North-Western European Journal of Mathematics

Multifractional processes and nonlinear functionals of Gaussian random fields

Renaud Marty[1]

Received: July 7, 2021/Accepted: January 25, 2023/Online: March 10, 2023

Abstract

In this paper we consider sequences of nonlinear functionals of Gaussian random fields. We prove their convergence to multifractional processes which generalize Hermite processes.

Keywords: Limit theorems, multifractional processes, Gaussian random fields.

MSC: 60F17, 60G22, 60G60.

1 Introduction

This paper deals with sequences of processes defined from functionals of Gaussian random fields. Many works state that such sequences converge to various types of limits which depend on the form of the functionals and on statistical properties of the random fields. Consider for instance a stationary and Gaussian sequence $(X_n)_{n\in\mathbb{N}}$ of random variables with mean zero and variance 1. We assume that there exist $m \in \mathbb{N}^*$, $C > 0$, and $\alpha \in (0, \frac{1}{m})$ such that

$$\mathbb{E}(X_0 X_n) \sim \frac{C}{n^\alpha} \tag{1}$$

as $n \to \infty$. For every $N \in \mathbb{N}$ we define the process $S^N_{\phi,H}$ by

$$(S^N_{\phi,H}(t))_{t\geq 0} = \left(\frac{1}{N^H} \sum_{n=1}^{\lfloor Nt \rfloor} \phi(X_n) \right)_{t\geq 0} \tag{2}$$

where $H \in (\frac{1}{2}, 1)$ and $\phi : \mathbb{R} \to \mathbb{R}$ is a function such that

$$\int_{\mathbb{R}} (\phi(x))^2 e^{-x^2/2} dx < \infty \text{ and } \int_{\mathbb{R}} \phi(x) e^{-x^2/2} dx = 0.$$

[1]Université de Lorraine, CNRS, IECL, F-54000 Nancy, France

Assume that $\alpha = \frac{2-2H}{m}$ and that the Hermite rank of ϕ is equal to m. By Dobrushin and Major (1979) and Taqqu (1979), as $N \to \infty$, $S^N_{\phi,H}$ converges in distribution to a Hermite process $B_{m,H}$ defined by

$$(B_{m,H}(t))_{t\geq 0} = \left(c(m,H) \int_{D_m} dW_{x_1} \cdots dW_{x_m} \int_0^t \prod_{k=1}^{m} (\theta - x_k)_+^{\widetilde{H}-3/2} d\theta \right)_{t\geq 0} \tag{3}$$

where $\widetilde{H} = 1 + \frac{H-1}{m}$, $D_m = \{x_1 < x_2 < \cdots < x_m\}$, W is a Brownian motion, and $c(m,H)$ is a constant. The process $B_{m,H}$ can also be written as

$$(B_{m,H}(t))_{t\geq 0} = \left(\widehat{c}(m,H) \int_{\mathbb{R}^m} d\widehat{W}_{\xi_1} \cdots d\widehat{W}_{\xi_m} \frac{\exp(it(\xi_1 + \cdots + \xi_m)) - 1}{i|\xi_1 \cdots \xi_m|^{\widetilde{H}-1/2}(\xi_1 + \cdots + \xi_m)} \right)_{t\geq 0} \tag{4}$$

where $\widehat{W}$ is the Fourier transform of a Gaussian measure and $\widehat{c}(m,H)$ is a constant. It is a self-similar process with index H and its local Hölder exponent is H at every point. Moreover it is not Gaussian when $m > 1$, and $B_{1,H}$ is a fractional Brownian motion with Hurst index H.

The above result is extended in a multifractional framework in Cohen and Marty (2008) and Marty (2013). Let $(X_n(H))_{(n,H)\in\mathbb{N}\times(1/2,1)}$ be a Gaussian sequence of random fields with mean zero. We assume that for every $(H_1,H_2) \in (\frac{1}{2},1)^2$ there exists $R(H_1,H_2) > 0$ such that

$$\mathbb{E}(X_0(H_1)X_n(H_2)) \sim \frac{R(H_1,H_2)}{n^{2-H_1-H_2}} \tag{5}$$

as $n \to \infty$. Notice that if $H_1 = H_2$, then (5) is similar to (1). We consider a continuous function $h : \mathbb{R}_+ \to (\frac{1}{2},1)$ and, for every $N \in \mathbb{N}$, we define the process S^N_h by

$$(S^N_h(t))_{t\geq 0} = \left(\sum_{n=1}^{\lfloor Nt \rfloor} \frac{X_n(h(n/N))}{N^{h(n/N)}} \right)_{t\geq 0} \tag{6}$$

By Cohen and Marty (2008) and under additional assumptions, as $N \to \infty$ the finite-dimensional distributions of S^N_h converge to those of a Gaussian process S_h with mean zero and such that for all t_1 and t_2,

$$\mathbb{E}(S_h(t_1)S_h(t_2)) = \int_0^{t_1} \int_0^{t_2} \frac{R(h(\max\{\theta,\sigma\}), h(\min\{\theta,\sigma\}))}{|\theta - \sigma|^{2-h(\theta)-h(\sigma)}} d\theta d\sigma.$$

This is a multifractional extension of the result of Dobrushin and Major (1979) and Taqqu (1979) in the special case where $\phi(x) = x$ for all $x \in \mathbb{R}$. If h is constant, then S_h is a fractional Brownian motion, namely a Hermite process with $m = 1$. If h is non-constant, then S_h is multifractional like, for instance, the multifractional

Brownian motions (defined in Benassi, Jaffard, and Roux (1997) and Peltier and Lévy Véhel (1995)).

Now consider for every $N \in \mathbb{N}^*$ the process $S_{\phi,h}^N$ defined by

$$(S_{\phi,h}^N(t))_{t\geq 0} = \left(\sum_{n=1}^{\lfloor Nt \rfloor} \frac{\phi\left(X_n\left(\tilde{h}(n/N)\right)\right)}{N^{h(n/N)}} \right)_{t\geq 0} \tag{7}$$

where

$$\tilde{h} = 1 + \frac{h-1}{m}.$$

We assume that there exist a function $g : (H,\xi) \mapsto g(H,\xi)$ and a symmetric compact set $K \subset \mathbb{R}$ such that the field $X = (X_n(H))_{(n,H)\in\mathbb{N}\times(1/2,1)}$ of (7) is

$$X_n(H) = \int_K e^{in\xi} \frac{g(H,\xi)}{|\xi|^{H-1/2}} d\widehat{W}(\xi). \tag{8}$$

Under additional assumptions on g, X satisfies (5) with

$$R(H_1,H_2) = g(H_1,0)g(H_2,0) \int_{\mathbb{R}} \frac{e^{i\xi}}{|\xi|^{H_1+H_2-1}} d\xi$$

Notice that in this case,

$$R(H_1,H_2) = R(H_2,H_1). \tag{9}$$

The main result of Marty (2013) states that, as $N \to \infty$, the process $S_{\phi,h}^N$ converges in distribution in $\mathcal{D}([0,\infty),\mathbb{R})$ to a multifractional process $S_{m,h}$ defined by

$$(S_{m,h}(t))_{t\geq 0} = \left(\int_{\mathbb{R}^m} d\widehat{W}_{\xi_1} \cdots d\widehat{W}_{\xi_m} \int_0^t \widetilde{g}(\theta) \frac{\exp(i\theta(\xi_1 + \cdots + \xi_m))}{|\xi_1 \cdots \xi_m|^{\widetilde{h}(\theta)-1/2}} d\theta \right)_{t\geq 0} \tag{10}$$

where $\widetilde{g}$ is a deterministic function. This is an extension in a non-Gaussian setting of the result of Cohen and Marty (2008). Moreover, the process $S_{m,h}$ generalizes $B_{m,H}$, namely, if there exists $H \in (\frac{1}{2},1)$ such that $h(x) = H$ for all x, then $S_{m,h} = B_{m,H}$.

In this paper we extend the results mentioned above. We consider fields X such that for all $H_1 \neq H_2$,

$$R(H_1,H_2) \neq R(H_2,H_1),$$

in contrast with Marty (2013). With such fields X, we prove the convergence of sequences defined as in (7) to a process of the form

$$\left(\int_{D_m} dW_{x_1} \cdots dW_{x_m} \int_0^t f(\theta) \prod_{k=1}^m (\theta - x_k)_+^{\widetilde{h}(\theta)-3/2} d\theta \right)_{t\geq 0} \tag{11}$$

where f is a deterministic function. The limit is multifractional and generalizes Hermite processes, even though it generally differs from $S_{m,h}$ defined by (10) and studied in Marty (2013). In addition, we prove the convergence when $N \to \infty$ of

$$\left(\sum_{n=1}^{\lfloor Nt \rfloor} \frac{\phi\left(X_n\left(\tilde{h}\left(n/N^\beta\right)\right)\right)}{N^{h\left(n/N^\beta\right)}} \right)_{t \geq 0} \tag{12}$$

for all $\beta \in \mathbb{R} \backslash \{1\}$. In this case, the limit is a Hermite process with Hurst index depending on the sign of $\beta - 1$.

The paper is organized as follows. In Section 2 we present the setting and state the main result of the paper. In Section 3 we analyze a multifractional process of the form (11). Section 4 is devoted to the proof of the main result.

2 Setting and main result

Our main result is the convergence of a sequence defined from a functional of a Gaussian field with long-range dependence. This random field is presented in Section 2.1 and two examples are given in Section 2.2. The main result is stated in Section 2.3.

2.1 Gaussian fields with long-range dependence

We consider a random field $X = \{X(t,H)\}_{(t,H)\in\mathbb{R}_+\times(\frac{1}{2},1)}$ such that for all $(t,H) \in \mathbb{R}_+ \times (\frac{1}{2}, 1)$,

$$X(t,H) = \int_{-\infty}^{t} a(t-x,H)\,dW_x \tag{13}$$

where W is a Brownian motion and $a : \mathbb{R}_+ \times (\frac{1}{2}, 1) \to \mathbb{R}_+^*$ is a continuous function such that $\int_0^\infty a(x,H)^2 dx = 1$ for all H. As a consequence, X is a Gaussian field with mean zero and covariance function r which can be written as

$$r(t_1,H_1,t_2,H_2) = \mathbb{E}(X(t_1,H_1)X(t_2,H_2)) = \int_{-\infty}^{\min\{t_1,t_2\}} a(t_1-x,H_1)a(t_2-x,H_2)\,dx$$

for all $(t_1,H_1,t_2,H_2) \in (\mathbb{R}_+ \times (\frac{1}{2},1))^2$. This implies that for all (t,H), $r(t,H,t,H) = \mathbb{E}(X(t,H)^2) = 1$, and that for every H the process $t \mapsto X(t,H)$ is stationary, because for all (t_1,t_2),

$$\mathbb{E}(X(t_1,H)X(t_2,H)) = \int_0^\infty a(x,H)a(|t_1-t_2|+x,H)\,dx.$$

In addition to (13), we assume that there exists a continuous function $A:(\frac{1}{2},1)\to\mathbb{R}_+^*$ such that for every compact set $K\subset(\frac{1}{2},1)$,

$$\lim_{x\to\infty}\sup_{H\in K}|x^{3/2-H}a(x,H)-A(H)|=0. \tag{14}$$

We deduce the following property of the covariance which implies that X satisfies the assumptions considered in Cohen and Marty (2008).

Lemma 1 – *For every compact set $K\subset(\frac{1}{2},1)^2$,*

$$\lim_{|t_1-t_2|\to\infty}\sup_{(H_1,H_2)\in K}\left||t_1-t_2|^{2-H_1-H_2}r(t_1,H_1,t_2,H_2)-R(t_1,H_1,t_2,H_2)\right|=0. \tag{15}$$

where $R:(\mathbb{R}_+\times(\frac{1}{2},1))^2\to\mathbb{R}_+^*$ is a continuous function such that for all $(t_1,H_1,t_2,H_2)\in(\mathbb{R}_+\times(\frac{1}{2},1))^2$,

$$\begin{aligned}R(t_1,H_1,t_2,H_2)=&1_{t_1\geq t_2}A(H_1)A(H_2)\int_0^\infty(1+x)^{H_1-3/2}x^{H_2-3/2}dx\\&+1_{t_1<t_2}A(H_1)A(H_2)\int_0^\infty(1+x)^{H_2-3/2}x^{H_1-3/2}dx.\end{aligned} \tag{16}$$

Notice that if we set for all (H_1,H_2),

$$C(H_1,H_2)=A(H_1)A(H_2)\int_0^\infty(1+x)^{H_1-3/2}x^{H_2-3/2}dx$$

then

$$C(H_1,H_2)\neq C(H_2,H_1)$$

when $H_1\neq H_2$. Indeed, if we assume that $H_1>H_2$, then

$$\begin{aligned}\int_0^\infty(1+x)^{H_1-3/2}x^{H_2-3/2}dx&=\int_0^\infty\left(1+\frac{1}{x}\right)^{H_1-3/2}x^{H_1+H_2-3}dx\\&>\int_0^\infty\left(1+\frac{1}{x}\right)^{H_2-3/2}x^{H_1+H_2-3}dx\\&=\int_0^\infty(1+x)^{H_2-3/2}x^{H_1-3/2}dx.\end{aligned} \tag{17}$$

Hence, the framework of this paper is different from that of Marty (2013).

Proof. (Lemma 1) We assume $t_1 > t_2$. We have

$$\begin{aligned}
&r(t_1,H_1,t_2,H_2)\\
&=\int_{-\infty}^{t_2}(a(t_1-x,H_1)-(t_1-x)^{H_1-3/2}A(H_1))(a(t_2-x,H_2)-(t_2-x)^{H_2-3/2}A(H_2))dx\\
&\quad+\int_{-\infty}^{t_2}(a(t_1-x,H_1)-(t_1-x)^{H_1-3/2}A(H_1))(t_2-x)^{H_2-3/2}A(H_2)dx\\
&\quad+\int_{-\infty}^{t_2}(t_1-x)^{H_1-3/2}A(H_1)(a(t_2-x,H_2)-(t_2-x)^{H_2-3/2}A(H_2))dx\\
&\quad+\int_{-\infty}^{t_2}(t_1-x)^{H_1-3/2}A(H_1)(t_2-x)^{H_2-3/2}A(H_2)dx.
\end{aligned}\tag{18}$$

We set $\delta = t_1 - t_2$. By the change of variable $x \to t_2 - x$ we get

$$\begin{aligned}
&r(t_1,H_1,t_2,H_2)\\
&=\int_0^{\infty}(a(\delta+x,H_1)-(\delta+x)^{H_1-3/2}A(H_1))(a(x,H_2)-x^{H_2-3/2}A(H_2))dx\\
&\quad+\int_0^{\infty}(a(\delta+x,H_1)-(\delta+x)^{H_1-3/2}A(H_1))x^{H_2-3/2}A(H_2)dx\\
&\quad+\int_0^{\infty}(\delta+x)^{H_1-3/2}A(H_1)(a(x,H_2)-x^{H_2-3/2}A(H_2))dx\\
&\quad+\int_0^{\infty}(\delta+x)^{H_1-3/2}A(H_1)x^{H_2-3/2}A(H_2)dx.
\end{aligned}\tag{19}$$

As a consequence, by the change of variable $x \to \delta x$,

$$\begin{aligned}
\delta^{2-H_1-H_2}r(t_1,H_1,t_2,H_2)-A(H_1)A(H_2)\int_0^{\infty}(1+x)^{H_1-3/2}x^{H_2-3/2}dx&\\
=R_1(\delta,H_1,H_2)+R_2(\delta,H_1,H_2)+R_3(\delta,H_1,H_2)&
\end{aligned}\tag{20}$$

where

$$\begin{aligned}
R_1(\delta,H_1,H_2)=\delta^{3-H_1-H_2}\int_0^{\infty}&(a(\delta+\delta x,H_1)-(\delta+\delta x)^{H_1-3/2}A(H_1))\\
&\times(a(\delta x,H_2)-(\delta x)^{H_2-3/2}A(H_2))dx,
\end{aligned}$$

$$R_2(\delta,H_1,H_2)=\delta^{3/2-H_1}A(H_2)\int_0^{\infty}(a(\delta+\delta x,H_1)-(\delta+\delta x)^{H_1-3/2}A(H_1))x^{H_2-3/2}dx$$

and

$$R_3(\delta,H_1,H_2)=\delta^{3/2-H_2}A(H_1)\int_0^{\infty}(1+x)^{H_1-3/2}(a(\delta x,H_2)-(\delta x)^{H_2-3/2}A(H_2))dx$$

We have

$$R_3(\delta, H_1, H_2) = A(H_1) \int_0^\infty (1+x)^{H_1-3/2} x^{H_2-3/2} ((\delta x)^{3/2-H_2} a(\delta x, H_2) - A(H_2)) dx. \tag{21}$$

Let K be a compact set. Because of (14), for every $\nu > 0$ there exist two positive constants μ and c such that

$$\begin{aligned}\sup_{(H_1,H_2)\in K} |R_3(\delta, H_1, H_2)| \leq & \nu \sup_{(H_1,H_2)} \left| A(H_1) \int_{\mu/\delta}^\infty (1+x)^{H_1-3/2} x^{H_2-3/2} dx \right| \\ & + c \sup_{(H_1,H_2)} \left| A(H_1) \int_0^{\mu/\delta} (1+x)^{H_1-3/2} x^{H_2-3/2} dx \right|\end{aligned} \tag{22}$$

so that

$$\lim_{\delta\to\infty} \sup_{(H_1,H_2)\in K} |R_3(\delta, H_1, H_2)| = 0. \tag{23}$$

Similarly, we prove that

$$\lim_{\delta\to\infty} \sup_{(H_1,H_2)\in K} |R_1(\delta, H_1, H_2)| = \lim_{\delta\to\infty} \sup_{(H_1,H_2)\in K} |R_2(\delta, H_1, H_2)| = 0. \tag{24}$$

This completes the proof. □

2.2 Examples

We give two examples of fields satisfying the assumptions presented in Section 2.1.

Consider that X_1 is defined for all t and H by

$$X_1(t, H) = \frac{1}{C_1(H)} \int_{-\infty}^t \left((t-x)_+^{H-1/2} - (t-1-x)_+^{H-1/2} \right) dW_x \tag{25}$$

where

$$C_1(H) = \sqrt{\int_0^\infty \left((x)_+^{H-1/2} - (x-1)_+^{H-1/2} \right)^2 dx.}$$

For every H, $X_1(\cdot, H) : t \mapsto X_1(t, H)$ is the fractional Gaussian noise (see Taqqu (1979) for instance), namely the process of the increments $B_H(\cdot) - B_H(\cdot - 1)$ of the fractional Brownian motion B_H such that for all t,

$$B_H(t) = \frac{1}{C_1(H)} \int_{-\infty}^t \left((t-x)_+^{H-1/2} - (-x)_+^{H-1/2} \right) dW_x. \tag{26}$$

Then for all (t,H),

$$X_1(t,H) = \int_{-\infty}^{t} a_1(t-x,H)dW_x$$

where for all $x \geq 0$,

$$a_1(x,H) = \frac{1}{C_1(H)}\Big((x)_+^{H-1/2} - (x-1)_+^{H-1/2}\Big) = \frac{x^{H-1/2}}{C_1(H)}\left(1-\left(1-\frac{1}{x}\right)_+^{H-1/2}\right). \tag{27}$$

Moreover, for all $u \in (-1,1)$,

$$1-(1-u)^{H-1/2} = \left(H-\frac{1}{2}\right)u + \left(H-\frac{1}{2}\right)\left(\frac{3}{2}-H\right)\rho(u,H) \tag{28}$$

where

$$\rho(u,H) := \int_0^u (u-v)(1-v)^{H-5/2}dv \tag{29}$$

which satisfies

$$0 \leq \rho(u,H) \leq (1-u)^{H-5/2}\int_0^u (u-v)dv = (1-u)^{H-5/2}\frac{u^2}{2}. \tag{30}$$

From (27), (28), and (30), we deduce that for all $x > 2$,

$$\left|a_1(x,H) - x^{H-3/2}\frac{(H-1/2)}{C_1(H)}\right| \leq \frac{(H-1/2)(3/2-H)}{C_1(H)2^{H-5/2}x^{5/2-H}}.$$

Hence, the field defined X_1 by (25) satisfies (14) with the function $A(H) = \frac{H-1/2}{C_1(H)}$.

We give a second example. Consider the field X_2 defined for all t and H by

$$X_2(t,H) = \frac{1}{C_2(H)}\int_{-\infty}^{t}\left((t-x)_+^{H-1/2} - e^{-(t-x)}\int_0^{t-x} e^{\xi}\xi^{H-1/2}d\xi\right)dW_x \tag{31}$$

where

$$C_2(H) = \sqrt{\int_0^{\infty}\left(x_+^{H-1/2} - e^{-x}\int_0^x e^{\xi}\xi^{H-1/2}d\xi\right)^2 dx}. \tag{32}$$

For every H, $t \mapsto X_2(t,H)$ is the stationary fractional Ornstein-Uhlenbeck process (see Cheridito, Kawaguchi, and Maejima (2003) for instance) such that

$$X_2(t,H) = B_H(t) - e^{-t}\int_{-\infty}^{t} e^{\theta}B_H(\theta)d\theta$$

where B_H is the fractional Brownian motion defined by (26) with the constant $C_2(H)$ instead of $C_1(H)$. In this case, for all (t,H),

$$X_2(t,H) = \int_{-\infty}^{t} a_2(t-x,H)dW_x$$

where for all $x \geq 0$,

$$a_2(x,H) = \frac{1}{C_2(H)}\left(x_+^{H-1/2} - e^{-x}\int_0^x e^{\xi}\xi^{H-1/2}d\xi\right).$$

From the following calculations we deduce that the field X_2 defined by (31) satisfies (14) with the function $A(H) = (H-1/2)/C_2(H)$. For all (x,H), by the change of variable $\xi \to x-\xi$,

$$\begin{aligned}
x_+^{H-1/2} - e^{-x}\int_0^x e^{\xi}\xi^{H-1/2}d\xi =& e^{-x}x^{H-1/2} + e^{-x}\int_0^x e^{\xi}(x^{H-1/2} - \xi^{H-1/2})d\xi \\
=& e^{-x}x^{H-1/2} + x^{H-1/2}\int_0^x e^{-\xi}\left(1-\left(1-\frac{\xi}{x}\right)^{H-1/2}\right)d\xi \\
=& e^{-x}x^{H-1/2} + x^{H-1/2}\int_{x/2}^x e^{-\xi}\left(1-\left(1-\frac{\xi}{x}\right)^{H-1/2}\right)d\xi \\
&+ x^{H-1/2}\int_0^{x/2} e^{-\xi}\left(1-\left(1-\frac{\xi}{x}\right)^{H-1/2}\right)d\xi.
\end{aligned} \tag{33}$$

For every compact subset $K \subset (\frac{1}{2},1)$,

$$\lim_{x\to\infty}\sup_{H\in K}\left|x^{3/2-H}\frac{e^{-x}x^{H-1/2}}{C_2(H)}\right| = \lim_{x\to\infty} xe^{-x}\sup_{H\in K}\left|\frac{1}{C_2(H)}\right| = 0. \tag{34}$$

By the change of variable $\xi \to x\xi$, for all (x,H),

$$\begin{aligned}
\left|x^{H-1/2}\int_{x/2}^x e^{-\xi}\left(1-\left(1-\frac{\xi}{x}\right)^{H-1/2}\right)d\xi\right| &= x^{H+1/2}\int_{1/2}^1 e^{-x\xi}\left(1-(1-\xi)^{H-1/2}\right)d\xi \\
&\leq x^{H+1/2}e^{-x/2}\int_{1/2}^1 \left(1-(1-\xi)^{H-1/2}\right)d\xi \\
&\leq x^{H+1/2}e^{-x/2}.
\end{aligned}$$

Then

$$\begin{aligned}
&\lim_{x\to\infty}\sup_{H\in K}\left|x^{3/2-H}\frac{x^{H-1/2}}{C_2(H)}\int_{x/2}^x e^{-\xi}\left(1-\left(1-\frac{\xi}{x}\right)^{H-1/2}\right)d\xi\right| \\
&\leq \lim_{x\to\infty} x^2e^{-x/2}\sup_{H\in K}\left|\frac{1}{C_2(H)}\right| = 0.
\end{aligned} \tag{35}$$

By (28),

$$x^{H-1/2}\int_0^{x/2} e^{-\xi}\left(1-\left(1-\frac{\xi}{x}\right)^{H-1/2}\right)d\xi = x^{H-1/2}\left(H-\frac{1}{2}\right)\left(\frac{3}{2}-H\right)\int_0^{x/2} e^{-\xi}\rho\left(\frac{\xi}{x},H\right)d\xi + x^{H-1/2}\left(H-\frac{1}{2}\right)\int_0^{x/2} e^{-\xi}\frac{\xi}{x}d\xi \tag{36}$$

where ρ is defined by (29). From (30) we deduce that

$$\left|\int_0^{x/2} e^{-\xi}\rho\left(\frac{\xi}{x},H\right)d\xi\right| \leq \int_0^{x/2} e^{-\xi}\left(1-\frac{\xi}{x}\right)^{H-5/2}\frac{\xi^2}{x^2}d\xi \leq \frac{x^{-2}}{2^{H-5/2}}\int_0^{x/2} e^{-\xi}\xi^2 d\xi \leq \frac{x^{-2}}{2^{H-7/2}}.$$

Hence,

$$\begin{aligned}&\lim_{x\to\infty}\sup_{H\in K}\left|x^{3/2-H}x^{H-1/2}\frac{(H-1/2)(3/2-H)}{C_2(H)}\int_0^{x/2} e^{-\xi}\rho\left(\frac{\xi}{x},H\right)d\xi\right| \\ &\leq \lim_{x\to\infty} x^{-1}\sup_{H\in K}\left|\frac{(H-1/2)(3/2-H)}{2^{H-7/2}C_2(H)}\right| = 0.\end{aligned} \tag{37}$$

Finally,

$$x^{H-1/2}\left(H-\frac{1}{2}\right)\int_0^{x/2} e^{-\xi}\frac{\xi}{x}d\xi = x^{H-3/2}\left(H-\frac{1}{2}\right)\left(1-e^{-x/2}+\frac{x}{2}e^{-x/2}\right),$$

which implies that

$$\lim_{x\to\infty}\sup_{H\in K}\left|x^{3/2-H}x^{H-1/2}\frac{(H-1/2)}{C_2(H)}\int_0^{x/2} e^{-\xi}\frac{\xi}{x}d\xi - \frac{(H-1/2)}{C_2(H)}\right| = 0. \tag{38}$$

By (33), (34), (35), (36), (37), and (38) we conclude that X_2 satisfies (14) with the function $A(H) = \frac{H-1/2}{C_2(H)}$.

2.3 Main result

Let $\Phi \in L^2(\mathbb{R}, e^{-x^2/2}dx)$ be a function with Hermite rank $m \in \mathbb{N}^*$, which means that

$$\Phi = \sum_{k=m}^{\infty}\frac{\phi_k}{k!}P_k \tag{39}$$

where the convergence of $\sum_{k=m}^{\infty}$ is defined from the norm $\|\cdot\|_{L^2(\mathbb{R},e^{-x^2/2}dx)}$, P_k is the Hermite polynomial of degree k for every $k \geq m$, namely

$$P_k : x \mapsto P_k(x) = (-1)^k e^{x^2/2} \frac{d^k}{dx^k} e^{-x^2/2},$$

and

$$\phi_k = \langle \Phi, P_k \rangle_{L^2(\mathbb{R},e^{-x^2/2}dx)} = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \Phi(x) P_k(x) e^{-x^2/2} dx.$$

Let $h : \mathbb{R}_+ \to (\frac{1}{2}, 1)$ be a continuous function and for all $\theta \in \mathbb{R}_+$,

$$\widetilde{h}(\theta) = 1 + \frac{h(\theta) - 1}{m} \in \Big(1 - \frac{1}{2m}, 1\Big). \tag{40}$$

We set $h^- = \min h$ and $h^+ = \max h$, and assume that $[h^-, h^+] \subset (\frac{1}{2}, 1)$.

Let X be a random field defined as in Section 2.1, namely a Gaussian field such that (13) and (14) are satisfied. We fix a real interval I such that $I \subset [0, \infty)$. For all $N \in \mathbb{N}^*$ and $t \in I$, we consider

$$\Sigma_{\Phi,h}^N(t) = \sum_{n=1}^{\lfloor Nt \rfloor} \frac{\Phi(X(n, \widetilde{h}(n/N^\beta)))}{N^{h(n/N^\beta)}} \tag{41}$$

where $\beta \in \mathbb{R}$. Moreover, if $\beta < 1$, we assume that $\lim_{\theta \to \infty} h(\theta)$ exists and belongs to $(\frac{1}{2}, 1)$. We denote this limit by $h(\infty)$.

The main result states the convergence of the process $\Sigma_{\Phi,h}^N$ as $N \to \infty$. The proof is postponed to Section 4.

Theorem 1 – *As $N \to \infty$, the process $\Sigma_{\Phi,h}^N$ converges in distribution in $\mathcal{D}(I, \mathbb{R})$ to the process $\Sigma_{m,h}$ defined by*

$$(\Sigma_{m,h}(t))_{t \in I} = \left(\int_{D_m} dW_{x_1} \cdots dW_{x_m} \int_0^t G(\theta, x_1, \cdots, x_m) d\theta \right)_{t \in I} \tag{42}$$

where $D_m = \{x_1 < x_2 < \cdots < x_m\}$, W is a Brownian motion, and for all $(\theta, x_1, \cdots, x_m) \in [0, \infty) \times D_m$,

$$G(\theta, x_1, \cdots, x_m) = \begin{cases} \phi_m A\big(\widetilde{h}(0)\big)^m \prod_{k=1}^m (\theta - x_k)_+^{\widetilde{h}(0) - 3/2} & \text{if } \beta > 1, \\ \phi_m A\big(\widetilde{h}(\theta)\big)^m \prod_{k=1}^m (\theta - x_k)_+^{\widetilde{h}(\theta) - 3/2} & \text{if } \beta = 1, \\ \phi_m A\big(\widetilde{h}(\infty)\big)^m \prod_{k=1}^m (\theta - x_k)_+^{\widetilde{h}(\infty) - 3/2} & \text{if } \beta < 1. \end{cases} \tag{43}$$

If $\beta \neq 1$, then the limit $\Sigma_{m,h}$ is a Hermite process as defined in Dobrushin and Major (1979) and Taqqu (1979) with Hurst index equal to $h(0)$ or $h(\infty)$ depending on the sign of $\beta - 1$.

If $\beta = 1$ and h is a non-constant function, then the limit $\Sigma_{m,h}$ is a multifractional process as those we analyze in Section 3.

We conclude this section with a continuous version of Theorem 1. For every $\varepsilon > 0$, we define the process $\widetilde{\Sigma}^{\varepsilon}_{\Phi,h}$ by

$$\begin{aligned}\widetilde{\Sigma}^{\varepsilon}_{\Phi,h}(t) &= \int_0^t \varepsilon^{h(\varepsilon^{\beta-1}\theta)-1}\Phi(X(\varepsilon^{-1}\theta, \widetilde{h}(\varepsilon^{\beta-1}\theta)))d\theta \\ &= \int_0^{t/\varepsilon} \varepsilon^{h(\varepsilon^{\beta}\theta)}\Phi(X(\theta, \widetilde{h}(\varepsilon^{\beta}\theta)))d\theta.\end{aligned}$$

The following theorem states the convergence of $\widetilde{\Sigma}^{\varepsilon}_{\Phi,h}$ as $\varepsilon \to 0$.

Theorem 2 – *As $\varepsilon \to 0$, the process $\widetilde{\Sigma}^{\varepsilon}_{\Phi,h}$ converges in distribution in $\mathcal{C}(I,\mathbb{R})$ to the process $\Sigma_{m,h}$ defined as in Theorem 1.*

The proof of Theorem 2 is omitted because it is similar to that of Theorem 1 in a continuous setting.

3 A multifractional process

In this section we analyze a stochastic process which has the form of $\Sigma_{m,h}$ (see Theorem 1). In particular we prove that it satisfies multifractional properties.

We fix $m \in \mathbb{N}^*$ and for all $t \geq 0$ we define

$$Y(t) = \int_{D_m} dW_{x_1} \cdots dW_{x_m} \int_0^t f(\theta) \prod_{k=1}^{m} (\theta - x_k)_+^{\varphi(\theta)-3/2} d\theta \tag{44}$$

where $D_m = \{x_1 < x_2 < \cdots < x_m\}$, W is a Brownian motion (see Itô (1951) for the definition of the multiple Wiener integrals), and $f : \mathbb{R}_+ \to \mathbb{R}_+^*$ and $\varphi : \mathbb{R}_+ \to (1-\frac{1}{2m}, 1)$ are continuous functions. The process Y is well defined by Lemma 2 which is stated below and is used throughout the paper.

Lemma 2 – *For every $(t, \gamma_1, \cdots, \gamma_m) \in \mathbb{R}_+ \times (1 - \frac{1}{2m}, 1)^m$,*

$$\int_{\mathbb{R}^m} dx_1 \cdots dx_m \left(\int_0^t \prod_{k=1}^{m} (\theta - x_k)_+^{\gamma_k - 3/2} d\theta \right)^2 < \infty. \tag{45}$$

Proof. By the Fubini theorem and changes of variables,

$$\begin{aligned}
&\int_{\mathbb{R}^m} dx_1 \cdots dx_m \left(\int_0^t (\theta - x_k)_+^{\gamma_k - 3/2} d\theta \right)^2 \\
&\quad = \int_0^t \int_0^t d\theta d\sigma \int_{\mathbb{R}^m} dx_1 \cdots dx_m \prod_{k=1}^m ((\theta - x_k)_+ (\sigma - x_k)_+)^{\gamma_k - 3/2} \qquad (46) \\
&\quad = \int_0^t \int_0^t d\theta d\sigma |\theta - \sigma|^{2\sum_j \gamma_j - 2} \prod_{k=1}^m \int_0^\infty dx((1+x)x)^{\gamma_k - 3/2}.
\end{aligned}$$

Since $\gamma_k \in (1 - \frac{1}{2m}, 1)$ for all k, the upper bound of (46) is finite. □

The following lemma states the continuity of Y.

Lemma 3 – *The sample paths of Y are continuous almost surely.*

Proof. For all $s < t$ such that $|t - s|$ is small enough,

$$\begin{aligned}
&\mathbb{E}((Y(t) - Y(s))^2) \\
&\quad = \int_{D_m} dx_1 \cdots dx_m \left(\int_s^t f(\theta) \prod_{k=1}^m (\theta - x_k)_+^{\varphi(\theta) - 3/2} d\theta \right)^2 \\
&\quad \leq \int_s^t d\theta \int_s^t d\sigma \int_{\mathbb{R}^m} dx_1 \cdots dx_m f(\theta) f(\sigma) \prod_{k=1}^m (\theta - x_k)_+^{\varphi(\theta) - 3/2} (\sigma - x_k)_+^{\varphi(\sigma) - 3/2}
\end{aligned}$$

By changes of variables,

$$\begin{aligned}
\mathbb{E}((Y(t) - Y(s))^2) \leq \sup_{[s,t]} |f|^2 \int_0^1 d\theta \int_0^1 d\sigma (t-s)^{2 + m(\varphi((t-s)\theta + s) + \varphi((t-s)\sigma + s) - 2)} \\
\times |\theta - \sigma|^{m(\varphi((t-s)\theta + s) + \varphi((t-s)\sigma + s) - 2)} \qquad (47) \\
\times \left(\int_0^\infty dx (1+x)^{\varphi((t-s)\theta + s) - 3/2} x^{\varphi((t-s)\sigma + s) - 3/2} \right)^m
\end{aligned}$$

We deduce that there exists a constant $C > 0$ independent of (s, t) such that

$$\mathbb{E}((Y(t) - Y(s))^2) \leq C \sup_{[s,t]} |f|^2 (t-s)^{2(1 + m(\inf_{[s,t]} \varphi - 1))}. \qquad (48)$$

Remark that $2(1 + m(\inf_{[s,t]} \varphi - 1)) > 1$. By the Kolmogorov continuity theorem, we deduce that Y has a modification with almost surely continuous sample paths. □

The multifractional properties of Y are stated in the following two theorems. The first one establishes that Y is locally self-similar.

Theorem 3 – *Let $t \geq 0$. If φ is Hölder-continuous then, as $\varepsilon \to 0$, the process*

$$\left(\frac{Y(t+\varepsilon u)-Y(t)}{\varepsilon^{1+m(\varphi(t)-1)}}\right)_{u\geq 0} \tag{49}$$

converges in distribution to

$$\left(f(t)\int_{D_m} dW_{x_1}\cdots dW_{x_m}\int_0^u \prod_{k=1}^m (\theta-x_k)_+^{\varphi(t)-3/2}d\theta\right)_{u\geq 0} \tag{50}$$

in $\mathcal{C}([0,\infty),\mathbb{R})$.

Proof. For all t and u,

$$\begin{aligned}
&\frac{Y(t+\varepsilon u)-Y(t)}{\varepsilon^{1+m(\varphi(t)-1)}}\\
&= \varepsilon^{-1-m(\varphi(t)-1)}\int_{D_m} dW_{x_1}\cdots dW_{x_m}\int_t^{t+\varepsilon u} f(\theta)\prod_{k=1}^m(\theta-x_k)_+^{\varphi(\theta)-3/2}d\theta\\
&= \varepsilon^{-m(\varphi(t)-1)}\int_{D_m} dW_{x_1}\cdots dW_{x_m}\int_0^{u} f(\varepsilon\theta+t)\prod_{k=1}^m(\varepsilon\theta+t-x_k)_+^{\varphi(\varepsilon\theta+t)-3/2}d\theta.
\end{aligned}$$

Then, by a change of variable in the stochastic integral, for every t the process defined by (49) is equal in distribution to

$$\left(\int_{D_m} dW_{x_1}\cdots dW_{x_m}\int_0^u \psi^\varepsilon(t,x,\theta)d\theta\right)_{u\geq 0} \tag{51}$$

where

$$\psi^\varepsilon(t,x,\theta) = \varepsilon^{m(\varphi(\varepsilon\theta+t)-\varphi(t))}f(\varepsilon\theta+t)\prod_{k=1}^m(\theta-x_k)_+^{\varphi(\varepsilon\theta+t)-3/2} \tag{52}$$

To prove the convergence of the finite-dimensional distributions, it suffices to show that

$$\lim_{\varepsilon\to 0}\int_{D_m} dx_1\cdots dx_m\left(\int_0^u d\theta\left(\psi^\varepsilon(t,x,\theta)-f(t)\prod_{k=1}^m(\theta-x_k)_+^{\varphi(t)-3/2}\right)\right)^2 = 0 \tag{53}$$

For all (t,x,θ), by the continuity of f and φ,

$$\lim_{\varepsilon\to 0}\left(\psi^\varepsilon(t,x,\theta)-f(t)\prod_{k=1}^m(\theta-x_k)_+^{\varphi(t)-3/2}\right) = 0.$$

Moreover, there exists a constant $C > 0$ such that for all (t, x, θ),

$$\left|\psi^{\varepsilon}(t,x,\theta) - f(t)\prod_{k=1}^{m}(\theta - x_k)_+^{\varphi(t)-3/2}\right| \leq C\prod_{k=1}^{m}\left((\theta - x_k)_+^{\min\varphi-3/2} - (\theta - x_k)_+^{\max\varphi-3/2}\right).$$

Hence, by the bounded convergence theorem and Lemma 2, we obtain (53). It remains to prove the tightness of the family of processes defined by (49). For all u and v,

$$\begin{aligned}
&\mathbb{E}\left(\left(\frac{Y(t+\varepsilon u) - Y(t)}{\varepsilon^{1+m(\varphi(t)-1)}} - \frac{Y(t+\varepsilon v) - Y(t)}{\varepsilon^{1+m(\varphi(t)-1)}}\right)^2\right)\\
&= \varepsilon^{-2-2m(\varphi(t)-1)}\int_{D_m} dx_1\cdots dx_m\left(\int_{t+\varepsilon v}^{t+\varepsilon u} f(\theta)\prod_{k=1}^{m}(\theta - x_k)_+^{\varphi(\theta)-3/2}d\theta\right)^2\\
&= \int_{D_m} dx_1\cdots dx_m\left(\int_v^u \psi^{\varepsilon}(t,x,\theta)d\theta\right)^2\\
&\leq \|f\|_\infty^2\int_{D_m} dx_1\cdots dx_m\left(\int_v^u d\theta\prod_{k=1}^{m}\left((\theta - x_k)_+^{\min\varphi-3/2} + (\theta - x_k)_+^{\max\varphi-3/2}\right)\right)^2.
\end{aligned}$$

Then there exists a constant $C > 0$ independent of (s, t) such that

$$\mathbb{E}\left(\left(\frac{Y(t+\varepsilon u) - Y(t)}{\varepsilon^{1+m(\varphi(t)-1)}} - \frac{Y(t+\varepsilon v) - Y(t)}{\varepsilon^{1+m(\varphi(t)-1)}}\right)^2\right) \leq C|u - v|^{2(1+m(\inf\varphi-1))}.$$

The tightness is proved (see Billingsley (1968) for instance), then Y is locally self-similar. □

The following results gives the local Hölder exponent.

Theorem 4 – *Let $t \geq 0$ and assume that $f(t) \neq 0$. The local Hölder exponent of Y at t is $1 + m(\varphi(t) - 1)$.*

Proof. By Theorem 3 and similar arguments to those of the proof of Proposition 10 of Peltier and Lévy Véhel (1995), it suffices to show that for every $p \in \mathbb{N}^*$ there exists $C_p > 0$ such that for all $s \leq t$ in a neighborhood of t_0,

$$\mathbb{E}((Y(t) - Y(s))^{2p}) \leq C_p(t - s)^{2p\inf_{[s,t]}(1+m(\varphi-1))}. \tag{54}$$

This is obtained from (48) and Nelson (1973). □

We conclude this analysis of Y with an important remark. Assume that $f = \alpha \circ \varphi$ where α is a strictly positive function, so that

$$Y(t) = \int_{D_m} dW_{x_1}\cdots dW_{x_m}\int_0^t \alpha(\varphi(\theta))\prod_{k=1}^{m}(\theta - x_k)_+^{\varphi(\theta)-3/2}d\theta \tag{55}$$

which is the form of the limit $\Sigma_{m,h}$ of Theorem 1 (Section 2.3). For non-constant functions φ, the processes defined by (55) are generally different from those of Marty (2013). This can be shown as follows. As in the proof of Theorem 3 (see also the proof of Proposition 1 of Cohen and Marty (2008)), we can establish that for all $s \neq t$,

$$\begin{aligned}&\lim_{\varepsilon\to 0}\varepsilon^{-2}\mathbb{E}((Y(t+\varepsilon)-Y(t))(Y(s+\varepsilon)-Y(s)))\\&\qquad=|t-s|^{\varphi(t)+\varphi(s)-2}\alpha(\varphi(s))\alpha(\varphi(t))R_Y(\varphi(\max\{s,t\}),\varphi(\min\{s,t\}))\end{aligned}$$

with for all (H_1,H_2),

$$R_Y(H_1,H_2)=\frac{1}{m!}\left(\int_0^{\infty}(1+x)^{H_1-3/2}x^{H_2-3/2}dx\right)^m.$$

Notice that $R_Y(H_1,H_2)\neq R_Y(H_2,H_1)$ if $H_1\neq H_2$ (see Section 2.1).

Now consider the process such that for all t,

$$Z(t)=\int_{\mathbb{R}^m}d\widehat{W}_{\xi_1}\cdots d\widehat{W}_{\xi_m}\int_0^t\beta(\varphi(\theta))\frac{\exp(i\theta(\xi_1+\cdots+\xi_m))}{|\xi_1\cdots\xi_m|^{\varphi(\theta)-1/2}}d\theta \tag{56}$$

where β is a continuous function and $\widehat{W}$ the Fourier transform of W. This is the form of the processes studied in Marty (2013). We can prove that for all $s\neq t$,

$$\begin{aligned}&\lim_{\varepsilon\to 0}\varepsilon^{-2}\mathbb{E}((Z(t+\varepsilon)-Z(t))(Z(s+\varepsilon)-Z(s)))\\&\qquad=|t-s|^{\varphi(t)+\varphi(s)-2}\beta(\varphi(s))\beta(\varphi(t))R_Z(\varphi(\max\{s,t\}),\varphi(\min\{s,t\}))\end{aligned}$$

with for all (H_1,H_2),

$$R_Z(H_1,H_2)=\left(\int_{-\infty}^{\infty}\exp(i\xi)|\xi|^{1-H_1-H_2}d\xi\right)^m.$$

The function R_Z is symmetric, in contrast with R_Y. Consider three points t_1, t_2 and $t_3\in\mathbb{R}_+$ such that $t_1<t_2<t_3$, and a function φ such that $\varphi(t_1)=\varphi(t_3)>\varphi(t_2)$. Since R_Z is symmetric,

$$R_Z(\varphi(\max\{t_1,t_2\}),\varphi(\min\{t_1,t_2\}))=R_Z(\varphi(\max\{t_3,t_2\}),\varphi(\min\{t_3,t_2\})). \tag{57}$$

As a consequence of (17),

$$\begin{aligned}R_Y(\varphi(\max\{t_1,t_2\}),\varphi(\min\{t_1,t_2\}))&=R_Y(\varphi(t_2),\varphi(t_1)))\\&<R_Y(\varphi(t_1),\varphi(t_2)))\\&=R_Y(\varphi(t_3),\varphi(t_2)))\\&=R_Y(\varphi(\max\{t_3,t_2\}),\varphi(\min\{t_3,t_2\})),\end{aligned}$$

which differs from (57). Finally, this proves the following result.

Theorem 5 – *There exist functions* $\varphi \in \mathcal{C}(\mathbb{R}_+, (\frac{1}{2}, 1))$ *such that for all* $\alpha \in \mathcal{C}((\frac{1}{2}, 1), \mathbb{R}_+^*)$ *and* $\beta \in \mathcal{C}((\frac{1}{2}, 1), \mathbb{R})$ *the processes* Y *and* Z *defined by* (55) *and* (56) *have different distributions.*

Hence, the processes given by (55) are generally different from those studied in Marty (2013).

4 Proof of Theorem 1

In this section we prove Theorem 1. It is a consequence of the tightness (Section 4.1) and of the convergence of the finite-dimensional distributions of $\Sigma_{\Phi,h}^N$ as $N \to \infty$ (Section 4.2). Technical lemmas are postponed to Section 4.3.

4.1 Tightness

The following lemma establishes the tightness of $(\Sigma_{\Phi,h}^N)_N$ in $\mathcal{D}(I, \mathbb{R})$.

Lemma 4 – *The sequence of processes* $(\Sigma_{\Phi,h}^N)_N$ *is tight in* $\mathcal{D}(I, \mathbb{R})$.

Proof. We only give the main idea of the proof because it is similar to Section 4.4 of Marty (2013). By the Cauchy-Schwarz inequality, the properties of the Hermite polynomials, and Lemma 1, we prove that there exists $C > 0$ such that for all N and for all (t_1, t_2, t_3) satisfying $t_1 < t_2 < t_3$ and $t_3 - t_1 < 1$ we have

$$\mathbb{E}(|\Sigma_{\Phi,h}^N(t_3) - \Sigma_{\Phi,h}^N(t_2)||\Sigma_{\Phi,h}^N(t_2) - \Sigma_{\Phi,h}^N(t_1)|) \leq C(t_3 - t_1)^{2h^-}.$$

Since $h > 1/2$ and by Theorem 15.6 of Billingsley (1968), we get the tightness property of $(\Sigma_{\Phi,h}^N)_N$. □

4.2 Convergence of the finite-dimensional distributions

In this subsection we prove the convergence of the finite-dimentional distributions of $\Sigma_{\Phi,h}^N$ as $N \to \infty$. For all n and N we set

$$h_n^N := h\left(\frac{n}{N^\beta}\right) \text{ and } \widetilde{h}_n^N := \widetilde{h}\left(\frac{n}{N^\beta}\right).$$

Lemma 5 – *Let* $d \in \mathbb{N}^*$ *and* $(t_1, \cdots, t_d) \in I^d$. *As* $N \to \infty$, $(\Sigma_{\Phi,h}^N(t_1), \cdots, \Sigma_{\Phi,h}^N(t_d))$ *converges in distribution to* $(\Sigma_{m,h}(t_1), \cdots, \Sigma_{m,h}(t_d))$ *in* $\mathbb{R}^d$.

Proof. By Lemma 6, it suffices to prove the convergence of the finite-dimensional distributions of the process Θ_1^N defined for all t by

$$\Theta_1^N(t) = \frac{\phi_m}{m!} \sum_{n=1}^{\lfloor Nt \rfloor} N^{-h_n^N} P_m(X(n, \widetilde{h}_n^N)) \tag{58}$$

Since P_m is the Hermite polynomial of degree m and $\mathbb{E}(X(t,H)^2) = 1$ for all (t,H),

$$P_m(X(t,H)) = m! \int_{D_m(t)} \prod_{k=1}^{m} a(t - x_k, H) dW_{x_1} \cdots dW_{x_m} \tag{59}$$

where $D_m(t) = \{x_1 < x_2 < \cdots < x_m \leq t\}$. As a consequence,

$$\begin{aligned}
\Theta_1^N(t) &= \frac{\phi_m}{m!} \sum_{n=1}^{\lfloor Nt \rfloor} N^{-h_n^N} P_m(X(n, \widetilde{h}_n^N)) \\
&= \phi_m \sum_{n=1}^{\lfloor Nt \rfloor} N^{-h_n^N} \int_{D_m(n)} \prod_{k=1}^{m} a(n - x_k, \widetilde{h}_n^N) dW_{x_1} \cdots dW_{x_m} \\
&= \phi_m \int_{D_m(n)} dW_{x_1} \cdots dW_{x_m} \sum_{n=1}^{\lfloor Nt \rfloor} N^{-h_n^N} \prod_{k=1}^{m} a(n - x_k, \widetilde{h}_n^N).
\end{aligned} \tag{60}$$

By a change of variable in the stochastic integral and the self-similarity property of W, Θ_1^N is equal in distribution to Θ_2^N defined for all t by

$$\begin{aligned}
\Theta_2^N(t) &= \phi_m \int_{D_m} dW_{x_1} \cdots dW_{x_m} \sum_{n=1}^{\lfloor Nt \rfloor} N^{-h_n^N + m/2} 1_{Nx_m < n} \prod_{k=1}^{m} a(n - Nx_k, \widetilde{h}_n^N) \\
&= \phi_m \int_{D_m} dW_{x_1} \cdots dW_{x_m} \sum_{n=1}^{\lfloor Nt \rfloor} N^{-h_n^N + m/2} \prod_{k=1}^{m} 1_{Nx_k < n} a(n - Nx_k, \widetilde{h}_n^N).
\end{aligned} \tag{61}$$

By Lemma 10, the convergence of the finite-dimensional distributions of Θ_2^N is equivalent to the convergence of those of Θ_3^N which is defined for all t by

$$\begin{aligned}
\Theta_3^N(t) &= \phi_m \int_{D_m} dW_{x_1} \cdots dW_{x_m} \sum_{n=1}^{\lfloor Nt \rfloor} N^{-1} A(\widetilde{h}_n^N)^m \prod_{k=1}^{m} \left(\frac{n}{N} - x_k \right)_+^{\widetilde{h}_n^N - 3/2} \\
&= \int_{D_m} dW_{x_1} \cdots dW_{x_m} \int_0^t G^N(\theta, x_1, \cdots, x_m) d\theta
\end{aligned} \tag{62}$$

with for all $\theta \in [0,t)$,

$$
\begin{aligned}
G^N(\theta, x_1, \cdots, x_m) &= \phi_m \sum_{n=1}^{\lfloor Nt \rfloor} 1_{((n-1)/N, n/N]}(\theta) A(\widetilde{h}_n^N)^m \prod_{k=1}^m \left(\frac{n}{N} - x_k\right)_+^{\widetilde{h}_n^N - 3/2} \\
&= \phi_m A(\widetilde{h}_{\lceil N\theta \rceil}^N)^m \prod_{k=1}^m \left(\frac{\lceil N\theta \rceil}{N} - x_k\right)_+^{\widetilde{h}_{\lceil N\theta \rceil}^N - 3/2}
\end{aligned}
\tag{63}
$$

where $\lceil \cdot \rceil$ is the ceiling function. Finally, by Lemma 11, the proof is completed. □

4.3 Technical lemmas

In this section we prove Lemmas 6, 10, and 11 which are used in the proof of the convergence of the finite-dimensional distributions of $\Sigma_{\Phi,h}^N$ (Lemma 5). The norm $\|\cdot\|_{L^2(\Omega,\mathbb{P})}$ is denoted by $\|\cdot\|$.

Lemma 6 – *For all t, $\lim_{N\to\infty} \left\|\Sigma_{\Phi,h}^N(t) - \Theta_1^N(t)\right\| = 0$.*

Proof. As a consequence of (15), the proof is similar to that of Lemma 4.5 of Marty (2013). □

Lemmas 7, 8, and 9 which follow are used in the proof of Lemma 10.

Lemma 7 – *For every $\eta > 0$ there exist two positive constants $C(\eta) > 0$ and $K(\eta) > 0$ such that for all $x_1 > 0$, $x_2 > 0$, ..., $x_m > 0$, and $H \in [h^-, h^+]$,*

$$
\begin{aligned}
&|a(x_1, H) \cdots a(x_m, H) - A(H)^m (x_1 \cdots x_m)^{H-3/2}| \\
&\qquad \le B(x_1 \cdots x_m)^{H-3/2} \sum_{k=1}^m (\eta 1_{x_k > K(\eta)} + C(\eta) 1_{x_k \le K(\eta)})
\end{aligned}
\tag{64}
$$

with

$$
B = \max_{k=1,\cdots,m} \left(\max_{(x,H)\in\mathbb{R}_+\times[h^-,h^+]} |x^{3/2-H} a(x,H)|^{k-1} \times \max_{H\in[h^-,h^+]} |A(H)|^{m-k-1} \right) < \infty.
$$

Proof. By (14), for every $\eta > 0$ there exists $K(\eta) > 0$ such that for all $x > K(\eta)$ and $H \in [h^-, h^+]$,

$$
|a(x,H) - A(H)x^{H-3/2}| \le x^{H-3/2}\eta. \tag{65}
$$

Moreover, for all $x \in [0, K(\eta)]$,

$$
\begin{aligned}
&|a(x,H) - A(H)x^{H-3/2}| \\
&\le \max_{(x,H)\in[0,K(\eta)]\times[h^-,h^+]} |a(x,H)| + x^{H-3/2} \max_{H\in[h^-,h^+]} |A(H)| \\
&\le x^{H-3/2}(1 + x^{3/2-H}) \left(\max_{(x,H)\in[0,K(\eta)]\times[h^-,h^+]} |a(x,H)| + \max_{H\in[h^-,h^+]} |A(H)| \right) \\
&\le C(\eta) x^{H-3/2}
\end{aligned}
\tag{66}
$$

with

$$C(\eta) = (1 + K(\eta)^{3/2-H})\left(\max_{(x,H)\in[0,K(\eta)]\times[h^-,h^+]}|a(x,H)| + \max_{H\in[h^-,h^+]}|A(H)|\right).$$

From (65) and (66) we deduce that for all $x \geq 0$ and $H \in [h^-, h^+]$,

$$|a(x,H) - A(H)x^{H-3/2}| \leq x^{H-3/2}(\eta 1_{x>K(\eta)} + C(\eta)1_{x\leq K(\eta)}). \tag{67}$$

Hence, for all $(x_1, \cdots, x_m, H)$,

$$\begin{aligned}
&|a(x_1,H)\cdots a(x_m,H) - A(H)^m(x_1\cdots x_m)^{H-3/2}|\\
\leq&\sum_{k=1}^m |a(x_1,H)\cdots a(x_{k-1},H)(a(x_k,H) - A(H)x_k^{H-3/2})A(H)^{m-k-1}(x_{k+1}\cdots x_m)^{H-3/2}|\\
\leq&(x_1\cdots x_m)^{H-3/2}\sum_{k=1}^m\left|\left(\prod_{j=1}^{k-1}x_j^{3/2-H}a(x_j,H)\right)(\eta 1_{x_k>K(\eta)} + C(\eta)1_{x_k\leq K(\eta)})A(H)^{m-k-1}\right|\\
\leq&B(x_1\cdots x_m)^{H-3/2}\sum_{k=1}^m(\eta 1_{x_k>K(\eta)} + C(\eta)1_{x_k\leq K(\eta)})
\end{aligned}$$

which completes the proof. □

Lemma 8 – *Let $t \geq 0$. For all $(\theta, x) \in [0,t]\times\mathbb{R}$,*

$$\lim_{N\to\infty}\left(\frac{\lceil N\theta\rceil}{N} - x\right)_+^{\widetilde{h}^N_{\lceil N\theta\rceil}-3/2} = \begin{cases} (\theta - x)_+^{\widetilde{h}(\theta)-3/2} & \text{if} \quad \beta = 1,\\ (\theta - x)_+^{\widetilde{h}(\infty)-3/2} & \text{if} \quad \beta < 1,\\ (\theta - x)_+^{\widetilde{h}(0)-3/2} & \text{if} \quad \beta > 1.\end{cases} \tag{68}$$

Moreover, for all $(N,\theta,x) \in \mathbb{N}^*\times[0,t]\times\mathbb{R}$,

$$\left(\frac{\lceil N\theta\rceil}{N} - x\right)_+^{\widetilde{h}^N_{\lceil N\theta\rceil}-3/2} \leq \begin{cases} (\theta - x)_+^{\sup_{[0,t]}\widetilde{h}-3/2} + (\theta - x)_+^{\inf_{[0,t]}\widetilde{h}-3/2} & \text{if} \quad \beta \geq 1,\\ (\theta - x)_+^{\sup_{[0,\infty)}\widetilde{h}-3/2} + (\theta - x)_+^{\inf_{[0,\infty)}\widetilde{h}-3/2} & \text{if} \quad \beta < 1.\end{cases} \tag{69}$$

Proof. For all θ and N,

$$\theta \leq \frac{\lceil N\theta\rceil}{N} < \theta + \frac{1}{N}.$$

Then (68) follows from the continuity of h. Moreover, for all x,

$$(\theta - x)_+ \leq \left(\frac{\lceil N\theta\rceil}{N} - x\right)_+.$$

Since $\widetilde{h}^N_{\lceil N\theta\rceil} - \frac{3}{2} < -\frac{1}{2} < 0$,

$$\left(\frac{\lceil N\theta\rceil}{N} - x\right)_+^{\widetilde{h}^N_{\lceil N\theta\rceil}-3/2} \leq (\theta - x)_+^{\widetilde{h}^N_{\lceil N\theta\rceil}-3/2}$$

and then we get (69). □

Lemma 9 – *Let $t \geq 0$ and F_t be the function such that for all $(\theta, x_1, \cdots, x_m) \in [0,t] \times \mathbb{R}^m$,*

$$F_t(\theta, x_1, \cdots, x_m) = \begin{cases} \prod_{k=1}^m ((\theta - x_k)_+^{\sup_{[0,t]} \widetilde{h}-3/2} + (\theta - x_k)_+^{\inf_{[0,t]} \widetilde{h}-3/2}) \text{ if } \beta \geq 1, \\ \prod_{k=1}^m ((\theta - x_k)_+^{\sup_{[0,\infty)} \widetilde{h}-3/2} + (\theta - x_k)_+^{\inf_{[0,\infty)} \widetilde{h}-3/2}) \text{ if } \beta < 1. \end{cases} \tag{70}$$

We have

$$\int_{\mathbb{R}^m} dx_1 \cdots dx_m \left(\int_0^t F_t(\theta, x_1, \cdots, x_m) d\theta\right)^2 < \infty. \tag{71}$$

Proof. It is a consequence of Lemma 2. □

Lemma 10 – *For all t, $\lim_{N\to\infty} \left\|\Theta_2^N(t) - \Theta_3^N(t)\right\| = 0$.*

Proof. Fix $\eta > 0$. By Lemma 7,

$$\begin{aligned}
&\sum_{n=1}^{\lfloor Nt\rfloor} \left| N^{-h_n^N + m/2} \prod_{k=1}^m a((n - Nx_k)_+, \widetilde{h}_n^N) - \frac{1}{N} A(\widetilde{h}_n^N)^m \prod_{k=1}^m \left(\frac{n}{N} - x_k\right)_+^{\widetilde{h}_n^N - 3/2} \right| \\
\leq & \frac{B}{N} \sum_{n=1}^{\lfloor Nt\rfloor} \prod_{k=1}^m \left(\frac{n}{N} - x_k\right)_+^{\widetilde{h}_n^N - 3/2} \sum_{j=1}^m (\eta 1_{n - Nx_j \geq K(\eta)} + C(\eta) 1_{n - Nx_j < K(\eta)}) \\
= & B \int_0^t d\theta \prod_{k=1}^m \left(\frac{\lceil N\theta\rceil}{N} - x_k\right)_+^{\widetilde{h}^N_{\lceil N\theta\rceil} - 3/2} \sum_{j=1}^m (\eta 1_{\lceil N\theta\rceil - Nx_j \geq K(\eta)} + C(\eta) 1_{\lceil N\theta\rceil - Nx_j < K(\eta)})
\end{aligned}$$

As a consequence,

$$\left\|\Theta_2^N(t) - \Theta_3^N(t)\right\|^2 \leq 2B^2 (\phi_m)^2 (\eta^2 U_1^N(t) + C(\eta)^2 U_2^N(t))$$

with

$$U_1^N(t) = \int_{\mathbb{R}^m} dx_1 \cdots dx_m \left(\int_0^t d\theta \sum_{j=1}^m 1_{\lceil N\theta\rceil/N - x_j \geq K(\eta)/N} \prod_{k=1}^m \left(\frac{\lceil N\theta\rceil}{N} - x_k\right)_+^{\widetilde{h}^N_{\lceil N\theta\rceil} - 3/2}\right)^2$$

and

$$U_2^N(t) = \int_{\mathbb{R}^m} dx_1 \cdots dx_m \left(\int_0^t d\theta \sum_{j=1}^m 1_{\lceil N\theta\rceil/N - x_j < K(\eta)/N} \prod_{k=1}^m \left(\frac{\lceil N\theta\rceil}{N} - x_k \right)_+^{\widetilde{h}^N_{\lceil N\theta\rceil} - 3/2} \right)^2$$

By Lemmas 8 and 9, and the bounded convergence theorem,

$$\lim_{N\to\infty} U_1^N(t) = \int_{\mathbb{R}^m} dx_1 \cdots dx_m \left(\int_0^t F_t(\theta, x_1 \cdots, x_m) d\theta \right)^2$$

and

$$\lim_{N\to\infty} U_2^N(t) = 0.$$

It follows that for all $\eta > 0$,

$$0 \le \limsup_{N\to\infty} \left\| \Theta_2^N(t) - \Theta_3^N(t) \right\|^2 \le 2\eta^2 (\phi_m)^2 \int_{\mathbb{R}^m} dx_1 \cdots dx_m \left(\int_0^t F_t(\theta, x_1 \cdots, x_m) d\theta \right)^2.$$

This gives

$$\lim_{N\to\infty} \left\| \Theta_2^N(t) - \Theta_3^N(t) \right\|^2 = 0$$

and completes the proof. □

Lemma 11 – *For all* t, $\lim_{N\to\infty} \left\| \Theta_3^N(t) - \Sigma_{m,h}(t) \right\| = 0$.

Proof. For all t,

$$\left\| \Theta_3^N(t) - \Sigma_{m,h}(t) \right\| \le \int_{\mathbb{R}^m} dx_1 \cdots dx_m \left(\int_0^t \left| G^N(\theta, x_1, \cdots, x_m) - G(\theta, x_1, \cdots, x_m) \right| d\theta \right)^2. \tag{72}$$

By (68), for almost every $(\theta, x_1, \cdots, x_m)$,

$$\lim_{N\to\infty} |G^N(\theta, x_1, \cdots, x_m) - G(\theta, x_1, \cdots, x_m)| = 0.$$

Moreover, by (69), there exists $C > 0$ such that for almost every $(\theta, x_1, \cdots, x_m)$,

$$|G^N(\theta, x_1, \cdots, x_m) - G(\theta, x_1, \cdots, x_m)| \le C F_t(\theta, x_1, \cdots, x_m)$$

where F_t is defined by (70). By Lemma 9 and the bounded convergence theorem, we deduce that

$$\lim_{N\to\infty} \int_{\mathbb{R}^m} dx_1 \cdots dx_m \left(\int_0^t |G^N(\theta, x_1, \cdots, x_m) - G(\theta, x_1, \cdots, x_m)| d\theta \right)^2 = 0,$$

which completes the proof with (72). □

Acknowledgments

The author thanks the anonymous referee for helpful comments.

References

Benassi, A., S. Jaffard, and D. Roux (1997). “Elliptic Gaussian random processes”. *Rev. Mat. Iberoamericana* **13**, pp. 19–90 (cit. on p. 33).

Billingsley, P. (1968). *Convergence of Probability Measures*. Wiley (cit. on pp. 45, 47).

Cheridito, P., H. Kawaguchi, and M. Maejima (2003). “Fractional Ornstein-Uhlenbeck processes”. *Electron. J. Probab.* **8** (cit. on p. 38).

Cohen, S. and R. Marty (2008). “Invariance principle, multifractional Gaussian processes and long-range dependence”. *Ann. Inst. H. Poincaré Probab. Statist.* **44**, pp. 475–489 (cit. on pp. 32, 33, 35, 46).

Dobrushin, R. L. and P. Major (1979). “Non-central limit theorems for non-linear functionals of Gaussian fields”. *Z. Wahrsch. Verw. Gebiete* **50**, pp. 27–52 (cit. on pp. 32, 42).

Itô, K. (1951). “Multiple Wiener integral”. *J. Math. Soc. Japan* **3**, pp. 157–169 (cit. on p. 42).

Marty, R. (2013). “From Hermite polynomials to multifractional processes”. *J. Appl. Probab.* **50**, pp. 323–343 (cit. on pp. 32–35, 46, 47, 49).

Nelson, E. (1973). “The free Markoff field”. *J. Funct. Anal.* **12**, pp. 211–227 (cit. on p. 45).

Peltier, R. F. and J. Lévy Véhel (1995). *Multifractional Brownian motion: definition and preliminary results*. Research Report RR-2645. INRIA (cit. on pp. 33, 45).

Taqqu, M. S. (1979). “Convergence of integrated processes of arbitrary Hermite rank”. *Z. Wahrsch. Verw. Gebiete* **50**, pp. 53–83 (cit. on pp. 32, 37, 42).

Contents