm=n}} \sigma(m)\sigma\left(\frac{l}{3}\right) = \sum_{\substack{(l,m)\in\mathbb{N}_0^2 \\ 3c\,l+dm=n}} \sigma(l)\sigma(m) = W_{(3c,d)}(n).
$$

We simultaneously apply apply the function τ to l and to m as arguments, respectively, to infer

$$
\sum_{\substack{(l,m)\in\mathbb{N}_0^2 \\ cl+dm=n}} \sigma\left(\frac{m}{3}\right)\sigma\left(\frac{l}{3}\right) = \sum_{\substack{(l,m)\in\mathbb{N}_0^2 \\ cl+dm=\frac{n}{3}}} \sigma(l)\sigma(m) = W_{(c,d)}\left(\frac{n}{3}\right).
$$

Finally, we bring all these evaluations together to obtain the stated result for $R_{(c,d)}(n)$. □

From this proof, we note that a formula for the number of representations of a positive integer n by the octonary quadratic form (4) depends on the evaluated convolution Sums for some given levels cd and $3cd$ with $c, d \in \mathbb{N}_0$.

As a consequence, we do consider only the levels $\alpha\beta$ which are divisible by 3; that is $\alpha\beta \equiv 0 \pmod 3$.

Determination of All Relevant $(c,d) \in \mathbb{N}_0^2$ for $R_{(c,d)}(n)$ for a Given Level $\alpha\beta \in \mathbb{N}_0$

The following method determine all pairs $(c,d) \in \mathbb{N}_0^2$ necessary for the determination of $R_{(c,d)}(n)$ for a given $\alpha\beta \in \mathbb{N}_0$ belonging to the above class. The following method is quasi similar to the one used in Subsection 4.1.

Let $\Delta = \frac{\alpha\beta}{3} = \frac{2^{\nu}\mho}{3}$. Let $P_3 = \{p_0 = 2^{\nu}\} \cup \bigcup_{j>2}\{p_j \mid p_j \text{ is a prime divisor of } \mho\}$. Let $\mathcal{P}(P_3)$ be the power set of P_3. Then for each $Q \in \mathcal{P}(P_3)$ we define $\mu(Q) = \prod_{p\in Q} p$. We set $\mu(Q) = 1$ if Q is an empty set. Let now Ω_3 be defined in a similar way as Ω_4 in Subsection 4.1; however, with Δ instead of Λ, i.e.,

$$\Omega_3 = \Big\{(\mu(Q_1), \mu(Q_2)) \,\Big|\, \text{there exist } Q_1, Q_2 \in \mathcal{P}(P_3) \text{ such that} \\ \gcd(\mu(Q_1), \mu(Q_2)) = 1 \text{ and } \mu(Q_1)\,\mu(Q_2) = \Delta\Big\}.$$

Note that $\Omega_3 \neq \emptyset$ since $(1, \Delta) \in \Omega_3$.

As an example, suppose again that $\alpha\beta = 2^3 \cdot 3 \cdot 5$. Then $\Delta = 2^3 \cdot 5$, $P_3 = \{2^3, 5\}$ and $\Omega_3 = \{(1, 40), (5, 8)\}$.

Proposition 2 – *Suppose that the level $\alpha\beta \in \mathbb{N}_0$ and $\alpha\beta \equiv 0 \pmod 3$. Suppose in addition that Ω_3 be defined as above. Then for all $n \in \mathbb{N}$ the set Ω_3 contains all pairs $(c,d) \in \mathbb{N}_0^2$ such that $R_{(c,d)}(n)$ can be obtained by applying $W_{(\alpha,\beta)}(n)$ and some other evaluated convolution sums.*

Proof. Similar to the proof of Proposition 1. □

We then infer the following:

Corollary 2 – *Let $n \in \mathbb{N}$, $\alpha\beta \in \mathbb{N}_0$ with $\alpha\beta \equiv 0 \pmod 3$ and Ω_3 be determined as above. Then for each $(c,d) \in \Omega_3$ we obtain*

$$R_{(c,d)}(n) = 12\sigma\left(\frac{n}{c}\right) - 36\sigma\left(\frac{n}{3c}\right) + 12\sigma\left(\frac{n}{d}\right) - 36\sigma\left(\frac{n}{3d}\right) \\ + 144\, W_{(c,d)}(n) + 1296\, W_{(c,d)}\left(\frac{n}{3}\right) - 432\,(W_{(3c,d)}(n) + W_{(c,3d)}(n)).$$

5 Sample of the Evaluation of the Convolution Sums when the Level Belongs to $\mathfrak{N}$

In this section, we give explicit formulae for the convolution sum $W_{(\alpha,\beta)}(n)$ when $\alpha\beta = 33$, 40 and 56. These levels belong to $\mathfrak{N}$. Hence, the primitive Dirichlet characters are trivial.

When we apply T. Miyake[25], we conclude that

$$M_4(\Gamma_0(11)) \subset M_4(\Gamma_0(33)) \tag{26}$$

$$M_4(\Gamma_0(5)) \subset M_4(\Gamma_0(10)) \subset M_4(\Gamma_0(20)) \subset M_4(\Gamma_0(40)) \tag{27}$$

$$M_4(\Gamma_0(8)) \subset M_4(\Gamma_0(40)). \tag{28}$$

$$M_4(\Gamma_0(7)) \subset M_4(\Gamma_0(14)) \subset M_4(\Gamma_0(28)) \subset M_4(\Gamma_0(56)) \tag{29}$$

$$M_4(\Gamma_0(8)) \subset M_4(\Gamma_0(56)). \tag{30}$$

This implies the same inclusion relation for the bases, the space of Eisenstein forms of weight 4 and the spaces of cusp forms of weight 4.

5.1 Bases of $E_4(\Gamma_0(\alpha\beta))$ and $S_4(\Gamma_0(\alpha\beta))$ for $\alpha\beta = 33, 40, 56$

We apply the dimension formulae in T. Miyake[26] or W. A. Stein[27] to deduce that

$$\dim(S_4(\Gamma_0(33))) = 10, \quad \dim(S_4(\Gamma_0(40))) = 14 \quad \text{and} \quad \dim(S_4(\Gamma_0(56))) = 20.$$

We use (19) to infer that

$$\dim(E_4(\Gamma_0(33))) = 4 \quad \text{and} \quad \dim(E_4(\Gamma_0(40))) = \dim(E_4(\Gamma_0(56))) = 8.$$

We apply Theorem 1 to determine as many elements of $S_4(\Gamma_0(33))$, $S_4(\Gamma_0(40))$ and $S_4(\Gamma_0(56))$ as possible. Then we apply Remark 1 **(r2)** when selecting basis elements of a given space of cusp forms as stated in the proof of Theorem 4 (b).

Corollary 3 – **(a)** *The sets* $\mathcal{B}_{E,33} = \{M(q^t) \mid t|33\}$, $\mathcal{B}_{E,40} = \{M(q^t) \mid t|40\}$ *and* $\mathcal{B}_{E,56} = \{M(q^t) \mid t|56\}$ *are bases of* $E_4(\Gamma_0(33))$, $E_4(\Gamma_0(40))$ *and* $E_4(\Gamma_0(56))$, *respectively.*

(b) *Let* $i, j, k \in \mathbb{N}_0$ *satisfy* $1 \le i \le 10$, $1 \le j \le 14$ *and* $1 \le k \le 20$.
Let $\delta_1 \in D(33)$ *and* $(r(i,\delta_1))_{i,\delta_1}$ *be the Table 5 of the powers of* $\eta(\delta_1 z)$.
Let $\delta_2 \in D(40)$ *and* $(r(j,\delta_2))_{j,\delta_2}$ *be the Table 6 of the powers of* $\eta(\delta_2 z)$.
Let $\delta_3 \in D(56)$ *and* $(r(k,\delta_3))_{k,\delta_3}$ *be the Table 7 of the powers of* $\eta(\delta_3 z)$.

[25] Miyake, 1989, *Modular Forms*, Lema 2.1.3, p. 41.
[26] Ibid., Thrm 2.5.2, p. 60.
[27] Stein, 2011, *Modular Forms, A Computational Approach*, Prop. 6.1, p. 91.

Let furthermore

$$\mathfrak{B}_{33,i}(q) = \prod_{\delta_1|33} \eta^{r(i,\delta_1)}(\delta_1 z), \quad \mathfrak{B}_{40,j}(q) = \prod_{\delta_2|40} \eta^{r(j,\delta_2)}(\delta_2 z),$$
$$\mathfrak{B}_{56,k}(q) = \prod_{\delta_3|56} \eta^{r(k,\delta_3)}(\delta_3 z)$$

be selected elements of $S_4(\Gamma_0(33))$, $S_4(\Gamma_0(40))$ *and* $S_4(\Gamma_0(56))$, *respectively.*
Then the sets

$$\mathcal{B}_{S,33} = \{\mathfrak{B}_{33,i}(q) \mid 1 \le i \le 10\}, \quad \mathcal{B}_{S,40} = \{\mathfrak{B}_{40,j}(q) \mid 1 \le j \le 14\},$$
$$\mathcal{B}_{S,56} = \{\mathfrak{B}_{56,k}(q) \mid 1 \le k \le 20\}$$

are bases of $S_4(\Gamma_0(33))$, $S_4(\Gamma_0(40))$ *and* $S_4(\Gamma_0(56))$, *respectively*

(c) *The sets* $\mathcal{B}_{M,33} = \mathcal{B}_{E,33} \cup \mathcal{B}_{S,33}$, $\mathcal{B}_{M,40} = \mathcal{B}_{E,40} \cup \mathcal{B}_{S,40}$ *and* $\mathcal{B}_{M,56} = \mathcal{B}_{E,56} \cup \mathcal{B}_{S,56}$ *constitute bases of* $M_4(\Gamma_0(33))$, $M_4(\Gamma_0(40))$ *and* $M_4(\Gamma_0(56))$, *respectively.*

By Remark 1 (r1), $\mathfrak{B}_{33,i}(q)$, $\mathfrak{B}_{40,j}(q)$ and $\mathfrak{B}_{56,k}(q)$ can be expressed in the form $\sum_{n=1}^{\infty} \mathfrak{b}_{33,i}(n)q^n$, $\sum_{n=1}^{\infty} \mathfrak{b}_{40,j}(n)q^n$ and $\sum_{n=1}^{\infty} \mathfrak{b}_{56,k}(n)q^n$, respectively.

We observe that

- by (26) the basis element $\mathfrak{B}_{33,2}(q)$ is in $S_4(\Gamma_0(11))$ and is the only one. In addition, $\mathfrak{B}_{33,6}(q) = \mathfrak{B}_{33,2}(q^2)$. Hence, $\mathfrak{b}_{33,6}(n) = \mathfrak{b}_{33,2}\left(\frac{n}{2}\right)$.
- the basis elements of $S_4(\Gamma_0(40))$ have been determined almost with respect to the inclusion relation (27), except the element $\mathfrak{B}_{40,5}(q)$ which results from the basis element of $S_4(\Gamma_0(8))$ according to (28).
- there is no element of $S_4(\Gamma_0(7))$ which occurs as an element of $S_4(\Gamma_0(56))$. This indicates that an element of $S_4(\Gamma_0(7))$ cannot be determined when using Theorem 1. The inclusion relations (29) and (30) preserve the bases.

Proof. It follows immediately from Theorem 4.

In case **(a)**: the result is obtained by setting $n = 1, 3, 11, 33$, $n = 1, 2, 4, 5, 8, 10, 20, 40$ and $n = 1, 2, 4, 7, 8, 14, 28, 56$, respectively.

In case **(b)**: the linear independence of the sets $\mathcal{B}_{S,33}$ and $\mathcal{B}_{S,56}$ is proved by applying case 2 in the proof of Theorem 4 (b) and by taking $n = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10$ and $n = 1, 2, 3, \ldots, 13, 14$, respectively. Finally $\mathcal{B}_{S,40}$ is linearly independent by case 1 in the proof of Theorem 4 (b) and by taking $n = 1, 2, 3, \ldots, 19, 20$.

Therefore, we obtain the stated result. □

5.2 Evaluation of $W_{(\alpha,\beta)}(n)$ when $\alpha\beta = 33, 40, 56$

Corollary 4 – *We have*

$$
\begin{aligned}
(L(q)-33\,L(q^{33}))^2 = 1024 + \sum_{n=1}^{\infty}\Big(&\frac{2300736}{1271}\,\sigma_3(n) - \frac{59459328}{77531}\,\sigma_3\Big(\frac{n}{3}\Big)\\
&+\frac{271016064}{1271}\,\sigma_3\Big(\frac{n}{11}\Big) - \frac{75206279808}{77531}\,\sigma_3\Big(\frac{n}{33}\Big) - \frac{348480}{1271}\,\mathfrak{b}_{33,1}(n)\\
&-\frac{14117760}{1271}\,\mathfrak{b}_{33,2}(n) - \frac{6573339072}{77531}\,\mathfrak{b}_{33,3}(n) - \frac{26803856448}{77531}\,\mathfrak{b}_{33,4}(n)\\
&-\frac{62014527936}{77531}\,\mathfrak{b}_{33,5}(n) - \frac{97134678144}{77531}\,\mathfrak{b}_{33,6}(n) - \frac{87378566400}{77531}\,\mathfrak{b}_{33,7}(n)\\
&-\frac{742808448}{1271}\,\mathfrak{b}_{33,8}(n) + \frac{44352}{1271}\,\mathfrak{b}_{33,9}(n) - \frac{4447872}{1271}\,\mathfrak{b}_{33,10}(n)\Big)q^n, \qquad (31)
\end{aligned}
$$

$$
\begin{aligned}
(3\,L(q^3)-11\,L(q^{11}))^2 = 64 + \sum_{n=1}^{\infty}\Big(&-\frac{348480}{1271}\,\sigma_3(n) + \frac{106313472}{77531}\,\sigma_3\Big(\frac{n}{3}\Big)\\
&-\frac{34793088}{1271}\,\sigma_3\Big(\frac{n}{11}\Big) + \frac{80875631232}{77531}\,\sigma_3\Big(\frac{n}{33}\Big) + \frac{348480}{1271}\,\mathfrak{b}_{33,1}(n)\\
&+\frac{3136320}{1271}\,\mathfrak{b}_{33,2}(n) + \frac{1346173632}{77531}\,\mathfrak{b}_{33,3}(n) + \frac{5361496704}{77531}\,\mathfrak{b}_{33,4}(n)\\
&+\frac{11895235776}{77531}\,\mathfrak{b}_{33,5}(n) + \frac{17925551424}{77531}\,\mathfrak{b}_{33,6}(n) + \frac{15428171520}{77531}\,\mathfrak{b}_{33,7}(n)\\
&+\frac{127847808}{1271}\,\mathfrak{b}_{33,8}(n) - \frac{44352}{1271}\,\mathfrak{b}_{33,9}(n) + \frac{4447872}{1271}\,\mathfrak{b}_{33,10}(n)\Big)q^n, \qquad (32)
\end{aligned}
$$

$$
\begin{aligned}
(L(q)-40\,L(q^{40}))^2 = 1521 + \sum_{n=1}^{\infty}\Big(&\frac{26800}{117}\,\sigma_3(n) + \frac{43520}{117}\,\sigma_3\Big(\frac{n}{2}\Big)\\
&+\frac{245120}{39}\,\sigma_3\Big(\frac{n}{4}\Big) - \frac{26800}{117}\,\sigma_3\Big(\frac{n}{5}\Big) - \frac{1766400}{13}\,\sigma_3\Big(\frac{n}{8}\Big) - \frac{127760}{117}\,\sigma_3\Big(\frac{n}{10}\Big)\\
&-\frac{357440}{39}\,\sigma_3\Big(\frac{n}{20}\Big) + \frac{6558720}{13}\,\sigma_3\Big(\frac{n}{40}\Big) + \frac{192224}{117}\,\mathfrak{b}_{40,1}(n)\\
&+\frac{439744}{117}\,\mathfrak{b}_{40,2}(n) + \frac{304832}{39}\,\mathfrak{b}_{40,3}(n) + \frac{1061120}{39}\,\mathfrak{b}_{40,4}(n)\\
&+\frac{41840}{3}\,\mathfrak{b}_{40,5}(n) - 15360\,\mathfrak{b}_{40,6}(n) - \frac{24320}{3}\,\mathfrak{b}_{40,7}(n) + \frac{1688320}{39}\,\mathfrak{b}_{40,8}(n)\\
&+116800\,\mathfrak{b}_{40,9}(n) - \frac{128000}{3}\,\mathfrak{b}_{40,10}(n) - \frac{485120}{3}\,\mathfrak{b}_{40,11}(n)\\
&-\frac{1130240}{3}\,\mathfrak{b}_{40,12}(n) - \frac{121280}{3}\,\mathfrak{b}_{40,13}(n) - 69120\,\mathfrak{b}_{40,14}(n)\Big)q^n, \qquad (33)
\end{aligned}
$$

$$
\begin{aligned}
(5\,L(q^5)-8\,L(q^8))^2 &= 9+\sum_{n=1}^{\infty}\Big(\frac{5920}{117}\,\sigma_3(n)-\frac{76000}{117}\,\sigma_3\Big(\frac{n}{2}\Big)-\frac{16960}{39}\,\sigma_3\Big(\frac{n}{4}\Big)\\
&\quad+\frac{668000}{117}\,\sigma_3\Big(\frac{n}{5}\Big)+\frac{721920}{13}\,\sigma_3\Big(\frac{n}{8}\Big)-\frac{8240}{117}\,\sigma_3\Big(\frac{n}{10}\Big)-\frac{95360}{39}\,\sigma_3\Big(\frac{n}{20}\Big)\\
&\quad-\frac{721920}{13}\,\sigma_3\Big(\frac{n}{40}\Big)-\frac{5920}{117}\,\mathfrak{b}_{40,1}(n)+\frac{22720}{117}\,\mathfrak{b}_{40,2}(n)-\frac{59200}{39}\,\mathfrak{b}_{40,3}(n)\\
&\quad+\frac{12800}{39}\,\mathfrak{b}_{40,4}(n)-\frac{38800}{3}\,\mathfrak{b}_{40,5}(n)+7680\,\mathfrak{b}_{40,6}(n)-\frac{47360}{3}\,\mathfrak{b}_{40,7}(n)\\
&\quad-\frac{505088}{39}\,\mathfrak{b}_{40,8}(n)-67520\,\mathfrak{b}_{40,9}(n)-\frac{12800}{3}\,\mathfrak{b}_{40,10}(n)+\frac{113920}{3}\,\mathfrak{b}_{40,11}(n)\\
&\quad+\frac{298240}{3}\,\mathfrak{b}_{40,12}(n)+\frac{63040}{3}\,\mathfrak{b}_{40,13}(n)+69120\,\mathfrak{b}_{40,14}(n)\Big)q^n, \qquad (34)
\end{aligned}
$$

$$
\begin{aligned}
(L(q)-56\,L(q^{56}))^2 &= 3025+\sum_{n=1}^{\infty}\Big(\frac{1284}{5}\,\sigma_3(n)-420\,\sigma_3\Big(\frac{n}{2}\Big)+\frac{31584}{5}\,\sigma_3\Big(\frac{n}{4}\Big)\\
&\quad-\frac{1764}{5}\,\sigma_3\Big(\frac{n}{7}\Big)-\frac{32256}{5}\,\sigma_3\Big(\frac{n}{8}\Big)-588\,\sigma_3\Big(\frac{n}{14}\Big)-\frac{51744}{5}\,\sigma_3\Big(\frac{n}{28}\Big)\\
&\quad+\frac{3687936}{5}\,\sigma_3\Big(\frac{n}{56}\Big)+\frac{11916}{5}\,\mathfrak{b}_{56,1}(n)+\frac{92604}{5}\,\mathfrak{b}_{56,2}(n)+29568\,\mathfrak{b}_{56,3}(n)\\
&\quad+\frac{1140216}{5}\,\mathfrak{b}_{56,4}(n)-411936\,\mathfrak{b}_{56,5}(n)+\frac{2557632}{5}\,\mathfrak{b}_{56,6}(n)\\
&\quad+223608\,\mathfrak{b}_{56,7}(n)+3998400\,\mathfrak{b}_{56,8}(n)+4042752\,\mathfrak{b}_{56,9}(n)\\
&\quad+\frac{145152}{5}\,\mathfrak{b}_{56,10}(n)-8064\,\mathfrak{b}_{56,11}(n)-48384\,\mathfrak{b}_{56,12}(n)+\frac{532224}{5}\,\mathfrak{b}_{56,14}(n)\\
&\quad+161280\,\mathfrak{b}_{56,15}(n)-\frac{225792}{5}\,\mathfrak{b}_{56,16}(n)+129024\,\mathfrak{b}_{56,17}(n)\\
&\quad+\frac{2515968}{5}\,\mathfrak{b}_{56,18}(n)+1354752\,\mathfrak{b}_{56,19}(n)-225792\,\mathfrak{b}_{56,20}(n)\Big)q^n, \qquad (35)
\end{aligned}
$$

$$
\begin{aligned}
(7\,L(q^7)-8\,L(q^8))^2 &= 1+\sum_{n=1}^{\infty}\Big(-\frac{308}{25}\,\sigma_3(n)+\frac{1876}{25}\,\sigma_3\Big(\frac{n}{2}\Big)-\frac{40096}{25}\,\sigma_3\Big(\frac{n}{4}\Big)\\
&\quad+\frac{285908}{25}\,\sigma_3\Big(\frac{n}{7}\Big)+\frac{409088}{25}\,\sigma_3\Big(\frac{n}{8}\Big)-\frac{27076}{25}\,\sigma_3\Big(\frac{n}{14}\Big)-\frac{60704}{25}\,\sigma_3\Big(\frac{n}{28}\Big)\\
&\quad-\frac{562688}{25}\,\sigma_3\Big(\frac{n}{56}\Big)+\frac{308}{25}\,\mathfrak{b}_{56,1}(n)+\frac{2436}{25}\,\mathfrak{b}_{56,2}(n)+\frac{11648}{25}\,\mathfrak{b}_{56,3}(n)\\
&\quad+\frac{121352}{25}\,\mathfrak{b}_{56,4}(n)-\frac{101472}{25}\,\mathfrak{b}_{56,5}(n)+\frac{288064}{25}\,\mathfrak{b}_{56,6}(n)-\frac{87864}{25}\,\mathfrak{b}_{56,7}(n)\\
&\quad+\frac{2190912}{25}\,\mathfrak{b}_{56,8}(n)+\frac{1821312}{25}\,\mathfrak{b}_{56,9}(n)+\frac{201984}{25}\,\mathfrak{b}_{56,10}(n)
\end{aligned}
$$

(Cont. next page)

$$+29568\,\mathfrak{b}_{56,11}(n)+\frac{284928}{5}\,\mathfrak{b}_{56,12}(n)-59136\,\mathfrak{b}_{56,13}(n)-\frac{1512192}{25}\,\mathfrak{b}_{56,14}(n)$$
$$+59136\,\mathfrak{b}_{56,15}(n)+\frac{6724096}{25}\,\mathfrak{b}_{56,16}(n)+118272\,\mathfrak{b}_{56,17}(n)$$
$$\left.+\frac{1616896}{25}\,\mathfrak{b}_{56,18}(n)+430080\,\mathfrak{b}_{56,19}(n)+53760\,\mathfrak{b}_{56,20}(n)\right)q^n. \tag{36}$$

Proof. These identities follow immediately when one sets $(\alpha,\beta)=(1,33)$, $(3,11)$, $(1,40)$, $(5,8)$, $(1,56)$, $(7,8)$ in Lemma 2. In case of $\alpha\beta=40$, we take all n in the following set $\{1,2,\dots,20,40,80\}$ to obtain a system of 22 linear equations with unknowns X_δ and Y_j, where $\delta\in D(40)$ and $1\leq j\leq 14$. □

We are now prepared to state and to prove our main result of this section.

Corollary 5 – *Let n be a positive integer. Then*

$$W_{(1,33)}(n)=-\frac{13859}{335544}\sigma_3(n)+\frac{51614}{2558523}\sigma_3\left(\frac{n}{3}\right)-\frac{7129}{1271}\sigma_3\left(\frac{n}{11}\right)$$
$$+\frac{60271327}{1860744}\sigma_3\left(\frac{n}{33}\right)+\left(\frac{1}{24}-\frac{1}{132}n\right)\sigma(n)+\left(\frac{1}{24}-\frac{1}{4}n\right)\sigma\left(\frac{n}{33}\right)$$
$$+\frac{55}{7626}\,\mathfrak{b}_{33,1}(n)+\frac{4085}{13981}\,\mathfrak{b}_{33,2}(n)+\frac{11412047}{5117046}\,\mathfrak{b}_{33,3}(n)+\frac{15511491}{1705682}\,\mathfrak{b}_{33,4}(n)$$
$$+\frac{35888037}{1705682}\,\mathfrak{b}_{33,5}(n)+\frac{28106099}{852841}\,\mathfrak{b}_{33,6}(n)+\frac{25283150}{852841}\,\mathfrak{b}_{33,7}(n)$$
$$+\frac{214933}{13981}\,\mathfrak{b}_{33,8}(n)-\frac{7}{7626}\,\mathfrak{b}_{33,9}(n)+\frac{117}{1271}\,\mathfrak{b}_{33,10}(n), \tag{37}$$

$$W_{(3,11)}(n)=\frac{55}{7626}\sigma_3(n)+\frac{12869}{620248}\sigma_3\left(\frac{n}{3}\right)+\frac{15089}{10168}\sigma_3\left(\frac{n}{11}\right)-\frac{6382231}{232593}\sigma_3\left(\frac{n}{33}\right)$$
$$+\left(\frac{1}{24}-\frac{1}{44}n\right)\sigma\left(\frac{1}{3}n\right)+\left(\frac{1}{24}-\frac{1}{12}n\right)\sigma\left(\frac{n}{11}\right)-\frac{55}{7626}\,\mathfrak{b}_{33,1}(n)-\frac{165}{2542}\,\mathfrak{b}_{33,2}(n)$$
$$-\frac{2337107}{5117046}\,\mathfrak{b}_{33,3}(n)-\frac{1551359}{852841}\,\mathfrak{b}_{33,4}(n)-\frac{6883817}{1705682}\,\mathfrak{b}_{33,5}(n)$$
$$-\frac{943053}{155062}\,\mathfrak{b}_{33,6}(n)-\frac{4464170}{852841}\,\mathfrak{b}_{33,7}(n)-\frac{3363}{1271}\,\mathfrak{b}_{33,8}(n)$$
$$+\frac{7}{7626}\,\mathfrak{b}_{33,9}(n)-\frac{117}{1271}\,\mathfrak{b}_{33,10}(n), \tag{38}$$

$$W_{(1,40)}(n)=\frac{1}{4212}\sigma_3(n)-\frac{17}{2106}\sigma_3\left(\frac{n}{2}\right)-\frac{383}{2808}\sigma_3\left(\frac{n}{4}\right)+\frac{335}{67392}\sigma_3\left(\frac{n}{5}\right)$$
$$+\frac{115}{39}\sigma_3\left(\frac{n}{8}\right)+\frac{1597}{67392}\sigma_3\left(\frac{n}{10}\right)+\frac{1117}{5616}\sigma_3\left(\frac{n}{20}\right)-\frac{34}{13}\sigma_3\left(\frac{n}{40}\right)$$
$$+\left(\frac{1}{24}-\frac{1}{160}n\right)\sigma(n)+\left(\frac{1}{24}-\frac{1}{4}n\right)\sigma\left(\frac{n}{40}\right)-\frac{6007}{168480}\,\mathfrak{b}_{40,1}(n)$$

(Cont. next page)

$$
\begin{aligned}
&- \frac{6871}{84240}\,\mathfrak{b}_{40,2}(n) - \frac{4763}{28080}\,\mathfrak{b}_{40,3}(n) - \frac{829}{1404}\,\mathfrak{b}_{40,4}(n) - \frac{523}{1728}\,\mathfrak{b}_{40,5}(n) \\
&+ \frac{1}{3}\,\mathfrak{b}_{40,6}(n) + \frac{19}{108}\,\mathfrak{b}_{40,7}(n) - \frac{1319}{1404}\,\mathfrak{b}_{40,8}(n) - \frac{365}{144}\,\mathfrak{b}_{40,9}(n) + \frac{25}{27}\,\mathfrak{b}_{40,10}(n) \\
&+ \frac{379}{108}\,\mathfrak{b}_{40,11}(n) + \frac{883}{108}\,\mathfrak{b}_{40,12}(n) + \frac{379}{432}\,\mathfrak{b}_{40,13}(n) + \frac{3}{2}\,\mathfrak{b}_{40,14}(n), \qquad (39)
\end{aligned}
$$

$$
\begin{aligned}
W_{(5,8)}(n) = &- \frac{37}{33696}\,\sigma_3(n) + \frac{475}{33696}\,\sigma_3\left(\frac{n}{2}\right) + \frac{53}{5616}\,\sigma_3\left(\frac{n}{4}\right) + \frac{425}{67392}\,\sigma_3\left(\frac{n}{5}\right) \\
&- \frac{34}{39}\,\sigma_3\left(\frac{n}{8}\right) + \frac{103}{67392}\,\sigma_3\left(\frac{n}{10}\right) + \frac{149}{2808}\,\sigma_3\left(\frac{n}{20}\right) + \frac{47}{39}\,\sigma_3\left(\frac{n}{40}\right) \\
&+ \left(\frac{1}{24} - \frac{1}{32}n\right)\sigma\left(\frac{n}{5}\right) + \left(\frac{1}{24} - \frac{1}{20}n\right)\sigma\left(\frac{n}{8}\right) + \frac{37}{33696}\,\mathfrak{b}_{40,1}(n) \\
&- \frac{71}{16848}\,\mathfrak{b}_{40,2}(n) + \frac{185}{5616}\,\mathfrak{b}_{40,3}(n) - \frac{5}{702}\,\mathfrak{b}_{40,4}(n) + \frac{485}{1728}\,\mathfrak{b}_{40,5}(n) \\
&- \frac{1}{6}\,\mathfrak{b}_{40,6}(n) + \frac{37}{108}\,\mathfrak{b}_{40,7}(n) + \frac{1973}{7020}\,\mathfrak{b}_{40,8}(n) + \frac{211}{144}\,\mathfrak{b}_{40,9}(n) + \frac{5}{54}\,\mathfrak{b}_{40,10}(n) \\
&- \frac{89}{108}\,\mathfrak{b}_{40,11}(n) - \frac{233}{108}\,\mathfrak{b}_{40,12}(n) - \frac{197}{432}\,\mathfrak{b}_{40,13}(n) - \frac{3}{2}\,\mathfrak{b}_{40,14}(n), \qquad (40)
\end{aligned}
$$

$$
\begin{aligned}
W_{(1,56)}(n) = &- \frac{1}{3840}\,\sigma_3(n) + \frac{5}{768}\,\sigma_3\left(\frac{n}{2}\right) - \frac{47}{480}\,\sigma_3\left(\frac{n}{4}\right) + \frac{7}{1280}\,\sigma_3\left(\frac{n}{7}\right) + \frac{1}{10}\,\sigma_3\left(\frac{n}{8}\right) \\
&+ \frac{7}{768}\,\sigma_3\left(\frac{n}{14}\right) + \frac{77}{480}\,\sigma_3\left(\frac{n}{28}\right) + \frac{7}{30}\,\sigma_3\left(\frac{n}{56}\right) + \left(\frac{1}{24} - \frac{1}{224}n\right)\sigma(n) \\
&+ \left(\frac{1}{24} - \frac{1}{4}n\right)\sigma\left(\frac{n}{56}\right) - \frac{331}{8960}\,\mathfrak{b}_{56,1}(n) - \frac{7717}{26880}\,\mathfrak{b}_{56,2}(n) - \frac{11}{24}\,\mathfrak{b}_{56,3}(n) \\
&- \frac{6787}{1920}\,\mathfrak{b}_{56,4}(n) + \frac{613}{96}\,\mathfrak{b}_{56,5}(n) - \frac{1903}{240}\,\mathfrak{b}_{56,6}(n) - \frac{1331}{384}\,\mathfrak{b}_{56,7}(n) \\
&- \frac{2975}{48}\,\mathfrak{b}_{56,8}(n) - \frac{188}{3}\,\mathfrak{b}_{56,9}(n) - \frac{9}{20}\,\mathfrak{b}_{56,10}(n) + \frac{1}{8}\,\mathfrak{b}_{56,11}(n) \\
&+ \frac{3}{4}\,\mathfrak{b}_{56,12}(n) - \frac{33}{20}\,\mathfrak{b}_{56,14}(n) - \frac{5}{2}\,\mathfrak{b}_{56,15}(n)\frac{7}{10}\,\mathfrak{b}_{56,16}(n) \\
&- 2\,\mathfrak{b}_{56,17}(n) - \frac{39}{5}\,\mathfrak{b}_{56,18}(n) - 21\,\mathfrak{b}_{56,19}(n) + \frac{7}{2}\,\mathfrak{b}_{56,20}(n), \qquad (41)
\end{aligned}
$$

$$
\begin{aligned}
W_{(7,8)}(n) = &\ \frac{11}{57600}\,\sigma_3(n) - \frac{67}{57600}\,\sigma_3\left(\frac{n}{2}\right) + \frac{179}{7200}\,\sigma_3\left(\frac{n}{4}\right) + \frac{289}{57600}\,\sigma_3\left(\frac{n}{7}\right) \\
&- \frac{7}{450}\,\sigma_3\left(\frac{n}{8}\right) + \frac{967}{57600}\,\sigma_3\left(\frac{n}{14}\right) + \frac{271}{7200}\,\sigma_3\left(\frac{n}{28}\right) + \frac{157}{450}\,\sigma_3\left(\frac{n}{56}\right) \\
&+ \left(\frac{1}{24} - \frac{1}{32}n\right)\sigma\left(\frac{n}{7}\right) + \left(\frac{1}{24} - \frac{1}{28}n\right)\sigma\left(\frac{n}{8}\right) - \frac{11}{57600}\,\mathfrak{b}_{56,1}(n) \\
&- \frac{29}{19200}\,\mathfrak{b}_{56,2}(n) - \frac{13}{1800}\,\mathfrak{b}_{56,3}(n) - \frac{2167}{28800}\,\mathfrak{b}_{56,4}(n) + \frac{151}{2400}\,\mathfrak{b}_{56,5}(n)
\end{aligned}
$$

(Cont. next page)

$$
\begin{aligned}
&- \frac{643}{3600}\,\mathfrak{b}_{56,6}(n) + \frac{523}{9600}\,\mathfrak{b}_{56,7}(n) - \frac{11411}{8400}\,\mathfrak{b}_{56,8}(n) - \frac{1581}{1400}\,\mathfrak{b}_{56,9}(n) \\
&- \frac{263}{2100}\,\mathfrak{b}_{56,10}(n) - \frac{11}{24}\,\mathfrak{b}_{56,11}(n) - \frac{53}{60}\,\mathfrak{b}_{56,12}(n) + \frac{11}{12}\,\mathfrak{b}_{56,13}(n) \\
&+ \frac{1969}{2100}\,\mathfrak{b}_{56,14}(n) - \frac{11}{12}\,\mathfrak{b}_{56,15}(n) - \frac{13133}{3150}\,\mathfrak{b}_{56,16}(n) - \frac{11}{6}\,\mathfrak{b}_{56,17}(n) \\
&- \frac{1579}{1575}\,\mathfrak{b}_{56,18}(n) - \frac{20}{3}\,\mathfrak{b}_{56,19}(n) - \frac{5}{6}\,\mathfrak{b}_{56,20}(n). \qquad (42)
\end{aligned}
$$

Proof. These identities follow from Theorem 5 when we set $(\alpha,\beta) = (1,33),(3,11),(1,40),(5,8),(1,56),(7,8)$. □

6 Sample of the Evaluation of the Convolution Sums when the Level is in $\mathbb{N}_0 \setminus \mathfrak{N}$

Explicit formulae for the convolution sums $W_{(1,45)}(n)$, $W_{(5,9)}(n)$, $W_{(1,50)}(n)$, $W_{(2,25)}(n)$, $W_{(1,54)}(n)$ and $W_{(2,27)}(n)$ are provided in this section. Since these levels belong to $\mathbb{N}_0 \setminus \mathfrak{N}$, the primitive Dirichlet characters are non-trivial.

The two convolution sums $W_{(1,50)}(n)$ and $W_{(2,25)}(n)$ are worth mentioning due to the fact that the positive divisors of 50 which are associated with the Dirichlet character for the formation of a basis of the space of Eisenstein forms constitute the entire set of positive divisors of 50.

6.1 Bases of $E_4(\Gamma_0(\alpha\beta))$ and $S_4(\Gamma_0(\alpha\beta))$ when $\alpha\beta = 45, 50, 54$

The dimension formulae for the space of cusp forms as given in T. Miyake's book[28] and W. A. Stein's book[29] and (18) are applied to compute

$$
\begin{aligned}
&\dim(E_4(\Gamma_0(45))) = 8, && \dim(S_4(\Gamma_0(45))) = 14, \\
&\dim(E_4(\Gamma_0(50))) = 12, && \dim(S_4(\Gamma_0(50))) = 17, \\
&\dim(E_4(\Gamma_0(54))) = 12, && \dim(S_4(\Gamma_0(54))) = 21.
\end{aligned}
$$

We use Theorem 1 to determine many eta quotients which are elements of the spaces $S_4(\Gamma_0(45))$, $S_4(\Gamma_0(50))$ and $S_4(\Gamma_0(54))$, respectively.

Let $D(45)$, $D(50)$ and $D(54)$ denote the sets of positive divisors of 45, 50 and 54, respectively.

[28] Miyake, 1989, *Modular Forms*, Thrm 2.5.2, p. 60.
[29] Stein, 2011, *Modular Forms, A Computational Approach*, Prop. 6.1, p. 91.

We observe that

$$M_4(\Gamma_0(5)) \subset M_4(\Gamma_0(15)) \subset M_4(\Gamma_0(45)) \tag{43}$$
$$M_4(\Gamma_0(9)) \subset M_4(\Gamma_0(45)) \tag{44}$$
$$M_4(\Gamma_0(5)) \subset M_4(\Gamma_0(25)) \subset M_4(\Gamma_0(50)) \tag{45}$$
$$M_4(\Gamma_0(5)) \subset M_4(\Gamma_0(10)) \subset M_4(\Gamma_0(50)). \tag{46}$$
$$M_4(\Gamma_0(6)) \subset M_4(\Gamma_0(18)) \subset M_4(\Gamma_0(54)) \tag{47}$$
$$M_4(\Gamma_0(9)) \subset M_4(\Gamma_0(18)) \subset M_4(\Gamma_0(54)) \tag{48}$$
$$M_4(\Gamma_0(9)) \subset M_4(\Gamma_0(27)) \subset M_4(\Gamma_0(54)) \tag{49}$$

Corollary 6 – ***(a)*** *Let $n \in \mathbb{N}$, $\chi(n) = (\frac{-4}{n})$ and $\psi(n) = (\frac{-3}{n})$ be primitive Dirichlet characters such that $\chi(n)$ is not an annihilator of $E_4(\Gamma_0(45))$ and $E_4(\Gamma_0(54))$, and $\psi(n)$ is not an annihilator of $E_4(\Gamma_0(50))$. Then the sets*

$$\mathcal{B}_{E,45} = \{M(q^t) \mid t|45\} \cup \{M_{(\frac{-4}{n})}(q^s) \mid s = 1, 3\},$$
$$\mathcal{B}_{E,50} = \{M(q^t) \mid t|50\} \cup \{M_{(\frac{-3}{n})}(q^s) \mid s \in D(50)\} \text{ and}$$
$$\mathcal{B}_{E,54} = \{M(q^t) \mid t|54\} \cup \{M_{(\frac{-4}{n})}(q^s) \mid s = 1, 3, 9, 27\}$$

constitute bases of $E_4(\Gamma_0(45))$, $E_4(\Gamma_0(50))$ and $E_4(\Gamma_0(54))$, respectively.

(b) *Let $1 \le i \le 14$, $1 \le j \le 17$ and $1 \le k \le 21$ be positive integers.*
Let $\delta_1 \in D(45)$ and $(r(i,\delta_1))_{i,\delta_1}$ be the Table 8 of the powers of $\eta(\delta_1 z)$.
Let $\delta_2 \in D(50)$ and $(r(j,\delta_2))_{j,\delta_2}$ be the Table 9 of the powers of $\eta(\delta_2 z)$.
Let $\delta_3 \in D(54)$ and $(r(k,\delta_3))_{k,\delta_3}$ be the Table 13 of the powers of $\eta(\delta_3 z)$.
Let furthermore

$$\mathfrak{B}_{45,i}(q) = \prod_{\delta_1|45} \eta^{r(i,\delta_1)}(\delta_1 z), \quad \mathfrak{B}_{50,j}(q) = \prod_{\delta_2|50} \eta^{r(j,\delta_2)}(\delta_2 z) \quad \text{and}$$
$$\mathfrak{B}_{54,k}(q) = \prod_{\delta_3|54} \eta^{r(k,\delta_3)}(\delta_3 z)$$

be selected elements of $S_4(\Gamma_0(45))$, $S_4(\Gamma_0(50))$ and $S_4(\Gamma_0(54))$, respectively.
The sets

$$\mathcal{B}_{S,45} = \{\mathfrak{B}_{45,i}(q) \mid 1 \le i \le 14\}, \quad \mathcal{B}_{S,50} = \{\mathfrak{B}_{50,j}(q) \mid 1 \le j \le 17\} \quad \text{and}$$
$$\mathcal{B}_{S,54} = \{\mathfrak{B}_{54,k}(q) \mid 1 \le k \le 21\}$$

are bases of $S_4(\Gamma_0(45))$, $S_4(\Gamma_0(50))$ and $S_4(\Gamma_0(54))$, respectively.

(c) *The sets*

$$\mathcal{B}_{M,45} = \mathcal{B}_{E,45} \cup \mathcal{B}_{S,45}, \quad \mathcal{B}_{M,50} = \mathcal{B}_{E,50} \cup \mathcal{B}_{S,50} \quad \textit{and}$$
$$\mathcal{B}_{M,54} = \mathcal{B}_{E,54} \cup \mathcal{B}_{S,54}$$

constitute bases of $M_4(\Gamma_0(45))$, $M_4(\Gamma_0(50))$ *and* $M_4(\Gamma_0(54))$, *respectively.*

By Remark 1 (r1), each $\mathfrak{B}_{\alpha\beta,i}(q)$ is expressible in the form $\sum_{n=1}^{\infty} \mathfrak{b}_{\alpha\beta,i}(n)q^n$.

Proof. Let $n \in \mathbb{N}$. It holds that $45 = 3^2 \times 5$ and $54 = 2 \times 3^3$. Since $\gcd(4,3) = 1$, it holds that the primitive Dirichlet character $\chi(n) = (\frac{-4}{n})$ is not an annihilator of $E_4(\Gamma_0(3^2))$ and $E_4(\Gamma_0(3^3))$. Hence, $\chi(n) = (\frac{-4}{n})$ is not an annihilator of $E_4(\Gamma_0(45))$ and $E_4(\Gamma_0(54))$. Similarly, since $\gcd(3,5) = 1$, the primitive Dirichlet character $\psi(n) = (\frac{-3}{n})$ is not an annihilator of the space $E_4(\Gamma_0(5^2))$. Therefore, $\psi(n) = (\frac{-3}{n})$ is not an annihilator of $E_4(\Gamma_0(50))$.

We only give the proof for $\mathcal{B}_{M,45} = \mathcal{B}_{E,45} \cup \mathcal{B}_{S,45}$ since the other cases are proved similarly. In the case of $\mathcal{B}_{E,50}$, the applicable primitive Dirichlet character

$$\left(\frac{-3}{n}\right) = \begin{cases} -1 & \text{if } n \equiv 2 \pmod 3, \\ 0 & \text{if } \gcd(3,n) \neq 1, \\ 1 & \text{if } n \equiv 1 \pmod 3. \end{cases} \tag{50}$$

(a) Suppose that $x_\delta, z_1, z_3 \in \mathbb{C}$ with $\delta|45$. Let

$$\sum_{\delta|45} x_\delta \, M(q^\delta) + z_1 \, M_{(\frac{-4}{n})}(q) + z_3 \, M_{(\frac{-4}{n})}(q^3) = 0.$$

We observe that

$$\left(\frac{-4}{n}\right) = \begin{cases} -1 & \text{if } n \equiv 3 \pmod 4, \\ 0 & \text{if } \gcd(4,n) \neq 1, \\ 1 & \text{if } n \equiv 1 \pmod 4. \end{cases} \tag{51}$$

and recall that for all $0 \neq a \in \mathbb{Z}$ it holds that $(\frac{a}{0}) = 0$. Since the conductor of the Dirichlet character $(\frac{-4}{n})$ is 4, we infer from (5) that $C_0 = 0$. We then deduce

$$\sum_{\delta|45} x_\delta + \sum_{i=1}^{\infty} \left(240 \sum_{\delta|45} \sigma_3\left(\frac{n}{\delta}\right) x_\delta + \left(\frac{-4}{n}\right) \sigma_3(n) z_1 + \left(\frac{-4}{n}\right) \sigma_3\left(\frac{n}{3}\right) z_3 \right) q^n = 0.$$

Then we equate the coefficients of q^n for $n \in D(45)$ plus for example $n = 2, 7$ to obtain a system of 8 linear equations whose unique solution is $x_\delta = z_1 = z_3 = 0$ with $\delta \in D(45)$. So, the set $\mathcal{B}_E$ is linearly independent. Hence, the set $\mathcal{B}_E$ is a basis of $E_4(\Gamma_0(45))$.

(b) Suppose that $x_i \in \mathbb{C}$ with $1 \leq i \leq 14$. Let $\sum_{i=1}^{14} x_i\, \mathfrak{B}_{45,i}(q) = 0$. Then

$$\sum_{i=1}^{14} x_i \sum_{n=1}^{\infty} \mathfrak{b}_{45,i}(n) q^n = \sum_{n=1}^{\infty} \Big(\sum_{i=1}^{14} \mathfrak{b}_{45,i}(n)\, x_i \Big) q^n = 0.$$

So, we equate the coefficients of q^n for $1 \leq n \leq 14$ to obtain a system of 14 linear equations whose unique solution is $x_i = 0$ for all $1 \leq i \leq 14$. It follows that the set $\mathcal{B}_S$ is linearly independent. Hence, the set $\mathcal{B}_S$ is a basis of $S_4(\Gamma_0(45))$.

(c) Since $M_4(\Gamma_0(45)) = E_4(\Gamma_0(45)) \oplus S_4(\Gamma_0(45))$, the result follows from (a) and (b).

□

6.2 Evaluation of $W_{(\alpha,\beta)}(n)$ when $\alpha\beta = 45, 50, 54$

In this section, the evaluation of the convolution sum $W_{(\alpha,\beta)}(n)$ is discussed for $(\alpha,\beta) = (1,45)$, $(5,9)$, $(1,50)$, $(2,25)$, $(1,54)$ and $(2,27)$.

Corollary 7 – *It holds that*

$$\begin{aligned}
(5\,L(q^5) - 9\,L(q^9))^2 = 16 + \sum_{n=1}^{\infty} \Big(& -\frac{120}{13}\sigma_3(n) - \frac{51960}{923}\sigma_3\Big(\frac{n}{3}\Big) + \frac{75000}{13}\sigma_3\Big(\frac{n}{5}\Big) \\
& + \frac{1296000}{71}\sigma_3\Big(\frac{n}{9}\Big) - \frac{5089800}{923}\sigma_3\Big(\frac{n}{15}\Big) + \frac{5184000}{71}\sigma_3\Big(\frac{n}{45}\Big) - \frac{19344}{1349}\,\mathfrak{b}_{45,1}(n) \\
& + \frac{239256}{1349}\,\mathfrak{b}_{45,2}(n) + \frac{10760952}{17537}\,\mathfrak{b}_{45,3}(n) + \frac{762672}{1349}\,\mathfrak{b}_{45,4}(n) \\
& - \frac{2459904}{1349}\,\mathfrak{b}_{45,5}(n) + \frac{1247280}{1349}\,\mathfrak{b}_{45,6}(n) + \frac{5755968}{1349}\,\mathfrak{b}_{45,7}(n) \\
& + \frac{370080}{71}\,\mathfrak{b}_{45,8}(n) + \frac{2503632}{1349}\,\mathfrak{b}_{45,9}(n) + \frac{302400}{71}\,\mathfrak{b}_{45,10}(n) \\
& - \frac{14389920}{1349}\,\mathfrak{b}_{45,11}(n) + \frac{413352}{17537}\,\mathfrak{b}_{45,12}(n) + \frac{11760}{1349}\,\mathfrak{b}_{45,13}(n) \Big) q^n.
\end{aligned} \tag{52}$$

$$\begin{aligned}
(L(q) - 45\,L(q^{45}))^2 = 1936 + \sum_{n=1}^{\infty} \Big(& -\frac{120}{13}\sigma_3(n) - \frac{51960}{923}\sigma_3\Big(\frac{n}{3}\Big) + \frac{75000}{13}\sigma_3\Big(\frac{n}{5}\Big) \\
& + \frac{1296000}{71}\sigma_3\Big(\frac{n}{9}\Big) - \frac{5089800}{923}\sigma_3\Big(\frac{n}{15}\Big) + \frac{5184000}{71}\sigma_3\Big(\frac{n}{45}\Big) - \frac{19344}{1349}\,\mathfrak{b}_{45,1}(n) \\
& + \frac{239256}{1349}\,\mathfrak{b}_{45,2}(n) + \frac{10760952}{17537}\,\mathfrak{b}_{45,3}(n) + \frac{762672}{1349}\,\mathfrak{b}_{45,4}(n)
\end{aligned}$$

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$$- \frac{2459904}{1349}\,\mathfrak{b}_{45,5}(n) + \frac{1247280}{1349}\,\mathfrak{b}_{45,6}(n) + \frac{5755968}{1349}\,\mathfrak{b}_{45,7}(n)$$
$$+ \frac{370080}{71}\,\mathfrak{b}_{45,8}(n) + \frac{2503632}{1349}\,\mathfrak{b}_{45,9}(n) + \frac{302400}{71}\,\mathfrak{b}_{45,10}(n)$$
$$- \frac{14389920}{1349}\,\mathfrak{b}_{45,11}(n) + \frac{413352}{17537}\,\mathfrak{b}_{45,12}(n) + \frac{11760}{1349}\,\mathfrak{b}_{45,13}(n)\Bigg)q^n. \tag{53}$$

$$(2\,L(q^2) - 25\,L(q^{25}))^2 = 529 + \sum_{n=1}^{\infty}\Bigg(\frac{810}{13}\,\sigma_3(n) + \frac{11460}{13}\,\sigma_3\Big(\frac{n}{2}\Big) - \frac{3210}{13}\,\sigma_3\Big(\frac{n}{5}\Big)$$
$$- 660\,\sigma_3\Big(\frac{n}{10}\Big) + \frac{1890000}{13}\,\sigma_3\Big(\frac{n}{25}\Big) - \frac{240000}{13}\,\sigma_3\Big(\frac{n}{50}\Big) - \frac{810}{13}\,\mathfrak{b}_{50,1}(n)$$
$$+ \frac{6714}{13}\,\mathfrak{b}_{50,2}(n) - 1620\,\mathfrak{b}_{50,3}(n) - 4230\,\mathfrak{b}_{50,4}(n) - \frac{178950}{13}\,\mathfrak{b}_{50,5}(n)$$
$$- 20250\,\mathfrak{b}_{50,6}(n) + 810\,\mathfrak{b}_{50,7}(n) - 13050\,\mathfrak{b}_{50,8}(n) + 12420\,\mathfrak{b}_{50,9}(n)$$
$$- \frac{68400}{13}\,\mathfrak{b}_{50,10}(n) - 4500\,\mathfrak{b}_{50,11}(n) - 36000\,\mathfrak{b}_{50,12}(n) - 21150\,\mathfrak{b}_{50,13}(n)$$
$$+ 1800\,\mathfrak{b}_{50,14}(n) - 15000\,\mathfrak{b}_{50,15}(n) + 20700\,\mathfrak{b}_{50,16}(n) + 28800\,\mathfrak{b}_{50,17}(n)\Bigg)q^n. \tag{54}$$

$$(L(q) - 50\,L(q^{50}))^2 = 2401 + \sum_{n=1}^{\infty}\Bigg(\frac{7590}{13}\,\sigma_3(n) + \frac{13500}{13}\,\sigma_3\Big(\frac{n}{2}\Big) - \frac{6870}{13}\,\sigma_3\Big(\frac{n}{5}\Big)$$
$$- \frac{23100}{13}\,\sigma_3\Big(\frac{n}{10}\Big) - \frac{60000}{13}\,\sigma_3\Big(\frac{n}{25}\Big) + \frac{7560000}{13}\,\sigma_3\Big(\frac{n}{50}\Big) + \frac{38772}{13}\,\mathfrak{b}_{50,1}(n)$$
$$+ \frac{639792}{13}\,\mathfrak{b}_{50,2}(n) - 7020\,\mathfrak{b}_{50,3}(n) - 20250\,\mathfrak{b}_{50,4}(n) - \frac{721050}{13}\,\mathfrak{b}_{50,5}(n)$$
$$- 116550\,\mathfrak{b}_{50,6}(n) - 2250\,\mathfrak{b}_{50,7}(n) - 123750\,\mathfrak{b}_{50,8}(n) + 99900\,\mathfrak{b}_{50,9}(n)$$
$$- \frac{910800}{13}\,\mathfrak{b}_{50,10}(n) - 38700\,\mathfrak{b}_{50,11}(n) - 309600\,\mathfrak{b}_{50,12}(n) + 13950\,\mathfrak{b}_{50,13}(n)$$
$$- 88200\,\mathfrak{b}_{50,14}(n) - 129000\,\mathfrak{b}_{50,15}(n) - 6300\,\mathfrak{b}_{50,16}(n) + 28800\,\mathfrak{b}_{50,17}(n)\Bigg)q^n. \tag{55}$$

$$(L(q) - 54\,L(q^{54}))^2 = 2809 + \sum_{n=1}^{\infty}\Bigg(\frac{86736}{305}\,\sigma_3(n) + \frac{1016064}{305}\,\sigma_3\Big(\frac{n}{2}\Big)$$
$$+ \frac{3796704}{305}\,\sigma_3\Big(\frac{n}{3}\Big) - \frac{32553792}{305}\,\sigma_3\Big(\frac{n}{6}\Big) - \frac{62804160}{61}\,\sigma_3\Big(\frac{n}{9}\Big)$$

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$$
\begin{aligned}
&+\frac{306688896}{305}\,\sigma_3\Big(\frac{n}{18}\Big)+\frac{61725888}{61}\,\sigma_3\Big(\frac{n}{27}\Big)-\frac{68024448}{305}\,\sigma_3\Big(\frac{n}{54}\Big)\\
&+\frac{689184}{305}\,\mathfrak{b}_{54,1}(n)+\frac{417024}{61}\,\mathfrak{b}_{54,2}(n)-\frac{1963008}{61}\,\mathfrak{b}_{54,4}(n)\\
&-\frac{746496}{61}\,\mathfrak{b}_{54,5}(n)-\frac{34450272}{305}\,\mathfrak{b}_{54,6}(n)-\frac{8259840}{61}\,\mathfrak{b}_{54,7}(n)\\
&-\frac{12462336}{61}\,\mathfrak{b}_{54,8}(n)-\frac{13347072}{305}\,\mathfrak{b}_{54,9}(n)-\frac{32368896}{61}\,\mathfrak{b}_{54,10}(n)\\
&-\frac{2664576}{61}\,\mathfrak{b}_{54,11}(n)+\frac{30642624}{305}\,\mathfrak{b}_{54,12}(n)+\frac{5889024}{61}\,\mathfrak{b}_{54,13}(n)\\
&-\frac{148252032}{61}\,\mathfrak{b}_{54,14}(n)+\frac{1919808}{305}\,\mathfrak{b}_{54,15}(n)-\frac{17943552}{61}\,\mathfrak{b}_{54,16}(n)\\
&+\frac{18009216}{61}\,\mathfrak{b}_{54,17}(n)-\frac{418176}{61}\,\mathfrak{b}_{54,18}(n)+\frac{946944}{61}\,\mathfrak{b}_{54,19}(n)\\
&-\frac{4686336}{61}\,\mathfrak{b}_{54,20}(n)+\frac{41682816}{305}\,\mathfrak{b}_{54,21}(n)\Big)q^n \qquad (56)
\end{aligned}
$$

$$
\begin{aligned}
(2\,L(q^2)-27\,L(q^{27}))^2 &= 625+\sum_{n=1}^{\infty}\Big(-\frac{2592}{61}\,\sigma_3(n)+\frac{196416}{61}\,\sigma_3\Big(\frac{n}{2}\Big)\\
&-\frac{41777}{2745}\,\sigma_3\Big(\frac{n}{3}\Big)-\frac{146803358}{2745}\,\sigma_3\Big(\frac{n}{6}\Big)-\frac{20326}{45}\,\sigma_3\Big(\frac{n}{9}\Big)\\
&-\frac{2031846244}{2745}\,\sigma_3\Big(\frac{n}{18}\Big)+\frac{851427}{5}\,\sigma_3\Big(\frac{n}{27}\Big)+\frac{235058418}{305}\,\sigma_3\Big(\frac{n}{54}\Big)\\
&+\frac{2592}{61}\,\mathfrak{b}_{54,1}(n)-\frac{21504}{61}\,\mathfrak{b}_{54,2}(n)+\frac{3657617}{2745}\,\mathfrak{b}_{54,3}(n)\\
&-\frac{1009152}{61}\,\mathfrak{b}_{54,4}(n)+\frac{311040}{61}\,\mathfrak{b}_{54,5}(n)+\mathfrak{b}_{54,6}(n)-\frac{4112640}{61}\,\mathfrak{b}_{54,7}(n)\\
&-\frac{9248256}{61}\,\mathfrak{b}_{54,8}(n)+\frac{29090321}{915}\,\mathfrak{b}_{54,9}(n)-\frac{19927296}{61}\,\mathfrak{b}_{54,10}(n)\\
&+\frac{1057536}{61}\,\mathfrak{b}_{54,11}(n)-\frac{5350010}{61}\,\mathfrak{b}_{54,12}(n)+\frac{6822144}{61}\,\mathfrak{b}_{54,13}(n)\\
&-\frac{93125376}{61}\,\mathfrak{b}_{54,14}(n)+\frac{6842758}{183}\,\mathfrak{b}_{54,15}(n)-\frac{10934784}{61}\,\mathfrak{b}_{54,16}(n)\\
&+\frac{559872}{61}\,\mathfrak{b}_{54,17}(n)+\frac{297216}{61}\,\mathfrak{b}_{54,18}(n)+\frac{5965056}{61}\,\mathfrak{b}_{54,19}(n)\\
&+\frac{1741824}{61}\,\mathfrak{b}_{54,20}(n)-\frac{6843124}{183}\,\mathfrak{b}_{54,21}(n)\Big)q^n \qquad (57)
\end{aligned}
$$

Proof. We give the proof for the case where $\alpha = 5$ and $\beta = 9$. The proof for the other cases can be done similarly.

This follows immediately when one sets $\alpha = 5$ and $\beta = 9$ in Lemma 2. However, we briefly show the proof for $(5\,L(q^5) - 9\,L(q^9))^2$ as an example. One obtains

$$\begin{aligned}(5\,L(q^5) - 9\,L(q^9))^2 &= \sum_{\delta|45} x_\delta M(q^\delta) + z_1\, M_{(\frac{-4}{n})}(q) + z_3\, M_{(\frac{-4}{n})}(q^3) + \sum_{j=1}^{14} y_j\, \mathfrak{B}_{45,j}(q) \\ &= \sum_{\delta|45} x_\delta + \sum_{i=1}^{\infty} \Bigg(\sum_{\delta|45} 240\,\sigma_3\Big(\frac{n}{\delta}\Big)\, x_\delta + \Big(\frac{-4}{n}\Big)\sigma_3(n)\, z_1 \\ &\qquad + \Big(\frac{-4}{n}\Big)\sigma_3\Big(\frac{n}{3}\Big)\, z_3 + \sum_{j=1}^{14} \mathfrak{b}_{45,j}(n)\, y_j \Bigg) q^n. \end{aligned} \tag{58}$$

Since the conductor of the Dirichlet character $(\frac{-4}{n})$ is 4, from (5) we have $C_0 = 0$. Now when we equate the right hand side of (58) with that of (12), and when we take the coefficients of q^n for which $1 \le n \le 15$ and $n = 17, 19, 21, 23, 25, 27, 45$ for example, we obtain a system of linear equations with a unique solution. Hence, we obtain the stated result. □

Now we state and prove our main result of this subsection.

Corollary 8 – *Let n be a positive integer. Then*

$$\begin{aligned} W_{(5,9)}(n) &= \frac{1}{5616}\sigma_3(n) + \frac{433}{398736}\,\sigma_3\Big(\frac{n}{3}\Big) + \frac{25}{5616}\sigma_3\Big(\frac{n}{5}\Big) + \frac{13}{568}\sigma_3\Big(\frac{n}{9}\Big) \\ &+ \frac{42415}{398736}\sigma_3\Big(\frac{n}{15}\Big) - \frac{100}{71}\sigma_3\Big(\frac{n}{45}\Big) + \Big(\frac{1}{24} - \frac{1}{36}n\Big)\sigma\Big(\frac{n}{5}\Big) + \Big(\frac{1}{24} - \frac{1}{20}n\Big)\sigma\Big(\frac{n}{9}\Big) \\ &+ \frac{403}{1456920}\,\mathfrak{b}_{45,1}(n) - \frac{3323}{971280}\,\mathfrak{b}_{45,2}(n) - \frac{448373}{37879920}\,\mathfrak{b}_{45,3}(n) \\ &- \frac{15889}{1456920}\,\mathfrak{b}_{45,4}(n) + \frac{6406}{182115}\,\mathfrak{b}_{45,5}(n) - \frac{5197}{291384}\,\mathfrak{b}_{45,6}(n) \\ &- \frac{3331}{40470}\,\mathfrak{b}_{45,7}(n) - \frac{257}{2556}\,\mathfrak{b}_{45,8}(n) - \frac{52159}{1456920}\,\mathfrak{b}_{45,9}(n) - \frac{35}{426}\,\mathfrak{b}_{45,10}(n) \\ &+ \frac{3331}{16188}\,\mathfrak{b}_{45,11}(n) - \frac{5741}{12626640}\,\mathfrak{b}_{45,12}(n) - \frac{49}{291384}\,\mathfrak{b}_{45,13}(n). \end{aligned} \tag{59}$$

$$\begin{aligned} W_{(1,45)}(n) &= \frac{217}{1872}\sigma_3(n) + \frac{433}{398736}\sigma_3\Big(\frac{n}{3}\Big) - \frac{625}{5616}\sigma_3\Big(\frac{n}{5}\Big) - \frac{25}{71}\sigma_3\Big(\frac{n}{9}\Big) \\ &+ \frac{42415}{398736}\sigma_3\Big(\frac{n}{15}\Big) - \frac{587}{568}\sigma_3\Big(\frac{n}{45}\Big) + \Big(\frac{1}{24} - \frac{1}{36}n\Big)\sigma\Big(\frac{n}{5}\Big) + \Big(\frac{1}{24} - \frac{1}{20}n\Big)\sigma\Big(\frac{n}{9}\Big) \\ &+ \frac{403}{1456920}\,\mathfrak{b}_{45,1}(n) - \frac{3323}{971280}\,\mathfrak{b}_{45,2}(n) - \frac{448373}{37879920}\,\mathfrak{b}_{45,3}(n) \\ &- \frac{15889}{1456920}\,\mathfrak{b}_{45,4}(n) + \frac{6406}{182115}\,\mathfrak{b}_{45,5}(n) - \frac{5197}{291384}\,\mathfrak{b}_{45,6}(n) - \frac{3331}{40470}\,\mathfrak{b}_{45,7}(n) \end{aligned}$$

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$$- \frac{257}{2556}\, \mathfrak{b}_{45,8}(n) - \frac{52159}{1456920}\, \mathfrak{b}_{45,9}(n) - \frac{35}{426}\, \mathfrak{b}_{45,10}(n) + \frac{3331}{16188}\, \mathfrak{b}_{45,11}(n) - \frac{5741}{12626640}\, \mathfrak{b}_{45,12}(n) - \frac{49}{291384}\, \mathfrak{b}_{45,13}(n). \quad (60)$$

$$\begin{aligned} W_{(2,25)}(n) = &-\frac{9}{8320}\,\sigma_3(n) + \frac{17}{12480}\,\sigma_3\Big(\frac{n}{2}\Big) + \frac{107}{24960}\,\sigma_3\Big(\frac{n}{5}\Big) + \frac{11}{960}\,\sigma_3\Big(\frac{n}{10}\Big) \\ &+ \frac{25}{312}\,\sigma_3\Big(\frac{n}{25}\Big) + \frac{25}{78}\,\sigma_3\Big(\frac{n}{50}\Big) + \Big(\frac{1}{24} - \frac{1}{100}n\Big)\sigma\Big(\frac{n}{2}\Big) + \Big(\frac{1}{24} - \frac{1}{8}n\Big)\sigma\Big(\frac{n}{25}\Big) \\ &+ \frac{9}{8320}\,\mathfrak{b}_{50,1}(n) - \frac{373}{41600}\,\mathfrak{b}_{50,2}(n) + \frac{9}{320}\,\mathfrak{b}_{50,3}(n) + \frac{47}{640}\,\mathfrak{b}_{50,4}(n) \\ &+ \frac{1193}{4992}\,\mathfrak{b}_{50,5}(n) + \frac{45}{128}\,\mathfrak{b}_{50,6}(n) - \frac{9}{640}\,\mathfrak{b}_{50,7}(n) + \frac{29}{128}\,\mathfrak{b}_{50,8}(n) \\ &- \frac{69}{320}\,\mathfrak{b}_{50,9}(n) + \frac{19}{208}\,\mathfrak{b}_{50,10}(n) + \frac{5}{64}\,\mathfrak{b}_{50,11}(n) + \frac{5}{8}\,\mathfrak{b}_{50,12}(n) + \frac{47}{128}\,\mathfrak{b}_{50,13}(n) \\ &- \frac{1}{32}\,\mathfrak{b}_{50,14}(n) + \frac{25}{96}\,\mathfrak{b}_{50,15}(n) - \frac{23}{64}\,\mathfrak{b}_{50,16}(n) + \frac{1}{2}\,\mathfrak{b}_{50,17}(n) \end{aligned} \quad (61)$$

$$\begin{aligned} W_{(1,50)}(n) = &-\frac{149}{24960}\,\sigma_3(n) - \frac{15}{832}\,\sigma_3\Big(\frac{n}{2}\Big) + \frac{229}{24960}\,\sigma_3\Big(\frac{n}{5}\Big) + \frac{77}{2496}\,\sigma_3\Big(\frac{n}{10}\Big) \\ &+ \frac{25}{312}\,\sigma_3\Big(\frac{n}{25}\Big) + \frac{25}{78}\,\sigma_3\Big(\frac{n}{50}\Big) + \Big(\frac{1}{24} - \frac{1}{200}n\Big)\sigma(n) + \Big(\frac{1}{24} - \frac{1}{4}n\Big)\sigma\Big(\frac{n}{50}\Big) \\ &- \frac{1277}{41600}\,\mathfrak{b}_{50,1}(n) - \frac{243}{1664}\,\mathfrak{b}_{50,2}(n) + \frac{39}{320}\,\mathfrak{b}_{50,3}(n) + \frac{45}{128}\,\mathfrak{b}_{50,4}(n) \\ &+ \frac{4807}{4992}\,\mathfrak{b}_{50,5}(n) + \frac{259}{128}\,\mathfrak{b}_{50,6}(n) + \frac{5}{128}\,\mathfrak{b}_{50,7}(n) + \frac{275}{128}\,\mathfrak{b}_{50,8}(n) \\ &- \frac{111}{64}\,\mathfrak{b}_{50,9}(n) + \frac{253}{208}\,\mathfrak{b}_{50,10}(n) + \frac{43}{64}\,\mathfrak{b}_{50,11}(n) + \frac{43}{8}\,\mathfrak{b}_{50,12}(n) - \frac{31}{128}\,\mathfrak{b}_{50,13}(n) \\ &+ \frac{49}{32}\,\mathfrak{b}_{50,14}(n) + \frac{215}{96}\,\mathfrak{b}_{50,15}(n) + \frac{7}{64}\,\mathfrak{b}_{50,16}(n) - \frac{1}{2}\,\mathfrak{b}_{50,17}(n), \end{aligned} \quad (62)$$

$$\begin{aligned} W_{(1,54)}(n) = &-\frac{47}{65880}\,\sigma_3(n) - \frac{49}{915}\,\sigma_3\Big(\frac{n}{2}\Big) - \frac{13183}{65880}\,\sigma_3\Big(\frac{n}{3}\Big) + \frac{18839}{10980}\,\sigma_3\Big(\frac{n}{6}\Big) \\ &+ \frac{12115}{732}\,\sigma_3\Big(\frac{n}{9}\Big) - \frac{88741}{5490}\,\sigma_3\Big(\frac{n}{18}\Big) - \frac{3969}{244}\,\sigma_3\Big(\frac{n}{27}\Big) + \frac{18099}{1220}\,\sigma_3\Big(\frac{n}{54}\Big) \\ &+ \Big(\frac{1}{24} - \frac{1}{216}n\Big)\sigma(n) + \Big(\frac{1}{24} - \frac{1}{4}n\Big)\sigma\Big(\frac{n}{54}\Big) - \frac{2393}{65880}\,\mathfrak{b}_{54,1}(n) \\ &- \frac{181}{1647}\,\mathfrak{b}_{54,2}(n) + \frac{284}{549}\,\mathfrak{b}_{54,4}(n) + \frac{12}{61}\,\mathfrak{b}_{54,5}(n) + \frac{13291}{7320}\,\mathfrak{b}_{54,6}(n) \\ &+ \frac{1195}{549}\,\mathfrak{b}_{54,7}(n) + \frac{601}{183}\,\mathfrak{b}_{54,8}(n) + \frac{1931}{2745}\,\mathfrak{b}_{54,9}(n) + \frac{1561}{183}\,\mathfrak{b}_{54,10}(n) \\ &+ \frac{257}{366}\,\mathfrak{b}_{54,11}(n) - \frac{5911}{3660}\,\mathfrak{b}_{54,12}(n) - \frac{284}{183}\,\mathfrak{b}_{54,13}(n) + \frac{14299}{366}\,\mathfrak{b}_{54,14}(n) \end{aligned}$$

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$$
\begin{aligned}
&-\frac{1111}{10980}\,\mathfrak{b}_{54,15}(n)+\frac{2596}{549}\,\mathfrak{b}_{54,16}(n)-\frac{579}{122}\,\mathfrak{b}_{54,17}(n)+\frac{121}{1098}\,\mathfrak{b}_{54,18}(n)\\
&-\frac{137}{549}\,\mathfrak{b}_{54,19}(n)+\frac{226}{183}\,\mathfrak{b}_{54,20}(n)-\frac{12061}{5490}\,\mathfrak{b}_{54,21}(n) \qquad (63)
\end{aligned}
$$

$$
\begin{aligned}
W_{(2,27)}(n) &= \frac{1}{1464}\,\sigma_3(n)-\frac{359}{9882}\,\sigma_3\Big(\frac{n}{2}\Big)+\frac{41777}{170760960}\,\sigma_3\Big(\frac{n}{3}\Big)+\frac{73401679}{85380480}\,\sigma_3\Big(\frac{n}{6}\Big)\\
&+\frac{10163}{1399680}\,\sigma_3\Big(\frac{n}{9}\Big)+\frac{507961561}{42690240}\,\sigma_3\Big(\frac{n}{18}\Big)+\frac{2597}{34560}\,\sigma_3\Big(\frac{n}{27}\Big)\\
&-\frac{13058801}{1054080}\,\sigma_3\Big(\frac{n}{54}\Big)+\Big(\frac{1}{24}-\frac{1}{108}n\Big)\sigma\Big(\frac{n}{2}\Big)+\Big(\frac{1}{24}-\frac{1}{8}n\Big)\sigma\Big(\frac{n}{27}\Big)\\
&-\frac{1}{1464}\,\mathfrak{b}_{54,1}(n)+\frac{28}{4941}\,\mathfrak{b}_{54,2}(n)-\frac{3657617}{170760960}\,\mathfrak{b}_{54,3}(n)+\frac{146}{549}\,\mathfrak{b}_{54,4}(n)\\
&-\frac{5}{61}\,\mathfrak{b}_{54,5}(n)-\frac{1}{62208}\,\mathfrak{b}_{54,6}(n)+\frac{595}{549}\,\mathfrak{b}_{54,7}(n)+\frac{446}{183}\,\mathfrak{b}_{54,8}(n)\\
&-\frac{29090321}{56920320}\,\mathfrak{b}_{54,9}(n)+\frac{961}{183}\,\mathfrak{b}_{54,10}(n)-\frac{17}{61}\,\mathfrak{b}_{54,11}(n)+\frac{2675005}{1897344}\,\mathfrak{b}_{54,12}(n)\\
&-\frac{329}{183}\,\mathfrak{b}_{54,13}(n)+\frac{1497}{61}\,\mathfrak{b}_{54,14}(n)-\frac{3421379}{5692032}\,\mathfrak{b}_{54,15}(n)+\frac{1582}{549}\,\mathfrak{b}_{54,16}(n)\\
&-\frac{9}{61}\,\mathfrak{b}_{54,17}(n)-\frac{43}{549}\,\mathfrak{b}_{54,18}(n)-\frac{863}{549}\,\mathfrak{b}_{54,19}(n)\\
&-\frac{28}{61}\,\mathfrak{b}_{54,20}(n)+\frac{1710781}{2846016}\,\mathfrak{b}_{54,21}(n) \qquad (64)
\end{aligned}
$$

Proof. It follows immediately when we set $(\alpha,\beta)=(5,9),(1,45),(2,25),(1,50),(1,54),(2,27)$ in Theorem 5. □

7 Some Known Convolution Sums Revisited

In this section, we revisit some known convolution sums to illustrate our approach. While doing so, we observe that the results of these convolution sum for the levels 11, 12, 15, 16, 18, 25, 27, 32 and 36 are improved.

The basis elements of the space of cusp forms for each of these levels can be expressed as noted in Remark 1 (r1).

The dimension formulae for the space of cusp forms as given in T. Miyake's book[30] or W. A. Stein's book[31] and (18) are applied in the following to compute the dimension of the space of Eisenstein series and that of the space of cusp forms, respectively.

[30] Miyake, 1989, *Modular Forms*, Thrm 2.5.2, p. 60.
[31] Stein, 2011, *Modular Forms, A Computational Approach*, Prop. 6.1, p. 91.

7.1 Convolution Sums for Levels $\alpha\beta = 10, 11, 12, 15, 24$

These levels belong to $\mathfrak{N}$; consequently, their primitive Dirichlet characters are trivial.

We revisit the convolution sums established by

- E. Royer[32], and S. Cooper and D. Ye[33] for $\alpha\beta = 10$,
- E. Royer[34] for $\alpha\beta = 11$,
- A. Alaca et al.[35] for $\alpha\beta = 12$, 24, and
- B. Ramakrishnan and B. Sahu[36] for $\alpha\beta = 15$.

The obtained results in each case are immediate corollaries of Theorem 5 and improve the previous ones since we use the exact number of basis elements of the space of cusp forms in case of $\alpha\beta = 12$, 24.

Since $\alpha\beta = 10 = 2\cdot 5$ and because of (27) it holds that $\mathfrak{B}_{40,2}(q) = \mathfrak{B}_{40,1}(q^2)$, and therefore $\mathfrak{b}_{40,2}(n) = \mathfrak{b}_{40,1}\left(\frac{n}{2}\right)$. Our third basis element of the space $S_4(\Gamma_0(10))$ is different from the one used by D. Ye[37], which explains the difference in the two results. However, since the change of basis is an automorphism, both results are the same.

In addition to the basis element $\mathfrak{B}_{33,2}(q)$ of the space $S_4(\Gamma_0(11))$, we use the eta-quotient $\mathfrak{B}'_{33,1}(q) = \eta^2(z)\eta^2(11z) = \sum_{n=1}^{\infty} \mathfrak{b}'_{33,1}(n)q^n$ which is a basis element of $S_2(\Gamma_0(11))$.

B. Ramakrishnan and B. Sahu[38] achieve the evaluation of the convolution sums for $\alpha\beta = 15$ using a basis which contains one cusp form of weight 2. We consider the following η-quotients as basis elements of the space $S_4(\Gamma_0(15))$. These η-quotients are cusp form of weight 4.

$$\mathfrak{B}_{15,1}(q) = \eta^4(z)\eta^4(5z) \qquad \mathfrak{B}_{15,2}(q) = \eta^2(z)\eta^2(3z)\eta^2(5z)\eta^2(15z)$$
$$\mathfrak{B}_{15,3}(q) = \eta^4(3z)\eta^4(15z) \quad \mathfrak{B}_{15,4}(q) = \frac{\eta^3(z)\eta(3z)\eta^7(15z)}{\eta^3(5z)}.$$

---

[32] Royer, 2007, "Evaluating convolution sums of divisor function by quasi modular forms", Thrm 1.1.

[33] Cooper and Ye, 2014, "Evaluation of the convolution sums $\sum_{l+20m=n}\sigma(l)\sigma(m)$, $\sum_{4l+5m=n}\sigma(l)\sigma(m)$ and $\sum_{2l+5m=n}\sigma(l)\sigma(m)$", Thrm 2.1.

[34] Royer, 2007, "Evaluating convolution sums of divisor function by quasi modular forms", Thrm 1.3.

[35] A. Alaca, Ş. Alaca, and Williams, 2006, "Evaluation of the convolution sums $\sum_{l+12m=n}\sigma(l)\sigma(m)$ and $\sum_{3l+4m=n}\sigma(l)\sigma(m)$";
A. Alaca, Ş. Alaca, and Williams, 2007b, "Evaluation of the convolution sums $\sum_{l+24m=n}\sigma(l)\sigma(m)$ and $\sum_{3l+8m=n}\sigma(l)\sigma(m)$".

[36] Ramakrishnan and Sahu, 2013, "Evaluation of the convolution sums $\sum_{l+15m=n}\sigma(l)\sigma(m)$ and $\sum_{3l+5m=n}\sigma(l)\sigma(m)$".

[37] Cooper and Ye, 2014, "Evaluation of the convolution sums $\sum_{l+20m=n}\sigma(l)\sigma(m)$, $\sum_{4l+5m=n}\sigma(l)\sigma(m)$ and $\sum_{2l+5m=n}\sigma(l)\sigma(m)$".

[38] Ramakrishnan and Sahu, 2013, "Evaluation of the convolution sums $\sum_{l+15m=n}\sigma(l)\sigma(m)$ and $\sum_{3l+5m=n}\sigma(l)\sigma(m)$".

The η-quotients whose exponents are displayed in Table 4 build a basis of $S_4(\Gamma_0(24))$. It is obvious to verify using Remark 1 (r2) that these elements of the space of cusp forms are linearly independent; hence, they build a basis of $S_4(\Gamma_0(24))$.

Note that the space $S_4(\Gamma_0(12))$ is a subspace of the space $S_4(\Gamma_0(24))$. However, our selected $\mathfrak{B}_{24,3}(q)$ is not an element of $S_4(\Gamma_0(12))$; hence, we will use the element $\mathfrak{B}'_{24,3}(q) = \eta^4(2z)\eta^{-2}(4z)\eta^6(12z)$ instead.

It holds that $\mathfrak{B}_{15,3}(q) = \mathfrak{B}_{15,1}(q^2)$, and $\mathfrak{B}_{24,2}(q) = \mathfrak{B}_{24,1}(q^2)$, $\mathfrak{B}_{24,4}(q) = \mathfrak{B}_{24,1}(q^4)$ and $\mathfrak{B}_{24,6}(q) = \mathfrak{B}_{24,3}(q^2)$. Therefore $\mathfrak{b}_{15,3}(n) = \mathfrak{b}_{15,1}\left(\frac{n}{2}\right)$, and $\mathfrak{b}_{24,2}(n) = \mathfrak{b}_{24,1}\left(\frac{n}{2}\right)$, $\mathfrak{b}_{24,4}(n) = \mathfrak{b}_{24,1}\left(\frac{n}{4}\right)$ and $\mathfrak{b}_{24,6}(n) = \mathfrak{b}_{24,3}\left(\frac{n}{2}\right)$.

Corollary 9 – *We have*

$$\begin{aligned}(L(q)-10\,L(q^{10}))^2 = 81 + \sum_{n=1}^{\infty}\Bigg(&\frac{2640}{13}\,\sigma_3(n) - \frac{1920}{13}\,\sigma_3\left(\frac{n}{2}\right) - \frac{12000}{13}\,\sigma_3\left(\frac{n}{5}\right)\\ &+ \frac{264000}{13}\,\sigma_3\left(\frac{n}{10}\right) + \frac{2976}{13}\,\mathfrak{b}_{40,1}(n) + \frac{14400}{13}\,\mathfrak{b}_{40,2}(n) - 960\,\mathfrak{b}_{40,3}(n)\Bigg)q^n, \end{aligned} \tag{65}$$

$$\begin{aligned}(2\,L(q^2)-5\,L(q^5))^2 = 9 + \sum_{n=1}^{\infty}\Bigg(&-\frac{480}{13}\,\sigma_3(n) + \frac{10560}{13}\,\sigma_3\left(\frac{n}{2}\right) + \frac{66000}{13}\,\sigma_3\left(\frac{n}{5}\right)\\ &- \frac{48000}{13}\,\sigma_3\left(\frac{n}{10}\right) + \frac{480}{13}\,\mathfrak{b}_{40,1}(n) - \frac{576}{13}\,\mathfrak{b}_{40,2}(n) + 960\,\mathfrak{b}_{40,3}(n)\Bigg)q^n, \end{aligned} \tag{66}$$

$$\begin{aligned}(L(q)-11\,L(q^{11}))^2 = 100 + \sum_{n=1}^{\infty}\Bigg(&\frac{6240}{49}\,\sigma_3(n) + \frac{5524320}{49}\,\sigma_3\left(\frac{n}{11}\right)\\ &+ \frac{17280}{49}\,\mathfrak{b}'_{33,1}(n) + \frac{77184}{49}\,\mathfrak{b}_{33,2}(n)\Bigg)q^n, \end{aligned} \tag{67}$$

$$\begin{aligned}(L(q)-12\,L(q^{12}))^2 = 121 + \sum_{n=1}^{\infty}\Bigg(&\frac{1056}{5}\,\sigma_3(n) - \frac{432}{5}\,\sigma_3\left(\frac{n}{2}\right) - \frac{1296}{5}\,\sigma_3\left(\frac{n}{3}\right)\\ &- \frac{2304}{5}\,\sigma_3\left(\frac{n}{4}\right) - \frac{3888}{5}\,\sigma_3\left(\frac{n}{6}\right) + \frac{152064}{5}\,\sigma_3\left(\frac{n}{12}\right) + \frac{1584}{5}\,\mathfrak{b}_{24,1}(n)\\ &+ \frac{4896}{5}\,\mathfrak{b}_{24,2}(n) + 864\,\mathfrak{b}'_{24,3}(n)\Bigg)q^n, \end{aligned} \tag{68}$$

$$\begin{aligned}(3\,L(q^3)-4\,L(q^4))^2 = 1 + \sum_{n=1}^{\infty}\Bigg(&-\frac{144}{5}\,\sigma_3(n) - \frac{432}{5}\,\sigma_3\left(\frac{n}{2}\right) + \frac{9504}{5}\,\sigma_3\left(\frac{n}{3}\right)\\ &+ \frac{16896}{5}\,\sigma_3\left(\frac{n}{4}\right) - \frac{3888}{5}\,\sigma_3\left(\frac{n}{6}\right) - \frac{20736}{5}\,\sigma_3\left(\frac{n}{12}\right) + \frac{144}{5}\,\mathfrak{b}_{24,1}(n)\\ &+ \frac{2016}{5}\,\mathfrak{b}_{24,2}(n) - 864\,\mathfrak{b}'_{24,3}(n)\Bigg)q^n, \end{aligned} \tag{69}$$

$$
\begin{aligned}
(L(q)-15\,L(q^{15}))^2 = 196 + \sum_{n=1}^{\infty}\Bigg(&\frac{2976}{13}\,\sigma_3(n) - \frac{3456}{13}\,\sigma_3\Big(\frac{n}{3}\Big) - \frac{144000}{13}\,\sigma_3\Big(\frac{n}{5}\Big)\\
&+ \frac{756000}{13}\,\sigma_3\Big(\frac{n}{15}\Big) + \frac{5760}{13}\,\mathfrak{b}_{15,1}(n) + 2304\,\mathfrak{b}_{15,2}(n)\\
&+ \frac{48384}{13}\,\mathfrak{b}_{15,3}(n) - 3456\,\mathfrak{b}_{15,4}(n)\Bigg)q^n,
\end{aligned} \tag{70}
$$

$$
\begin{aligned}
(3\,L(q^3)-5\,L(q^5))^2 = 4 + \sum_{n=1}^{\infty}\Bigg(&-\frac{576}{13}\,\sigma_3(n) + \frac{25056}{13}\,\sigma_3\Big(\frac{n}{3}\Big) + \frac{204000}{13}\,\sigma_3\Big(\frac{n}{5}\Big)\\
&- \frac{216000}{13}\,\sigma_3\Big(\frac{n}{15}\Big) + \frac{576}{13}\,\mathfrak{b}_{15,1}(n) + 576\,\mathfrak{b}_{15,2}(n)\\
&+ \frac{8640}{13}\,\mathfrak{b}_{15,3}(n) + 3456\,\mathfrak{b}_{15,4}(n)\Bigg)q^n,
\end{aligned} \tag{71}
$$

$$
\begin{aligned}
(L(q)-24\,L(q^{24}))^2 = 529 + \sum_{n=1}^{\infty}\Bigg(&240\,\sigma_3(n) - 288\,\sigma_3\Big(\frac{n}{2}\Big) - 144\,\sigma_3\Big(\frac{n}{3}\Big)\\
&- \frac{2016}{5}\,\sigma_3\Big(\frac{n}{4}\Big) - 144\,\sigma_3\Big(\frac{n}{6}\Big) + \frac{32256}{5}\,\sigma_3\Big(\frac{n}{8}\Big) - \frac{6624}{5}\,\sigma_3\Big(\frac{n}{12}\Big)\\
&+ \frac{612864}{5}\,\sigma_3\Big(\frac{n}{24}\Big) + 864\,\mathfrak{b}_{24,1}(n) + 3744\,\mathfrak{b}_{24,2}(n) + 3888\,\mathfrak{b}_{24,3}(n)\\
&+ \frac{35136}{5}\,\mathfrak{b}_{24,4}(n) - 6912\,\mathfrak{b}_{24,5}(n) + 8640\,\mathfrak{b}_{24,6}(n)\\
&- 6912\,\mathfrak{b}_{24,7}(n) + 6912\,\mathfrak{b}_{24,8}(n)\Bigg)q^n,
\end{aligned} \tag{72}
$$

$$
\begin{aligned}
(3\,L(q^3)-8\,L(q^8))^2 = 25 + \sum_{n=1}^{\infty}\Bigg(&-\frac{72}{5}\,\sigma_3(n) - \frac{216}{5}\,\sigma_3\Big(\frac{n}{2}\Big) + \frac{10152}{5}\,\sigma_3\Big(\frac{n}{3}\Big)\\
&- \frac{864}{5}\,\sigma_3\Big(\frac{n}{4}\Big) - \frac{1944}{5}\,\sigma_3\Big(\frac{n}{6}\Big) + \frac{72192}{5}\,\sigma_3\Big(\frac{n}{8}\Big) - \frac{7776}{5}\,\sigma_3\Big(\frac{n}{12}\Big)\\
&- \frac{41472}{5}\,\sigma_3\Big(\frac{n}{24}\Big) + \frac{72}{5}\,\mathfrak{b}_{24,1}(n) + \frac{1008}{5}\,\mathfrak{b}_{24,2}(n) - 864\,\mathfrak{b}_{24,3}(n)\\
&+ \frac{9792}{5}\,\mathfrak{b}_{24,4}(n) + 1728\,\mathfrak{b}_{24,5}(n) - 5184\,\mathfrak{b}_{24,6}(n) + 8640\,\mathfrak{b}_{24,7}(n)\Bigg)q^n.
\end{aligned} \tag{73}
$$

In the case of the evaluation of $W_{(1,1)}(n)$, we observe, using Lemma 1, that for all $\alpha \in \mathbb{N}_0$ it hods that

$$
0 = (\alpha\,L(q^{\alpha}) - \alpha\,L(q^{\alpha}))^2 \in M_4(\Gamma_0(\alpha^2)). \tag{74}
$$

Corollary 10 – *Let n be a positive integer. Then*

$$\forall\alpha\in\mathbb{N}_0\quad W_{(\alpha,\alpha)}(n)=W_{(1,1)}\left(\frac{n}{\alpha}\right)=\frac{5}{12}\sigma_3\left(\frac{n}{\alpha}\right)+\left(\frac{1}{12}-\frac{1}{2\alpha}n\right)\sigma\left(\frac{n}{\alpha}\right),\tag{75}$$

$$\begin{aligned}W_{(1,10)}(n)&=\frac{1}{312}\sigma_3(n)+\frac{1}{78}\sigma_3\left(\frac{n}{2}\right)+\frac{25}{312}\sigma_3\left(\frac{n}{5}\right)+\frac{25}{78}\sigma_3\left(\frac{n}{10}\right)+\left(\frac{1}{24}-\frac{1}{40}n\right)\sigma(n)\\&+\left(\frac{1}{24}-\frac{1}{4}n\right)\sigma\left(\frac{n}{10}\right)-\frac{31}{1560}\mathfrak{b}_{40,1}(n)-\frac{5}{52}\mathfrak{b}_{40,1}\left(\frac{n}{2}\right)+\frac{1}{12}\mathfrak{b}_{40,3}(n),\end{aligned}\tag{76}$$

$$\begin{aligned}W_{(2,5)}(n)&=\frac{1}{312}\sigma_3(n)+\frac{1}{78}\sigma_3\left(\frac{n}{2}\right)+\frac{25}{312}\sigma_3\left(\frac{n}{5}\right)+\frac{25}{78}\sigma_3\left(\frac{n}{10}\right)+\left(\frac{1}{24}-\frac{1}{20}n\right)\sigma\left(\frac{n}{2}\right)\\&+\left(\frac{1}{24}-\frac{1}{8}n\right)\sigma\left(\frac{n}{5}\right)-\frac{1}{312}\mathfrak{b}_{40,1}(n)+\frac{1}{260}\mathfrak{b}_{40,1}\left(\frac{n}{2}\right)-\frac{1}{12}\mathfrak{b}_{40,3}(n),\end{aligned}\tag{77}$$

$$\begin{aligned}W_{(1,11)}(n)&=\frac{5}{1464}\sigma_3(n)+\frac{605}{1464}\sigma_3\left(\frac{n}{11}\right)+\left(\frac{1}{24}-\frac{1}{44}n\right)\sigma(n)+\left(\frac{1}{24}-\frac{1}{4}n\right)\sigma\left(\frac{n}{11}\right)\\&-\frac{14615}{386496}\mathfrak{b}'_{33,1}(n)-\frac{90493}{386496}\mathfrak{b}_{33,2}(n),\end{aligned}\tag{78}$$

$$\begin{aligned}W_{(1,12)}(n)&=\frac{1}{480}\sigma_3(n)+\frac{1}{160}\sigma_3\left(\frac{n}{2}\right)+\frac{3}{160}\sigma_3\left(\frac{n}{3}\right)+\frac{1}{30}\sigma_3\left(\frac{n}{4}\right)+\frac{9}{160}\sigma_3\left(\frac{n}{6}\right)\\&+\frac{3}{10}\sigma_3\left(\frac{n}{12}\right)+\left(\frac{1}{24}-\frac{1}{48}n\right)\sigma(n)+\left(\frac{1}{24}-\frac{1}{4}n\right)\sigma\left(\frac{n}{12}\right)-\frac{11}{480}\mathfrak{b}_{24,1}(n)\\&-\frac{17}{240}\mathfrak{b}_{24,2}(n)-\frac{1}{16}\mathfrak{b}'_{24,3}(n),\end{aligned}\tag{79}$$

$$\begin{aligned}W_{(3,4)}(n)&=\frac{1}{480}\sigma_3(n)+\frac{1}{160}\sigma_3\left(\frac{n}{2}\right)+\frac{3}{160}\sigma_3\left(\frac{n}{3}\right)+\frac{1}{30}\sigma_3\left(\frac{n}{4}\right)+\frac{9}{160}\sigma_3\left(\frac{n}{6}\right)\\&+\frac{3}{10}\sigma_3\left(\frac{n}{12}\right)+\left(\frac{1}{24}-\frac{1}{16}n\right)\sigma\left(\frac{n}{3}\right)+\left(\frac{1}{24}-\frac{1}{12}n\right)\sigma\left(\frac{n}{4}\right)-\frac{1}{480}\mathfrak{b}_{24,1}(n)\\&-\frac{7}{240}\mathfrak{b}_{24,2}(n)+\frac{1}{16}\mathfrak{b}'_{24,3}(n),\end{aligned}\tag{80}$$

$$\begin{aligned}W_{(1,15)}(n)&=\frac{1}{1560}\sigma_3(n)+\frac{1}{65}\sigma_3\left(\frac{n}{3}\right)+\frac{25}{39}\sigma_3\left(\frac{n}{5}\right)-\frac{25}{104}\sigma_3\left(\frac{n}{15}\right)\\&+\left(\frac{1}{24}-\frac{1}{60}n\right)\sigma(n)+\left(\frac{1}{24}-\frac{1}{4}n\right)\sigma\left(\frac{n}{15}\right)-\frac{1}{39}\mathfrak{b}_{15,1}(n)-\frac{2}{15}\mathfrak{b}_{15,2}(n)\\&-\frac{14}{65}\mathfrak{b}_{15,3}(n)+\frac{1}{5}\mathfrak{b}_{15,4}(n),\end{aligned}\tag{81}$$

$$W_{(3,5)}(n) = \frac{1}{390}\sigma_3(n) + \frac{7}{520}\sigma_3\Big(\frac{n}{3}\Big) - \frac{175}{312}\sigma_3\Big(\frac{n}{5}\Big) + \frac{25}{26}\sigma_3\Big(\frac{n}{15}\Big)$$
$$+ \Big(\frac{1}{24} - \frac{1}{20}n\Big)\sigma\Big(\frac{n}{3}\Big) + \Big(\frac{1}{24} - \frac{1}{8}n\Big)\sigma\Big(\frac{n}{5}\Big) - \frac{1}{390}\,\mathfrak{b}_{15,1}(n) - \frac{1}{30}\,\mathfrak{b}_{15,2}(n)$$
$$- \frac{1}{26}\,\mathfrak{b}_{15,3}(n) - \frac{1}{5}\,\mathfrak{b}_{15,4}(n), \tag{82}$$

$$W_{(1,24)}(n) = +\frac{1}{96}\,\sigma_3\Big(\frac{n}{2}\Big) + \frac{1}{192}\,\sigma_3\Big(\frac{n}{3}\Big) + \frac{7}{480}\,\sigma_3\Big(\frac{n}{4}\Big) + \frac{1}{192}\,\sigma_3\Big(\frac{n}{6}\Big) - \frac{7}{30}\,\sigma_3\Big(\frac{n}{8}\Big)$$
$$+ \frac{23}{480}\,\sigma_3\Big(\frac{n}{12}\Big) + \frac{17}{10}\,\sigma_3\Big(\frac{n}{24}\Big) + \Big(\frac{1}{24} - \frac{1}{96}n\Big)\sigma(n) + \Big(\frac{1}{24} - \frac{1}{4}n\Big)\sigma\Big(\frac{n}{24}\Big)$$
$$- \frac{1}{32}\,\mathfrak{b}_{24,1}(n) - \frac{13}{96}\,\mathfrak{b}_{24,2}(n) - \frac{9}{64}\,\mathfrak{b}_{24,3}(n) - \frac{61}{240}\,\mathfrak{b}_{24,4}(n)$$
$$+ \frac{1}{4}\,\mathfrak{b}_{24,5}(n) - \frac{6}{16}\,\mathfrak{b}_{24,6}(n) + \frac{1}{4}\,\mathfrak{b}_{24,7}(n) - \frac{1}{4}\,\mathfrak{b}_{24,8}(n), \tag{83}$$

$$W_{(3,8)}(n) = \frac{1}{1920}\,\sigma_3(n) + \frac{1}{640}\,\sigma_3\Big(\frac{n}{2}\Big) + \frac{3}{640}\,\sigma_3\Big(\frac{n}{3}\Big) + \frac{1}{160}\,\sigma_3\Big(\frac{n}{4}\Big) + \frac{9}{640}\,\sigma_3\Big(\frac{n}{6}\Big)$$
$$+ \frac{1}{30}\,\sigma_3\Big(\frac{n}{8}\Big) + \frac{9}{160}\,\sigma_3\Big(\frac{n}{12}\Big) + \frac{3}{10}\,\sigma_3\Big(\frac{n}{24}\Big) + \Big(\frac{1}{24} - \frac{1}{32}n\Big)\sigma\Big(\frac{n}{3}\Big)$$
$$+ \Big(\frac{1}{24} - \frac{1}{12}n\Big)\sigma\Big(\frac{n}{8}\Big) - \frac{1}{1920}\,\mathfrak{b}_{24,1}(n) - \frac{7}{960}\,\mathfrak{b}_{24,2}(n) + \frac{1}{32}\,\mathfrak{b}_{24,3}(n)$$
$$- \frac{17}{240}\,\mathfrak{b}_{24,4}(n) - \frac{1}{16}\,\mathfrak{b}_{24,5}(n) + \frac{3}{16}\,\mathfrak{b}_{24,6}(n) - \frac{5}{16}\,\mathfrak{b}_{24,7}(n). \tag{84}$$

For example (75) is easily proved as follows. Due to (74) and applying (12) we have

$$0 = -1152\,\alpha^2\,W_{(\alpha,\alpha)}(n) + 480\,\alpha^2\,\sigma_3\Big(\frac{n}{\alpha}\Big) + 96\,\alpha\,(\alpha - 6\,n)\,\sigma\Big(\frac{n}{\alpha}\Big).$$

Therefore, we obtain (8). By setting $\alpha = 1$, one gets the result obtained by M. Besge[39], J. W. L. Glaisher[40] and S. Ramanujan[41].

7.2 Convolution Sums for Levels $\alpha\beta = 9, 16, 18, 25, 27, 32, 36$

These levels are in $\mathbb{N}_0 \setminus \mathfrak{N}$; therefore, the primitive Dirichlet character of each of them is non-trivial and has a conductor greater than one.

[39] Besge, 1885, "Extrait d'une lettre de M Besge à M Liouville".

[40] Glaisher, 1862, "On the square of the series in which the coefficients are the sums of the divisors of the exponents".

[41] Ramanujan, 1916, "On certain arithmetical functions".

We revisit the evaluation of the convolution sum

- $W_{(1,9)}(n)$ obtained by K. S. Williams[42],
- $W_{(1,16)}(n)$, $W_{(1,18)}(n)$ and $W_{(2,9)}(n)$ obtained by A. Alaca et al.[43],
- $W_{(1,25)}(n)$ obtained by E. X. W. Xia et al.[44],
- $W_{(1,27)}(n)$ and $W_{(1,32)}(n)$ obtained by Ş. Alaca and Y. Kesicioğlu[45], and
- $W_{(1,36)}(n)$ and $W_{(4,9)}(n)$ obtained by D. Ye[46].

The convolution sums for $\alpha\beta = 9$, 16, 18, 25 and 36 have been evaluated using a different technique. The evaluation of the convolution sum for $\alpha\beta = 27$ and 32 is done using almost the same approach as the one that we are generalizing; however, we are not able to replicate those results using the provided basis elements of the space of the cusp forms.

Our method leads to an improvement of the result of the evaluation of the convolution sums for $\alpha\beta = 16$, 18, 25 and 36 since we apply the right number of basis elements of the space of cusp forms corresponding to the level 16, 18 and 25.

Due to (44), using $\mathfrak{B}_{45,1}(q)$ as basis element of $S_4(\Gamma_0(9))$ and applying the same primitive Dirichlet character as for $E_4(\Gamma_0(45))$, one easily replicates the result for the convolution sum $W_{(1,9)}(n)$.

For the evaluation of the convolution sums of level $\alpha\beta = 16$ and 25, we compute

$$\begin{aligned}
&\dim(E_4(\Gamma_0(16))) = 6, \quad \dim(S_4(\Gamma_0(16))) = 3,\\
&\dim(E_4(\Gamma_0(25))) = 6, \quad \dim(S_4(\Gamma_0(25))) = 5.
\end{aligned}$$

In case of the evaluation of $W_{(1,16)}(n)$, we use

- the basis elements whose table of the exponent of the η-quotients is displayed in the first half of Table 12 and
- the primitive Dirichlet character (50).

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[42] Williams, 2005, "The convolution sum $\sum_{m<\frac{n}{9}} \sigma(m)\sigma(n-9m)$".

[43] A. Alaca, Ş. Alaca, and Williams, 2007a, "Evaluation of the convolution sums $\sum_{l+18m=n} \sigma(l)\sigma(m)$ and $\sum_{2l+9m=n} \sigma(l)\sigma(m)$";
A. Alaca, Ş. Alaca, and Williams, 2008, "The convolution sum $\sum_{m<\frac{n}{16}} \sigma(m)\sigma(n-16m)$".

[44] Xia, Tian, and Yao, 2014, "Evaluation of the convolution sum $\sum_{l+25m=n} \sigma(l)\sigma(m)$".

[45] Ş. Alaca and Kesicioğlu, 2016, "Evaluation of the convolution sums $\sum_{l+27m=n} \sigma(l)\sigma(m)$ and $\sum_{l+32m=n} \sigma(l)\sigma(m)$".

[46] Ye, 2015, "Evaluation of the convolution sums $\sum_{l+36m=n} \sigma(l)\sigma(m)$ and $\sum_{4l+9m=n} \sigma(l)\sigma(m)$".

For the evaluation of $W_{(1,25)}(n)$ we use the basis element

$$\mathfrak{B}'_{50,2}(q) = \eta^3(z)\eta^4(5z)\eta(25z) = \sum_{n\geq 1} \mathfrak{b}'_{50,2}(n)q^n$$

instead of $\mathfrak{B}_{50,2}(q)$ given in Table 9 and we apply the primitive Dirichlet character (50).

Now, in case of the convolution sums for $\alpha\beta = 18$ and 36, we have

$$M_4(\Gamma_0(6)) \subset M_4(\Gamma_0(12)) \subset M_4(\Gamma_0(36)) \tag{85}$$
$$M_4(\Gamma_0(9)) \subset M_4(\Gamma_0(18)) \subset M_4(\Gamma_0(36)). \tag{86}$$

Therefore, it suffices to consider the basis of $S_4(\Gamma_0(36))$, whose table of the exponent of the η-quotients is given in Table 10. Note that

$$\dim(E_4(\Gamma_0(18))) = 8, \quad \dim(S_4(\Gamma_0(18))) = 5,$$
$$\dim(E_4(\Gamma_0(36))) = 12, \quad \dim(S_4(\Gamma_0(36))) = 12.$$

In case of the spaces $E_4(\Gamma_0(18))$ and $E_4(\Gamma_0(36))$ the primitive Dirichlet character (51) is applicable.

Some of the elements of $S_4(\Gamma_0(36))$ used by D. Ye[47] are different from our basis elements of $S_4(\Gamma_0(36))$, which translates into a slightly different result of the convolution sums. However, we are not able to replicate the result obtained by D. Ye[48] since we are unable to replicate the linear independence of the elements of $S_4(\Gamma_0(36))$ provided by D. Ye[49] by applying (22).

In case of the convolution sum for $\alpha\beta = 27$ and 32, we are unable to replicate the linear independence of the elements of the spaces of cusp form $S_4(\Gamma_0(27))$ and $S_4(\Gamma_0(32))$ provided by Ş. Alaca and Y. Kesicioğlu[50] by applying (22). For the basis of the Eisenstein series $E_4(\Gamma_0(27))$ and $E_4(\Gamma_0(32))$, Ş. Alaca and Y. Kesicioğlu[51] consider the primitive Dirichlet characters (50) and (51), respectively.

When one applies these primitive Dirichlet characters for those spaces, one obviously finds that

$$M_{(\frac{-3}{n})}(q^s) = \sum_{n>0} \left(\frac{-3}{n}\right)\sigma_3\left(\frac{n}{s}\right)q^n = 0, \qquad \text{for all } 1 < s \in D(27)$$

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[47] Ye, 2015, "Evaluation of the convolution sums $\sum_{l+36m=n}\sigma(l)\sigma(m)$ and $\sum_{4l+9m=n}\sigma(l)\sigma(m)$".

[48] Ibid., Thrm 2.1.

[49] Ibid.

[50] Ş. Alaca and Kesicioğlu, 2016, "Evaluation of the convolution sums $\sum_{l+27m=n}\sigma(l)\sigma(m)$ and $\sum_{l+32m=n}\sigma(l)\sigma(m)$", Thrms 2.2 (a), 2.3 (a).

[51] Ibid.

and

$$M_{(\frac{-4}{n})}(q^s)=\sum_{n>0}\left(\frac{-4}{n}\right)\sigma_3\left(\frac{n}{s}\right)q^n=0,\qquad \text{for all } 1<s\in D(32).$$

Therefore, the primitive Dirichlet characters $\chi(n)=(\frac{-3}{n})$ and $\psi(n)=(\frac{-4}{n})$ annihilate $E_4(\Gamma_0(27))$ and $E_4(\Gamma_0(32))$, respectively. Hence, no non-empty subset of the set $\{M_{(\frac{-3}{n})}(q^s)|s\in D(27)\}$ together with the set $\{M(q^t)|t\in D(27)\}$ can build a set of basis elements of $E_4(\Gamma_0(27))$. Similarly, no non-empty subset of the set $\{M_{(\frac{-4}{n})}(q^s)|s\in D(32)\}$ together with the set $\{M(q^t)|t\in D(32)\}$ can build a set of basis elements of $E_4(\Gamma_0(32))$. Consequently, we are not able to replicate the result in Ş. Alaca and Y. Kesicioğlu[52].

We observe that

$$M_4(\Gamma_0(9))\subset M_4(\Gamma_0(27)) \tag{87}$$

$$M_4(\Gamma_0(8))\subset M_4(\Gamma_0(16))\subset M_4(\Gamma_0(32)). \tag{88}$$

We then compute

$$\dim(E_4(\Gamma_0(27)))=6,\quad \dim(S_4(\Gamma_0(27)))=6,$$
$$\dim(E_4(\Gamma_0(32)))=8,\quad \dim(S_4(\Gamma_0(32)))=8.$$

By Theorem 3, the primitive Dirichlet characters (51) and (50) are not the annihilators of the spaces $E_4(\Gamma_0(27))$ and $E_4(\Gamma_0(32))$, respectively. Then by Theorem 4 (a), the sets $\{M(q^t)|t\in D(27)\}\cup\{M_{(\frac{-4}{n})}(q^s)|s=1,3\}$ and $\{M(q^t)|t\in D(32)\}\cup\{M_{(\frac{-3}{n})}(q^s)|s=1,4\}$ are bases for the space of Eisenstein series $E_4(\Gamma_0(27))$ and $E_4(\Gamma_0(32))$, respectively.

The basis elements with the exponents in Table 11 and Table 12 are obtained as a result of the application of Theorem 1 (i)–(v′). One can easily verify using Remark 1 (r2) and Theorem 4 (b) that these basis elements of the spaces of cusp forms are linearly independent; therefore, they constitute a basis of $S_4(\Gamma_0(27))$ and $S_4(\Gamma_0(32))$, respectively.

We notice that $\mathfrak{B}_{27,3}(q)=\mathfrak{B}_{27,1}(q^3)$; hence, $\mathfrak{b}_{27,3}(n)=\mathfrak{b}_{27,1}\left(\frac{n}{3}\right)$; analogously, we observe that $\mathfrak{B}_{32,2}(q)=\mathfrak{B}_{32,1}(q^2)$, $\mathfrak{B}_{32,4}(q)=\mathfrak{B}_{32,1}(q^4)$ and $\mathfrak{B}_{32,6}(q)=\mathfrak{B}_{32,3}(q^2)$. Therefore, $\mathfrak{b}_{32,2}(n)=\mathfrak{b}_{32,1}\left(\frac{n}{2}\right)$, $\mathfrak{b}_{32,4}(n)=\mathfrak{b}_{32,1}\left(\frac{n}{4}\right)$ and $\mathfrak{b}_{32,6}(n)=\mathfrak{b}_{32,3}\left(\frac{n}{2}\right)$.

Corollary 11 – *It holds that*

$$\begin{aligned}(L(q)-9\,L(q^9))^2=64+\sum_{n=1}^{\infty}\Bigg(&192\,\sigma_3(n)-384\,\sigma_3\left(\frac{n}{3}\right)\\&+15552\,\sigma_3\left(\frac{n}{9}\right)+192\,\mathfrak{b}_{45,1}(n)\Bigg)q^n.\end{aligned} \tag{89}$$

[52] Ş. Alaca and Kesicioğlu, 2016, "Evaluation of the convolution sums $\sum_{l+27m=n}\sigma(l)\sigma(m)$ and $\sum_{l+32m=n}\sigma(l)\sigma(m)$", Thrms 2.2 (b), 2.3 (b).

$$(L(q)-16\,L(q^{16}))^2 = 225+\sum_{n=1}^{\infty}\Big(216\,\sigma_3(n)-72\,\sigma_3\Big(\frac{n}{2}\Big)-288\,\sigma_3\Big(\frac{n}{4}\Big)-1152\,\sigma_3\Big(\frac{n}{8}\Big)$$
$$+55296\,\sigma_3\Big(\frac{n}{16}\Big)+504\,\mathfrak{b}_{32,1}(n)+864\,\mathfrak{b}_{32,2}(n)+2304\,\mathfrak{b}_{32,3}(n)\Big)q^n. \tag{90}$$

$$(L(q)-18\,L(q^{18}))^2 = 289+\sum_{n=1}^{\infty}\Big(\frac{1104}{5}\,\sigma_3(n)-\frac{384}{5}\,\sigma_3\Big(\frac{n}{2}\Big)-\frac{768}{5}\,\sigma_3\Big(\frac{n}{3}\Big)$$
$$-\frac{3072}{5}\,\sigma_3\Big(\frac{n}{6}\Big)-\frac{7776}{5}\,\sigma_3\Big(\frac{n}{9}\Big)+\frac{357696}{5}\,\sigma_3\Big(\frac{n}{18}\Big)+\frac{2976}{5}\,\mathfrak{b}_{36,1}(n)$$
$$+\frac{8544}{5}\,\mathfrak{b}_{36,2}(n)+\frac{17952}{5}\,\mathfrak{b}_{36,3}(n)+\frac{53376}{5}\,\mathfrak{b}_{36,4}(n)-\frac{52992}{5}\,\mathfrak{b}_{36,5}(n)\Big)q^n, \tag{91}$$

$$(2\,L(q)-9\,L(q^{9}))^2 = 49+\sum_{n=1}^{\infty}\Big(-\frac{96}{5}\,\sigma_3(n)+\frac{4416}{5}\,\sigma_3\Big(\frac{n}{2}\Big)-\frac{768}{5}\,\sigma_3\Big(\frac{n}{3}\Big)$$
$$-\frac{3072}{5}\,\sigma_3\Big(\frac{n}{6}\Big)+\frac{89424}{5}\,\sigma_3\Big(\frac{n}{9}\Big)-\frac{31104}{5}\,\sigma_3\Big(\frac{n}{18}\Big)+\frac{96}{5}\,\mathfrak{b}_{36,1}(n)$$
$$-\frac{96}{5}\,\mathfrak{b}_{36,2}(n)+\frac{3552}{5}\,\mathfrak{b}_{36,3}(n)-\frac{4224}{5}\,\mathfrak{b}_{36,4}(n)+\frac{16128}{5}\,\mathfrak{b}_{36,5}(n)\Big)q^n, \tag{92}$$

$$(L(q)-25\,L(q^{25}))^2 = 576+\sum_{n=1}^{\infty}\Big(\frac{2880}{13}\,\sigma_3(n)-\frac{5760}{13}\,\sigma_3\Big(\frac{n}{5}\Big)+\frac{1800000}{13}\,\sigma_3\Big(\frac{n}{25}\Big)$$
$$+\frac{12096}{13}\,\mathfrak{b}_{50,1}(n)+5760\,\mathfrak{b}'_{50,2}(n)+17280\,\mathfrak{b}_{50,3}(n)$$
$$+28800\,\mathfrak{b}_{50,4}(n)+\frac{302400}{13}\,\mathfrak{b}_{50,5}(n)\Big)q^n. \tag{93}$$

$$(L(q)-27\,L(q^{27}))^2 = 676+\sum_{n=1}^{\infty}\Big(224\,\sigma_3(n)-128\,\sigma_3\Big(\frac{n}{3}\Big)-1152\,\sigma_3\Big(\frac{n}{9}\Big)$$
$$+163296\,\sigma_3\Big(\frac{n}{27}\Big)+1024\,\mathfrak{b}_{27,1}(n)+2304\,\mathfrak{b}_{27,2}(n)+2304\,\mathfrak{b}_{27,3}(n)$$
$$+10368\,\mathfrak{b}_{27,4}(n)+10368\,\mathfrak{b}_{27,5}(n)\Big)q^n, \tag{94}$$

$$
\begin{aligned}
(L(q)-32\,L(q^{32}))^2 &= 961+\sum_{n=1}^{\infty}\Bigg(246\,\sigma_3(n)-198\,\sigma_3\Big(\frac{n}{2}\Big)-576\,\sigma_3\Big(\frac{n}{8}\Big)\\
&\quad -2304\,\sigma_3\Big(\frac{n}{16}\Big)+233472\,\sigma_3\Big(\frac{n}{32}\Big)+1242\,\mathfrak{b}_{32,1}(n)+3024\,\mathfrak{b}_{32,2}(n)\\
&\quad +7488\,\mathfrak{b}_{32,3}(n)+4032\,\mathfrak{b}_{32,4}(n)+2304\,\mathfrak{b}_{32,5}(n)\\
&\quad +13824\,\mathfrak{b}_{32,6}(n)+13824\,\mathfrak{b}_{32,7}(n)\Bigg)q^n.
\end{aligned}
\tag{95}
$$

$$
\begin{aligned}
(L(q)-36\,L(q^{36}))^2 &= 1225+\sum_{n=1}^{\infty}\Bigg(\frac{1152}{5}\,\sigma_3(n)-\frac{1008}{5}\,\sigma_3\Big(\frac{n}{2}\Big)-\frac{384}{5}\,\sigma_3\Big(\frac{n}{3}\Big)\\
&\quad +\frac{13056}{5}\,\sigma_3\Big(\frac{n}{4}\Big)-\frac{288}{5}\,\sigma_3\Big(\frac{n}{6}\Big)-\frac{3888}{5}\,\sigma_3\Big(\frac{n}{9}\Big)-\frac{19968}{5}\,\sigma_3\Big(\frac{n}{12}\Big)\\
&\quad -\frac{11664}{5}\,\sigma_3\Big(\frac{n}{18}\Big)+\frac{1492992}{5}\,\sigma_3\Big(\frac{n}{36}\Big)+\frac{7248}{5}\,\mathfrak{b}_{36,1}(n)+3744\,\mathfrak{b}_{36,2}(n)\\
&\quad +\frac{19008}{5}\,\mathfrak{b}_{36,3}(n)+\frac{77664}{5}\,\mathfrak{b}_{36,4}(n)+\frac{14688}{5}\,\mathfrak{b}_{36,5}(n)+16416\,\mathfrak{b}_{36,6}(n)\\
&\quad +\frac{80064}{5}\,\mathfrak{b}_{36,7}(n)+\frac{84672}{5}\,\mathfrak{b}_{36,8}(n)-12960\,\mathfrak{b}_{36,9}(n)\\
&\quad +\frac{90624}{5}\,\mathfrak{b}_{36,10}(n)+5184\,\mathfrak{b}_{36,11}(n)+2592\,\mathfrak{b}_{36,12}(n)\Bigg)q^n.
\end{aligned}
\tag{96}
$$

$$
\begin{aligned}
(4\,L(q)-9\,L(q^{9}))^2 &= 25+\sum_{n=1}^{\infty}\Bigg(\frac{1152}{5}\,\sigma_3(n)-\frac{1008}{5}\,\sigma_3\Big(\frac{n}{2}\Big)-\frac{384}{5}\,\sigma_3\Big(\frac{n}{3}\Big)\\
&\quad +\frac{13056}{5}\,\sigma_3\Big(\frac{n}{4}\Big)+\frac{80928}{5}\,\sigma_3\Big(\frac{n}{6}\Big)-\frac{3888}{5}\,\sigma_3\Big(\frac{n}{9}\Big)-\frac{913344}{5}\,\sigma_3\Big(\frac{n}{12}\Big)\\
&\quad -505872\,\sigma_3\Big(\frac{n}{18}\Big)+\frac{29187648}{5}\,\sigma_3\Big(\frac{n}{36}\Big)+\frac{7248}{5}\,\mathfrak{b}_{36,1}(n)+3744\,\mathfrak{b}_{36,2}(n)\\
&\quad +\frac{19008}{5}\,\mathfrak{b}_{36,3}(n)+\frac{77664}{5}\,\mathfrak{b}_{36,4}(n)+\frac{14688}{5}\,\mathfrak{b}_{36,5}(n)+\frac{864}{5}\,\mathfrak{b}_{36,6}(n)\\
&\quad +\frac{80064}{5}\,\mathfrak{b}_{36,7}(n)+\frac{84672}{5}\,\mathfrak{b}_{36,8}(n)-12960\,\mathfrak{b}_{36,9}(n)+\frac{90624}{5}\,\mathfrak{b}_{36,10}(n)\\
&\quad +5184\,\mathfrak{b}_{36,11}(n)+2592\,\mathfrak{b}_{36,12}(n)\Bigg)q^n,
\end{aligned}
\tag{97}
$$

Proof. Similar to the proof of Corollary 7 on p. 132. □

Corollary 12 – *Let n be a positive integer. Then*

$$
\begin{aligned}
W_{(1,9)}(n) &= \frac{1}{216}\,\sigma_3(n)+\frac{1}{27}\,\sigma_3\Big(\frac{n}{3}\Big)+\frac{3}{8}\,\sigma_3\Big(\frac{n}{9}\Big)+\Big(\frac{1}{24}-\frac{1}{36}n\Big)\sigma(n)\\
&\quad +\Big(\frac{1}{24}-\frac{1}{4}n\Big)\sigma\Big(\frac{n}{9}\Big)-\frac{1}{54}\,\mathfrak{b}_{45,1}(n)
\end{aligned}
\tag{98}
$$

$$\begin{aligned}
W_{(1,16)}(n) &= \frac{1}{768}\,\sigma_3(n) + \frac{1}{256}\,\sigma_3\Big(\frac{n}{2}\Big) + \frac{1}{64}\,\sigma_3\Big(\frac{n}{4}\Big) + \frac{1}{16}\,\sigma_3\Big(\frac{n}{8}\Big) + \frac{1}{3}\,\sigma_3\Big(\frac{n}{16}\Big) \\
&\quad + \Big(\frac{1}{24} - \frac{1}{64}n\Big)\sigma(n) + \Big(\frac{1}{24} - \frac{1}{4}n\Big)\sigma\Big(\frac{n}{16}\Big) - \frac{7}{256}\,\mathfrak{b}_{32,1}(n) \\
&\quad - \frac{3}{64}\,\mathfrak{b}_{32,2}(n) - \frac{1}{8}\,\mathfrak{b}_{32,3}(n)
\end{aligned} \tag{99}$$

$$\begin{aligned}
W_{(1,18)}(n) &= \frac{1}{1080}\,\sigma_3(n) + \frac{1}{270}\,\sigma_3\Big(\frac{n}{2}\Big) + \frac{1}{135}\,\sigma_3\Big(\frac{n}{3}\Big) + \frac{4}{135}\,\sigma_3\Big(\frac{n}{6}\Big) + \frac{3}{40}\,\sigma_3\Big(\frac{n}{9}\Big) \\
&\quad + \frac{3}{10}\,\sigma_3\Big(\frac{n}{18}\Big) + \Big(\frac{1}{24} - \frac{1}{72}n\Big)\sigma(n) + \Big(\frac{1}{24} - \frac{1}{4}n\Big)\sigma\Big(\frac{n}{18}\Big) - \frac{31}{1080}\,\mathfrak{b}_{36,1}(n) \\
&\quad - \frac{89}{1080}\,\mathfrak{b}_{36,2}(n) - \frac{187}{1080}\,\mathfrak{b}_{36,3}(n) - \frac{139}{270}\,\mathfrak{b}_{36,4}(n) + \frac{23}{45}\,\mathfrak{b}_{36,5}(n)
\end{aligned} \tag{100}$$

$$\begin{aligned}
W_{(2,9)}(n) &= \frac{1}{1080}\,\sigma_3(n) + \frac{1}{270}\,\sigma_3\Big(\frac{n}{2}\Big) + \frac{1}{135}\,\sigma_3\Big(\frac{n}{3}\Big) + \frac{4}{135}\,\sigma_3\Big(\frac{n}{6}\Big) + \frac{3}{40}\,\sigma_3\Big(\frac{n}{9}\Big) \\
&\quad + \frac{3}{10}\,\sigma_3\Big(\frac{n}{18}\Big) + \Big(\frac{1}{24} - \frac{1}{36}n\Big)\sigma\Big(\frac{n}{2}\Big) + \Big(\frac{1}{24} - \frac{1}{8}n\Big)\sigma\Big(\frac{n}{9}\Big) - \frac{1}{1080}\,\mathfrak{b}_{36,1}(n) \\
&\quad + \frac{1}{1080}\,\mathfrak{b}_{36,2}(n) - \frac{37}{1080}\,\mathfrak{b}_{36,3}(n) + \frac{11}{270}\,\mathfrak{b}_{36,4}(n) - \frac{7}{45}\,\mathfrak{b}_{36,5}(n)
\end{aligned} \tag{101}$$

$$\begin{aligned}
W_{(1,25)}(n) &= \frac{1}{1560}\,\sigma_3(n) + \frac{1}{65}\,\sigma_3\Big(\frac{n}{5}\Big) + \frac{125}{312}\,\sigma_3\Big(\frac{n}{25}\Big) + \Big(\frac{1}{24} - \frac{1}{100}n\Big)\sigma(n) \\
&\quad + \Big(\frac{1}{24} - \frac{1}{4}n\Big)\sigma\Big(\frac{n}{25}\Big) - \frac{21}{650}\,\mathfrak{b}_{50,1}(n) - \frac{1}{5}\,\mathfrak{b}'_{50,2}(n) - \frac{3}{5}\,\mathfrak{b}_{50,3}(n) \\
&\quad - \mathfrak{b}_{50,4}(n) - \frac{21}{26}\,\mathfrak{b}_{50,5}(n)
\end{aligned} \tag{102}$$

$$\begin{aligned}
W_{(1,27)}(n) &= \frac{1}{1944}\,\sigma_3(n) + \frac{1}{243}\,\sigma_3\Big(\frac{n}{3}\Big) + \frac{1}{27}\,\sigma_3\Big(\frac{n}{9}\Big) + \frac{3}{8}\,\sigma_3\Big(\frac{n}{27}\Big) \\
&\quad + \Big(\frac{1}{24} - \frac{1}{108}n\Big)\sigma(n) + \Big(\frac{1}{24} - \frac{1}{4}n\Big)\sigma\Big(\frac{n}{27}\Big) - \frac{8}{243}\,\mathfrak{b}_{27,1}(n) \\
&\quad - \frac{2}{27}\,\mathfrak{b}_{27,2}(n) - \frac{2}{27}\,\mathfrak{b}_{27,3}(n) - \frac{1}{3}\,\mathfrak{b}_{27,4}(n) - \frac{1}{3}\,\mathfrak{b}_{27,5}(n)
\end{aligned} \tag{103}$$

$$\begin{aligned}
W_{(1,32)}(n) &= -\frac{1}{6144}\,\sigma_3(n) + \frac{11}{2048}\,\sigma_3\Big(\frac{n}{2}\Big) + \frac{1}{64}\,\sigma_3\Big(\frac{n}{8}\Big) + \frac{1}{16}\,\sigma_3\Big(\frac{n}{16}\Big) + \frac{1}{3}\,\sigma_3\Big(\frac{n}{32}\Big) \\
&\quad + \Big(\frac{1}{24} - \frac{1}{128}n\Big)\sigma(n) + \Big(\frac{1}{24} - \frac{1}{4}n\Big)\sigma\Big(\frac{n}{32}\Big) - \frac{69}{2048}\,\mathfrak{b}_{32,1}(n) - \frac{21}{256}\,\mathfrak{b}_{32,2}(n) \\
&\quad - \frac{13}{64}\,\mathfrak{b}_{32,3}(n) - \frac{7}{64}\,\mathfrak{b}_{32,4}(n) - \frac{1}{16}\,\mathfrak{b}_{32,5}(n) - \frac{3}{8}\,\mathfrak{b}_{32,6}(n) - \frac{3}{8}\,\mathfrak{b}_{32,7}(n)
\end{aligned} \tag{104}$$

$$
\begin{aligned}
W_{(1,36)}(n) = {} & \frac{1}{4320}\,\sigma_3(n) + \frac{7}{1440}\,\sigma_3\Big(\frac{n}{2}\Big) + \frac{1}{540}\,\sigma_3\Big(\frac{n}{3}\Big) - \frac{17}{270}\,\sigma_3\Big(\frac{n}{4}\Big) + \frac{1}{720}\,\sigma_3\Big(\frac{n}{6}\Big) \\
& + \frac{3}{160}\,\sigma_3\Big(\frac{n}{9}\Big) + \frac{13}{135}\,\sigma_3\Big(\frac{n}{12}\Big) + \frac{9}{160}\,\sigma_3\Big(\frac{n}{18}\Big) + \frac{3}{10}\,\sigma_3\Big(\frac{n}{36}\Big) \\
& + \Big(\frac{1}{24} - \frac{1}{144}n\Big)\sigma(n) + \Big(\frac{1}{24} - \frac{1}{4}n\Big)\sigma\Big(\frac{n}{36}\Big) - \frac{151}{4320}\,\mathfrak{b}_{36,1}(n) - \frac{13}{144}\,\mathfrak{b}_{36,2}(n) \\
& - \frac{11}{120}\,\mathfrak{b}_{36,3}(n) - \frac{809}{2160}\,\mathfrak{b}_{36,4}(n) - \frac{17}{240}\,\mathfrak{b}_{36,5}(n) - \frac{19}{48}\,\mathfrak{b}_{36,6}(n) - \frac{139}{360}\,\mathfrak{b}_{36,7}(n) \\
& - \frac{49}{120}\,\mathfrak{b}_{36,8}(n) + \frac{5}{16}\,\mathfrak{b}_{36,9}(n) - \frac{59}{135}\,\mathfrak{b}_{36,10}(n) - \frac{1}{8}\,\mathfrak{b}_{36,11}(n) - \frac{1}{16}\,\mathfrak{b}_{36,12}(n)
\end{aligned}
\tag{105}
$$

$$
\begin{aligned}
W_{(4,9)}(n) = {} & \frac{1}{4320}\,\sigma_3(n) + \frac{7}{1440}\,\sigma_3\Big(\frac{n}{2}\Big) + \frac{1}{540}\,\sigma_3\Big(\frac{n}{3}\Big) - \frac{17}{270}\,\sigma_3\Big(\frac{n}{4}\Big) - \frac{281}{720}\,\sigma_3\Big(\frac{n}{6}\Big) \\
& + \frac{3}{160}\,\sigma_3\Big(\frac{n}{9}\Big) + \frac{4757}{1080}\,\sigma_3\Big(\frac{n}{12}\Big) + \frac{1171}{96}\,\sigma_3\Big(\frac{n}{18}\Big) - \frac{15991}{120}\,\sigma_3\Big(\frac{n}{36}\Big) \\
& + \Big(\frac{1}{24} - \frac{1}{36}n\Big)\sigma\Big(\frac{n}{4}\Big) + \Big(\frac{1}{24} - \frac{1}{16}n\Big)\sigma\Big(\frac{n}{9}\Big) - \frac{151}{4320}\,\mathfrak{b}_{36,1}(n) - \frac{13}{144}\,\mathfrak{b}_{36,2}(n) \\
& - \frac{11}{120}\,\mathfrak{b}_{36,3}(n) - \frac{809}{2160}\,\mathfrak{b}_{36,4}(n) - \frac{17}{240}\,\mathfrak{b}_{36,5}(n) - \frac{1}{240}\,\mathfrak{b}_{36,6}(n) - \frac{139}{360}\,\mathfrak{b}_{36,7}(n) \\
& - \frac{49}{120}\,\mathfrak{b}_{36,8}(n) + \frac{5}{16}\,\mathfrak{b}_{36,9}(n) - \frac{59}{135}\,\mathfrak{b}_{36,10}(n) - \frac{1}{8}\,\mathfrak{b}_{36,11}(n) - \frac{1}{16}\,\mathfrak{b}_{36,12}(n)
\end{aligned}
\tag{106}
$$

Proof. Similar to the proof of Corollary 8 on p. 135. □

8 Formulae for the Number of Representations of a Positive Integer

We make use of the convolution sums evaluated in Section 5 and Section 6 among others to determine explicit formulae for the number of representations of a positive integer n by the octonary quadratic forms (3) and (4), respectively.

8.1 Representations by the Octonary Quadratic Forms (3)

We determine formulae for the number of representations of a positive integer n by the Octonary Quadratic Form (4). We mainly apply the evaluation of the convolution sums $W_{(1,33)}(n)$, $W_{(3,11)}(n)$, $W_{(5,9)}(n)$, $W_{(1,45)}(n)$, $W_{(1,54)}(n)$, $W_{(2,27)}(n)$ and other well-known convolution sums to determine these formulae. In order to do that, we recall that

- $33 = 3 \cdot 11$, which is of the restricted form in Section 4.2. Hence, from Proposition 2 we derive that $\Omega_3 = \{(1,11)\}$.
- $45 = 3^2 \cdot 5$. It then follows from Proposition 2 that $\Omega_3 = \{(3,5),(1,15)\}$.
- $54 = 3^2 \cdot 2$. It is then immediate from Proposition 2 that $\Omega_3 = \{(2,9),(1,18)\}$.

We then deduce the following result:

Corollary 13 – *Let $n \in \mathbb{N}$ and $c,d) = (1,11),(1,15),(3,5)$. Then*

$$R_{(1,11)}(n) = 12\sigma(n) - 36\sigma\Big(\frac{n}{3}\Big) + 12\sigma\Big(\frac{n}{11}\Big) - 36\sigma\Big(\frac{n}{33}\Big) + 144\,W_{(1,11)}(n) + 1296\,W_{(1,11)}\Big(\frac{n}{3}\Big) - 432\Big(W_{(3,11)}(n) + W_{(1,33)}(n)\Big).$$

$$R_{(1,15)}(n) = 12\sigma(n) - 36\sigma\Big(\frac{n}{3}\Big) + 12\sigma\Big(\frac{n}{15}\Big) - 36\sigma\Big(\frac{n}{45}\Big) + 144\,W_{(1,15)}(n) + 1296\,W_{(1,15)}\Big(\frac{n}{3}\Big) - 432\Big(W_{(1,5)}\Big(\frac{n}{3}\Big) + W_{(1,45)}(n)\Big).$$

$$R_{(3,5)}(n) = 12\sigma\Big(\frac{n}{3}\Big) - 36\sigma\Big(\frac{n}{9}\Big) + 12\sigma\Big(\frac{n}{5}\Big) - 36\sigma\Big(\frac{n}{15}\Big) + 144\,W_{(3,5)}(n) + 1296\,W_{(3,5)}\Big(\frac{n}{3}\Big) - 432\Big(W_{(1,5)}\Big(\frac{n}{3}\Big) + W_{(5,9)}(n)\Big).$$

$$R_{(1,18)}(n) = 12\sigma(n) - 36\sigma\Big(\frac{n}{3}\Big) + 12\sigma\Big(\frac{n}{18}\Big) - 36\sigma\Big(\frac{n}{54}\Big) + 144\,W_{(1,18)}(n) + 1296\,W_{(1,18)}\Big(\frac{n}{3}\Big) - 432\Big(W_{(1,6)}\Big(\frac{n}{3}\Big) + W_{(1,54)}(n)\Big).$$

$$R_{(2,9)}(n) = 12\sigma\Big(\frac{n}{2}\Big) - 36\sigma\Big(\frac{n}{6}\Big) + 12\sigma\Big(\frac{n}{9}\Big) - 36\sigma\Big(\frac{n}{27}\Big) + 144\,W_{(2,9)}(n) + 1296\,W_{(2,9)}\Big(\frac{n}{3}\Big) - 432\Big(W_{(2,3)}\Big(\frac{n}{3}\Big) + W_{(2,27)}(n)\Big).$$

Proof. We only consider the case $(c,d) = (1,11)$ since the other cases can be proved in a similar way.

It follows immediately from Theorem 7 with $(c,d) = (1,11)$. One can then make use of

- (78), (37) and (38) to simplify $R_{(1,11)}(n)$.
- Corollary 10, (59) and (60) to simplify $R_{(1,15)}(n)$ and $R_{(3,5)}(n)$. □

8.2 Representations by Octonary Quadratic Forms (4)

We give formulae for the number of representations of a positive integer n by the Octonary Quadratic Form (4). We apply among others the evaluation of the convolution sums $W_{(1,40)}(n)$, $W_{(1,56)}(n)$, $W_{(5,8)}(n)$ and $W_{(7,8)}(n)$. To achieve that, we recall that $40 = 2^3 \cdot 5$ and $56 = 2^3 \cdot 7$, which are of the restricted form in Section 4.1. Therefore, we apply Proposition 1 to conclude that $\Omega_4 = \{(1,10),(2,5)\}$ in case $\alpha\beta = 40$ and $\Omega_4 = \{(1,14),(2,7)\}$ in case $\alpha\beta = 56$.

Corollary 14 – *Let $n \in \mathbb{N}$. Then*

$$N_{(1,10)}(n) = 8\sigma(n) - 32\sigma\left(\frac{n}{4}\right) + 8\sigma\left(\frac{n}{10}\right) - 32\sigma\left(\frac{n}{40}\right) + 64\,W_{(1,10)}(n) + 1024\,W_{(1,10)}\left(\frac{n}{4}\right) - 256\left(W_{(2,5)}\left(\frac{n}{2}\right) + W_{(1,40)}(n)\right),$$

$$N_{(2,5)}(n) = 8\sigma\left(\frac{n}{2}\right) - 32\sigma\left(\frac{n}{8}\right) + 8\sigma\left(\frac{n}{5}\right) - 32\sigma\left(\frac{n}{20}\right) + 64\,W_{(2,5)}(n) + 1024\,W_{(2,5)}\left(\frac{n}{4}\right) - 256\left(W_{(5,8)}(n) + W_{(1,10)}\left(\frac{n}{2}\right)\right),$$

$$N_{(1,14)}(n) = 8\sigma(n) - 32\sigma\left(\frac{n}{4}\right) + 8\sigma\left(\frac{n}{14}\right) - 32\sigma\left(\frac{n}{56}\right) + 64\,W_{(1,14)}(n) + 1024\,W_{(1,14)}\left(\frac{n}{4}\right) - 256\left(W_{(2,7)}\left(\frac{n}{2}\right) + W_{(1,56)}(n)\right),$$

$$N_{(2,7)}(n) = 8\sigma\left(\frac{n}{2}\right) - 32\sigma\left(\frac{n}{8}\right) + 8\sigma\left(\frac{n}{7}\right) - 32\sigma\left(\frac{n}{28}\right) + 64\,W_{(2,7)}(n) + 1024\,W_{(2,7)}\left(\frac{n}{4}\right) - 256\left(W_{(7,8)}(n) + W_{(1,14)}\left(\frac{n}{2}\right)\right).$$

Proof. These formulae follow immediately from Theorem 6 when we set $(a,b) = (1,10)$, $(2,5)$, $(1,14)$, $(2,7)$, respectively. One can then use the result of

- S. Cooper and D. Ye[53], (76), (39) and (40) for the sake of simplification of $N_{(1,10)}$ and $N_{(2,5)}$.
- E. Royer[54], E. Ntienjem[55], (41) and (42) to simplify the formulae $N_{(1,14)}$ and $N_{(2,7)}$. □

[53] Cooper and Ye, 2014, "Evaluation of the convolution sums $\sum_{l+20m=n}\sigma(l)\sigma(m)$, $\sum_{4l+5m=n}\sigma(l)\sigma(m)$ and $\sum_{2l+5m=n}\sigma(l)\sigma(m)$", Thrm 2.1.

[54] Royer, 2007, "Evaluating convolution sums of divisor function by quasi modular forms", Thrms 1.7.

[55] Ntienjem, 2015, "Evaluation of the Convolution Sums $\sum_{\alpha l+\beta m=n}\sigma(l)\sigma(m)$, where (α,β) is in $\{(1,14),(2,7),(1,26),(2,13),(1,28),(4,7),(1,30),(2,15),(3,10),(5,6)\}$", Thrm 3.2.1.

9 Forgotten Formulae for the Number of Representations of a Positive Integer

In the following section, formulae for the number of representations, $N_{(a,b)}(n)$, of a positive integer n for $(a,b) = (1,1),(1,3),(1,6),(2,3)$, are determined as applications of the evaluation of the convolution sums $W_{(1,4)}(n)$ by J. G. Huard et al.[56], $W_{(1,12)}(n)$, $W_{(3,4)}(n)$, $W_{(1,24)}(n)$ and $W_{(3,8)}(n)$ by A. Alaca et al.[57]. These numbers of representations of a positive integer n are discovered due to Proposition 1. One rather considers (79), (80), (83) and (84) in the following result.

Corollary 15 – *Let $n \in \mathbb{N}$. Then*

$$\begin{aligned} N_{(1,1)}(n) &= 16\,\sigma(n) - 64\,\sigma\left(\frac{n}{4}\right) + 64\,W_{(1,1)}(n) + 1024\,W_{(1,1)}\left(\frac{n}{4}\right) - 512\,W_{(1,4)}(n) \\ &= 16\,\sigma_3(n) - 32\,\sigma_3\left(\frac{n}{2}\right) + 256\,\sigma_3\left(\frac{n}{4}\right) = r_8(n), \end{aligned}$$

$$\begin{aligned} N_{(1,3)}(n) = {} & \frac{8}{5}\,\sigma_3(n) - \frac{16}{5}\,\sigma_3\left(\frac{n}{2}\right) + \frac{72}{5}\,\sigma_3\left(\frac{n}{3}\right) + \frac{128}{5}\,\sigma_3\left(\frac{n}{4}\right) - \frac{144}{5}\,\sigma_3\left(\frac{n}{6}\right) \\ & + \frac{1152}{5}\,\sigma_3\left(\frac{n}{12}\right) + \frac{32}{5}\,\mathfrak{b}_{24,1}(n) + \frac{128}{5}\,\mathfrak{b}_{24,2}(n) \end{aligned}$$

$$\begin{aligned} N_{(1,6)}(n) = {} & \frac{8}{15}\,\sigma_3(n) - \frac{8}{3}\,\sigma_3\left(\frac{n}{2}\right) + \frac{52}{15}\,\sigma_3\left(\frac{n}{3}\right) - \frac{56}{15}\,\sigma_3\left(\frac{n}{4}\right) - \frac{4}{3}\,\sigma_3\left(\frac{n}{6}\right) + \frac{1408}{15}\,\sigma_3\left(\frac{n}{8}\right) \\ & - \frac{184}{15}\,\sigma_3\left(\frac{n}{12}\right) + \frac{2432}{15}\,\sigma_3\left(\frac{n}{24}\right) + \frac{112}{15}\,\mathfrak{b}_{24,1}(n) + \frac{32}{15}\,\mathfrak{b}_{24,1}\left(\frac{n}{2}\right) \\ & - \frac{128}{15}\,\mathfrak{b}_{24,1}\left(\frac{n}{4}\right) + \frac{104}{3}\,\mathfrak{b}_{24,2}(n) + 36\,\mathfrak{b}_{24,3}(n) + \frac{976}{15}\,\mathfrak{b}_{24,4}(n) \\ & - 64\,\mathfrak{b}_{24,5}(n) + 80\,\mathfrak{b}_{24,6}(n) - 64\,\mathfrak{b}_{24,7}(n) + 64\,\mathfrak{b}_{24,8}(n). \end{aligned}$$

$$\begin{aligned} N_{(2,3)}(n) = {} & \frac{2}{5}\,\sigma_3(n) - \frac{2}{5}\,\sigma_3\left(\frac{n}{2}\right) + \frac{18}{5}\,\sigma_3\left(\frac{n}{3}\right) - \frac{8}{5}\,\sigma_3\left(\frac{n}{4}\right) - \frac{18}{5}\,\sigma_3\left(\frac{n}{6}\right) + \frac{128}{5}\,\sigma_3\left(\frac{n}{8}\right) \\ & - \frac{72}{5}\,\sigma_3\left(\frac{n}{12}\right) + \frac{1152}{5}\,\sigma_3\left(\frac{n}{24}\right) - \frac{2}{5}\,\mathfrak{b}_{24,1}(n) + \frac{32}{15}\,\mathfrak{b}_{24,1}\left(\frac{n}{2}\right) \\ & - \frac{128}{15}\,\mathfrak{b}_{24,1}\left(\frac{n}{4}\right) + \frac{28}{15}\,\mathfrak{b}_{24,2}(n) - 8\,\mathfrak{b}_{24,3}(n) + \frac{272}{15}\,\mathfrak{b}_{24,4}(n) \\ & + 16\,\mathfrak{b}_{24,5}(n) - 48\,\mathfrak{b}_{24,6}(n) + 80\,\mathfrak{b}_{24,7}(n). \end{aligned}$$

[56] Huard et al., 2002, "Elementary evaluation of certain convolution sums involving divisor functions".

[57] A. Alaca, Ş. Alaca, and Williams, 2006, "Evaluation of the convolution sums $\sum_{l+12m=n}\sigma(l)\sigma(m)$ and $\sum_{3l+4m=n}\sigma(l)\sigma(m)$";

A. Alaca, Ş. Alaca, and Williams, 2007b, "Evaluation of the convolution sums $\sum_{l+24m=n}\sigma(l)\sigma(m)$ and $\sum_{3l+8m=n}\sigma(l)\sigma(m)$".

Proof. We set $(a,b) = (1,1)$, $(1,3)$, $(1,6)$, $(2,3)$ in Theorem 6 and we use (75), the convolution sums $W_{(1,3)}(n)$ and $W_{(1,4)}(n)$ proved by J. G. Huard et al.[58], $W_{(1,6)}(n)$ and $W_{(2,3)}(n)$ proved by Ş. Alaca and K. S. Williams[59], (79), (80), (83) and (84) to simplify and then obtain the stated results. □

10 Concluding Remark

The set of levels (also positive integers) $\mathbb{N}_0$ can be expressed as the disjoint union of the sets $\mathfrak{N}$ and $\mathbb{N}_0 \setminus \mathfrak{N}$. When assuming that a basis of the space of cusp forms is determined, we have evaluated convolution sums for levels which belong to $\mathfrak{N}$; at the same time we have evaluated convolution sums for levels which are contained in $\mathbb{N}_0 \setminus \mathfrak{N}$ making the same assumption. When we put altogether, we can say that for all natural numbers α and β, the convolution sums for levels $\alpha\beta$ are evaluated.

The determination of a basis of the space of cusp forms is tedious, especially when the level $\alpha\beta$ is large and the cardinality of the set of all positive divisors of $\alpha\beta$ is greater than 10. An effective approach to constructing a basis of the space of cusp forms of weight 4 for $\Gamma_0(\alpha\beta)$ is given in the proof of Theorem 4 (b). An efficient approach to building a basis of the space of cusp forms of weight 4 for $\Gamma_0(\alpha\beta)$ is a work in progress.

In subsection 3.1, we have given a criterion for the determination of a primitive Dirichlet character when evaluating a convolution sum of level $\alpha\beta \in \mathbb{N}_0 \setminus \mathfrak{N}$. It would be nice to weaken this criterion.

Tables

	1	**2**	**3**	**4**	**6**	**8**	**12**	**24**
1	2	2	2	0	2	0	0	0
2	0	2	0	2	2	0	2	0
3	0	0	0	0	4	0	4	0
4	0	0	0	2	0	2	2	2
5	0	2	0	−2	−2	2	6	2
6	0	0	0	4	0	−2	0	6
7	0	2	0	0	−2	−2	4	6
8	2	−5	−2	6	9	−5	−8	11

Table 4 – Power of η-quotients being basis elements of $S_4(\Gamma_0(24))$

[58] Huard et al., 2002, "Elementary evaluation of certain convolution sums involving divisor functions".

[59] Ş. Alaca and Williams, 2007, "Evaluation of the convolution sums $\sum_{l+6m=n} \sigma(l)\sigma(m)$ and $\sum_{2l+3m=n} \sigma(l)\sigma(m)$".

	1	**3**	**11**	**33**
1	0	8	0	0
2	4	0	4	0
3	3	1	3	1
4	2	2	2	2
5	1	3	1	3
6	0	4	0	4
7	−1	5	−1	5
8	−2	6	−2	6
9	6	0	0	2
10	4	−2	−2	8

Table 5 – Power of η-quotients being basis elements of $S_4(\Gamma_0(33))$

	1	**2**	**4**	**5**	**8**	**10**	**20**	**40**
1	4	0	0	4	0	0	0	0
2	0	4	0	0	0	4	0	0
3	2	0	0	−2	0	8	0	0
4	0	0	4	0	0	0	4	0
5	0	0	0	0	0	4	4	0
6	0	2	0	0	0	−2	8	0
7	2	−2	0	−2	0	2	8	0
8	0	0	0	0	4	0	0	4
9	0	0	0	0	2	4	−4	6
10	2	−2	2	2	−2	0	0	6
11	1	0	0	−1	1	2	−2	7
12	0	0	2	0	0	0	−2	8
13	0	4	0	0	−2	0	−4	10
14	0	2	−2	0	0	−2	2	8

Table 6 – Power of η-quotients being basis elements of $S_4(\Gamma_0(40))$

	1	**2**	**4**	**7**	**8**	**14**	**28**	**56**
1	5	−1	0	5	0	−1	0	0
2	2	2	0	2	0	2	0	0
3	6	−2	0	−2	0	6	0	0
4	0	2	2	0	0	2	2	0
5	0	0	2	0	0	4	2	0
6	0	6	−2	0	0	−2	6	0
7	0	4	−2	0	0	0	6	0
8	1	1	0	1	0	−3	8	0
9	0	1	1	0	0	−3	9	0
10	0	0	0	0	2	0	4	2
11	0	−2	8	0	−2	2	−4	6
12	0	0	6	0	−2	0	−2	6
13	0	0	3	0	−1	4	−5	7
14	0	0	4	0	−2	0	0	6
15	0	2	2	0	−2	−2	2	6
16	0	1	1	0	0	1	−3	8
17	0	3	−1	0	0	−1	−1	8
18	0	0	1	0	1	0	−3	9
19	0	1	0	0	−1	−3	4	7
20	−2	5	−3	2	0	−5	7	4

Table 7 – Power of η-quotients being basis elements of $S_4(\Gamma_0(56))$

	1	**3**	**5**	**9**	**15**	**45**
1	0	8	0	0	0	0
2	2	2	2	0	2	0
3	0	4	0	0	4	0
4	3	4	0	−1	0	2
5	2	0	2	2	0	2
6	4	0	1	0	0	3
7	1	0	1	3	0	3

8	3	0	0	1	0	4
9	5	0	−1	−1	0	5
10	0	3	0	−1	1	5
11	1	1	1	0	−1	6
12	4	0	4	0	0	0
13	2	0	2	0	4	0
14	0	−1	3	9	−3	0

Table 8 – Power of η-quotients being basis elements for $S_4(\Gamma_0(45))$

	1	**2**	**5**	**10**	**25**	**50**
1	4	0	4	0	0	0
2	0	4	0	4	0	0
3	2	0	4	0	2	0
4	1	0	4	0	3	0
5	0	0	4	0	4	0
6	0	2	0	4	0	2
7	0	4	2	0	−2	4
8	0	1	0	4	0	3
9	1	0	0	4	−1	4
10	0	0	0	4	0	4
11	0	2	2	0	−2	6
12	0	−1	0	4	0	5
13	0	1	2	0	−2	7
14	1	0	2	0	−3	8
15	0	0	2	0	−2	8
16	−1	0	6	−2	−5	10
17	0	−1	1	3	−5	10

Table 9 – Power of η-quotients being basis elements for $S_4(\Gamma_0(50))$

	1	**2**	**3**	**4**	**6**	**9**	**12**	**18**	**36**
1	0	0	8	0	0	0	0	0	0
2	0	0	0	0	8	0	0	0	0
3	0	0	2	0	2	2	0	2	0
4	0	0	3	0	1	−1	0	5	0
5	0	0	4	0	0	−4	0	8	0
6	0	0	0	0	2	0	2	2	2
7	0	0	0	0	3	0	−1	3	3
8	0	0	0	0	3	0	1	−1	5
9	0	0	0	0	4	0	−2	0	6
10	0	0	0	0	4	0	0	−4	8
11	0	0	0	0	5	0	−3	−3	9
12	−5	11	5	−5	−1	−2	0	0	5

Table 10 – Power of η-functions being basis elements of $S_4(\Gamma_0(36))$

	1	**3**	**9**	**27**
1	0	8	0	0
2	0	4	4	0
3	0	0	8	0
4	0	5	0	3
5	0	1	4	3
6	3	2	−3	6

Table 11 – Power of η-functions being basis elements of $S_4(\Gamma_0(27))$

	1	**2**	**4**	**8**	**16**	**32**
1	0	4	4	0	0	0
2	0	0	4	4	0	0
3	0	0	6	−2	4	0
4	0	0	0	4	4	0
5	0	2	1	0	3	2

6	0	0	0	6	−2	4
7	0	0	2	0	2	4
8	0	0	−4	6	2	4

Table 12 – Power of η-functions being basis elements of $S_4(\Gamma_0(32))$

	1	**2**	**3**	**6**	**9**	**18**	**27**	**54**
1	2	2	2	2	0	0	0	0
2	0	0	0	8	0	0	0	0
3	0	0	2	2	2	2	0	0
4	0	0	5	0	0	0	3	0
5	0	0	1	0	4	0	3	0
6	0	0	6	0	−4	0	6	0
7	0	0	2	0	0	0	6	0
8	0	0	0	5	0	0	0	3
9	0	0	0	0	2	2	2	2
10	0	0	0	1	0	4	0	3
11	0	0	−1	4	0	0	1	4
12	0	0	0	6	0	−4	0	6
13	0	0	0	1	2	−2	2	5
14	0	0	0	2	0	0	0	6
15	0	0	0	0	4	0	−4	8
16	0	0	0	1	3	−3	−1	8
17	0	0	1	0	0	0	−1	8
18	−4	5	6	−4	0	3	−2	4
19	0	0	0	1	4	−4	−4	11
20	0	0	0	2	2	−2	−6	12
21	0	0	0	0	3	1	−1	5

Table 13 – Power of η-functions being basis elements of $S_4(\Gamma_0(54))$

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Contents