North-Western European Journal of Mathematics

Stability of Jamison sequences under certain perturbations

João Paulos[1]

Received: September 14, 2018/Accepted: May 27, 2019/Online: July 24, 2019

Abstract

An increasing sequence of positive integers (n_k) is said to be Jamison if whenever T is a linear bounded operator on a complex separable Banach space, the following holds:

$$\sup_k \|T^{n_k}\| < \infty \Rightarrow \sigma_p(T) \cap S^1 \text{ is countable}$$

In this paper, we study certain perturbations on the set of Jamison sequences and prove a stability result.

Keywords: Jamison sequences, linear dynamics..

MSC: 47A10, 54A99.

1 Introduction

Throughout this paper, unless otherwise stated, X is a complex separable Banach space and $\mathcal{L}(X)$ is the Banach algebra of bounded linear operators on X. Furthemore, we denote the unit circle in $\mathbb{C}$, as $S^1 = \{z \in \mathbb{C} : |z| = 1\}$.

Definition 1 – Let (n_k) be an increasing sequence of positive integers and let $T \in \mathcal{L}(X)$. We say that T is partially power bounded (with respect to (n_k)) if:

$$\sup_k \|T^{n_k}\| < \infty$$

In particular, if T is partially power bounded with respect to $(n_k) = (k)$, then T is said to be power bounded.

Definition 2 – Let (n_k) be an increasing sequence of positive integers. We say that (n_k) is a Jamison sequence if whenever an operator $T \in \mathcal{L}(X)$ is partially power bounded with respect to (n_k), then $\sigma_p(T) \cap S^1$ is countable. Here, $\sigma_p(T)$ denotes the point spectrum of T, i.e $\sigma_p(T) = \{\lambda \in \mathbb{C} : \ker(T - \lambda) \neq \{0\}\}$.

[1]Chalmers University of Technology, Department of Mathematical Sciences, Chalmers Tvärgata 3, 412 96 Göteborg

Remark 1 – If a sequence (n_k) is not a Jamison sequence, we say that (n_k) is a non-Jamison sequence.

Notation 1 – Let (t_k) and (n_k) be two increasing sequences of positive integers. We define another sequence $(r_k)_{\frac{t_k}{n_k}}$, as follows ($[\cdot]$ *denotes the closest integer)*:

$$r_k := \left| t_k - \left[\frac{t_k}{n_k} \right] n_k \right|$$

In this paper we prove that the set of Jamison sequences is stable under certain perturbations. More concretely, our main result in its full generality is as follows (Corollary 4):

Theorem 1 – *Let (t_k) be a (non) Jamison sequence and suppose that (n_k) is an increasing sequence of positive integers. Then, if one of the following conditions hold, (n_k) is also a (non) Jamison sequence:*

1. $\sup_k\left(\frac{t_k}{n_k}\right) < \infty$ *and* $\sup_k (r_k)_{\frac{t_k}{n_k}} < \infty$

2. $\sup_k\left(\frac{n_k}{t_k}\right) < \infty$ *and* $\sup_k (r_k)_{\frac{n_k}{t_k}} < \infty$

The proof of our main result relies on a characterization of Jamison sequences proved in Badea and Grivaux (2007). This characterization has as a central tool, the introduction of a metric $d_{(n_k)}$ in S^1 associated to each increasing sequence of positive integers (n_k).

In section 2, we present a brief overview of previous results on Jamison sequences. Some of these results will be used further in this paper and others should illustrate the relation between the size of $\sigma_p(T) \cap S^1$ and the growth rate of the sequence $(\|T^{n_k}\|)$.

In section 3, we prove that $(S^1, d_{(n_k)})$ is never compact (Corollary 1) and after introducing all the remaining necessary results, we provide a short list of examples.

Finally, in section 4, we prove our main result.

2 Overview

In this section, we present a brief overview of previous results on Jamison sequences. The starting point is the following result by Jamison, which we slightly reformulate as follows[2]:

Theorem 2 – *The sequence $(n_k) = (k)$ is a Jamison sequence.*

[2] Jamison, 1965, "Eigenvalues of modulus 1".

Proof. We start by introducing an equivalence relation in S^1: for $z, w \in S^1$, $z \sim w$ if and only if there are integers i, j such that $z^i w^j = 1$. It is a well known fact that if z and w are not equivalent, then the subset $\{(z^n, w^n) : n \in \mathbb{N}\}$ is dense in $S^1 \times S^1$[3]. This fact together with the following lemma, implies our result.

Lemma 1 – *Let $T \in \mathcal{L}(X)$ be such that $\sup_k \|T^k\| = M < \infty$ and suppose that $\lambda_1, \lambda_2 \in \sigma_p(T) \cap S^1$ are not equivalent, with norm one eigenvectors x_1 and x_2, respectively. Then:*

$$\|x_1 - x_2\| \geq \frac{2}{M+1}$$

Proof. Since λ_1 and λ_2 are not equivalent, we can pick a sequence (n_k) such that $(\lambda_1^{n_k}, \lambda_2^{n_k}) \to (-1, 1)$. By triangular inequality, one has that:

$$\|x_1 - \lambda_1^{n_k} x_1\| \leq \|x_1 - x_2\| + \|x_2 - \lambda_2^{n_k} x_2\| + \|\lambda_2^{n_k} x_2 - \lambda_1^{n_k} x_1\|$$

The left hand side converges to $\|2x_1\| = 2$ and, on the other hand one has that $\|x_2 - \lambda_2^{n_k} x_2\| \to 0$ and that $\|\lambda_2^{n_k} x_2 - \lambda_1^{n_k} x_1\| \leq M\|x_1 - x_2\|$. □

Now, suppose that $\sup_k \|T^k\| < \infty$ and that, by contradiction, $\sigma_p(T) \cap S^1$ is uncountable. By Lemma 1, there is an uncountable and mutually disjoint collection of open balls $\{B(x, \frac{1}{M+1})\}$, which contradicts the separability of X. □

Remark 2 – The previous result does not hold if one drops the separability condition on X. Indeed, consider $X = l^2(I)$ for some uncountable index set I and let $T \in \mathcal{L}(X)$ be such that $T(x_i) := (\lambda_i x_i)$ for any uncountable subset $\{\lambda_i\}_{i \in I} \subset S^1$. It is clear that $\|T^n\| = 1$ however, by definition, $\sigma_p(T) \cap S^1$ is uncountable.

It follows from Theorem 2 that if $\sigma_p(T) \cap S^1$ is uncountable, then one has that $\sup_k \|T^k\| = \infty$. In this context, the following question is quite natural:

Let $\sigma_p(T) \cap S^1$ be uncountable. Does this imply that $\lim_k \|T^k\| = \infty$?

Under some extra hypothesis, the answer is positive. Indeed, in the following two theorems, we relate measure theoretic, topological and set theoretic properties of $\sigma_p(T) \cap S^1$ with the growth of $\{\|T^k\|\}$. A proof of these results was given in Ransford (2005).

Theorem 3 – *Let $T \in \mathcal{L}(X)$. Suppose that, either $\sigma_p(T) \cap S^1$ has positive Lebesgue measure or that $\sigma_p(T) \cap S^1$ is of second Baire category in S^1. Then, one has that $\lim_k \|T^k\| = \infty$.*

[3] Pontrjagin, 1939, *Topological Groups*, p. 150.

Theorem 4 – *Let $T \in \mathcal{L}(X)$ and suppose that $\sigma_p(T) \cap S^1$ is uncountable. Then, there is a subset $Z \subset \mathbb{N}$ of density zero such that*

$$\lim_{k \notin Z} \|T^k\| = \infty$$

However, it turns out that the answer to our initial question (in its full generality) is negative. This was established in Ransford and Roginskaya (2006):

Theorem 5 – *Let $B > 1$ and (γ_k) be a sequence such that $\gamma_k \to \infty$. Then, there exists a separable Banach space X, an operator $T \in \mathcal{L}(X)$ and an increasing sequence of positive integers (n_k) such that $\lim_k \|T^{n_k}\| \leq B$, $\frac{n_{k+1}}{n_k} \leq \gamma_k$ for sufficiently large k and yet, $\sigma_p(T) \cap S^1$ is uncountable.*

The next two lemmas are presented for further reference in this paper:

Lemma 2 – *Let X be any Banach space, not necessarily separable and $T \in \mathcal{L}(X)$. Let $\lambda_1, \lambda_2 \in \sigma_p(T)$ and suppose that x_1 and x_2 are norm one eigenvectors, corresponding respectively to λ_1 and λ_2. Then, for any $n \geq 1$ one has that:*

$$|\lambda_1^n - \lambda_2^n| \leq 2\|T^n\|\,\|x_1 - x_2\|$$

Lemma 3 – *Let X be a separable Banach space and let $T \in \mathcal{L}(X)$. Suppose that (n_k) is an increasing sequence of positive integers such that $\sup_k \|T^{n_k}\| < \infty$. Then, given $\epsilon > 0$, there is a countable subset $\{\mu_n\} \subset S^1$ such that:*

$$\sigma_p(T) \cap S^1 \subset \bigcup_l \mu_l E$$

where

$$E = \bigcap_k \{\lambda \in S^1 : |\lambda^{n_k} - 1| \leq \epsilon\}$$

A proof of the latter result was given in Ransford and Roginskaya (2006). For the sake of completeness, we finish this section with two more results that were proven in Ransford and Roginskaya (2006).

Lemma 4 – *Let (n_k) be an increasing sequence of positive integers and let $\epsilon \in (0,1)$. For each $k \geq 1$, define:*

$$E_k = \bigcap_{j=1}^{k} \{\lambda \in S^1 : |\lambda^{n_j} - 1| < \epsilon\}$$

Then, E_k is the union of N_k disjoint arcs, each of length at most $\frac{\pi\epsilon}{n_k}$ and moreover, $N_k \leq n_1 \prod_{j=2}^{k} \left\lfloor 1 + \epsilon \frac{n_j}{n_{j+1}} \right\rfloor$.

Theorem 6 – *Let (n_k) be an increasing sequence of positive integers and let $T \in \mathcal{L}(X)$ such that $\sup_k \|T^{n_k}\| < \infty$. If the sequence $\{\frac{n_{k+1}}{n_k}\}$ is bounded, then $\sigma_p(T) \cap S^1$ is countable.*

Proof. Since $\{\frac{n_{k+1}}{n_k}\}$ is bounded, one can choose some $\epsilon > 1$ such that for all k, $\frac{\epsilon n_{k+1}}{n_k} < 1$. By Lemma 4 it follows that E_k is the union of N_k arcs of length at most $\frac{\pi\epsilon}{n_k}$, with

$$N_k \leq n_1 \prod_{j=2}^{k} \left\lfloor 1 + \epsilon \frac{n_j}{n_{j+1}} \right\rfloor$$

It follows that E has at most n_1 points and thus, by Lemma 3, $\sigma_p(T) \cap S^1$ is countable. □

The previous result shows that there is a relation between the growth of (n_k) and the size of $\sigma_p(T) \cap S^1$, as a set. It is precisely this interplay that is explored in the next two sections and that motivated our main result.

3 Characterization of Jamison sequences

Henceforth, and without loss of generality, we will consider increasing sequences of positive integers (n_k) such that $n_1 = 1$.

To any such sequence, one can define a metric on S^1 by

$$d_{(n_k)}(\lambda, \mu) := \sup_k |\lambda^{n_k} - \mu^{n_k}| \qquad (\lambda, \mu \in S^1)$$

Note that it is always the case that $d \leq d_{(n_k)}$, where d is the usual Euclidean distance. Hence, $\mathcal{T} \subset T^{(n_k)}$, if $\mathcal{T}$ is the usual topology on S^1 inherited as a subspace of $\mathbb{R}^2$ and if $\mathcal{T}^{(n_k)}$ is the topology on S^1 induced by $d_{(n_k)}$. A natural question one may ask is if there is some sequence (n_k) such that $\mathcal{T} = \mathcal{T}^{(n_k)}$.

Theorem 7 – *There is no sequence (n_k) such that $\mathcal{T} = \mathcal{T}^{(n_k)}$.*

Proof. It is enough to prove that $1_{S^1} : (S^1, \mathcal{T}) \to (S^1, \mathcal{T}^{(n_k)})$ is not continuous, where 1_{S^1} is the identity map on S^1. Let $p = (0,1) \in S^1$ and let $\epsilon > 0$ be small enough so that if $y \in B_{d_{(n_k)}}(p, \epsilon)$, then $y = e^{i\tau}$ is such that $|\tau| < \frac{\pi}{10}$. Suppose that there is some $\delta > 0$ such that $B_d(p, \delta) \subseteq B_{d_{(n_k)}}(p, \epsilon)$ and pick some $z = e^{i\theta}$ with θ small enough so that $z \in B_d(p, \delta)$. Since (n_k) is increasing, choose some n_k such that $\frac{\pi}{2} > n_k\theta > \frac{\pi}{10}$. Then:

$$d_{(n_k)}(z, 1) \geq |z^{n_k} - 1| > \epsilon$$

Thus, such δ does not exist which proves that 1_{S^1} is not continuous. □

We conclude that, for any sequence (n_k), $\mathcal{T}$ is strictly coarser than $\mathcal{T}^{(n_k)}$. Another natural question one could ask is if there is any sequence (n_k) such that S^1, when endowed with $\mathcal{T}^{(n_k)}$, is compact.

In this context it is useful to recall that a compact and Hausdorff topology is a minimal element in the partial order of Hausdorff topologies (in a given set) with respect to inclusion.

Corollary 1 – *There is no sequence (n_k) such that $(S^1, \mathcal{T}^{(n_k)})$ is compact.*

Proof. Suppose that there is some sequence (n_k) such that $(S^1, \mathcal{T}^{(n_k)})$ is compact. Since $\mathcal{T} \subseteq \mathcal{T}^{(n_k)}$ this would imply that $\mathcal{T} = \mathcal{T}^{(n_k)}$ for some sequence (n_k), contradicting Theorem 7. □

The next result is of cornerstone importance. It was proven in Badea and Grivaux (2007) (Theorem 2.8) and it provides a characterization of Jamison sequences on which the proof of the main results will heavily rely.

Theorem 8 – *Let (n_k) be an increasing sequence of positive integers. The following are equivalent:*

1. *The sequence (n_k) is a Jamison sequence.*
2. *For every uncountable subset K of S^1, the metric space $(K, d_{(n_k)})$ is non separable.*
3. *For every uncountable subset K of S^1 there exists a positive ϵ such that K contains an uncountable ϵ-separated family for the distance $d_{(n_k)}$.*
4. *There exists a positive ϵ such that every uncountable subset K of S^1 contains an uncountable ϵ-separated family for the distance $d_{(n_k)}$.*
5. *There exists an $\epsilon > 0$ such that any two distinct points $\lambda, \mu \in S^1$ are ϵ-separated for the distance $d_{(n_k)}$.*

We now introduce some notation that will be used in what follows. For a fixed sequence (n_k) and $\epsilon > 0$ we define the following subset of S^1:

$$\Lambda_\epsilon^{(n_k)} := \left\{ \lambda \in S^1 : \sup_k |\lambda^{n_k} - 1| < \epsilon \right\}$$

As a consequence of Theorem 8, one can characterize a given increasing sequence of positive integers (n_k) in terms of the cardinality of the sets $\Lambda_\epsilon^{(n_k)}$. The following result appears in Badea and Grivaux (2007) (Corollary 2.11).

Theorem 9 – *Let (n_k) be an increasing sequence of positive integers. Then, (n_k) is a Jamison sequence if and only if there is some $\epsilon > 0$ such that $\Lambda_\epsilon^{(n_k)}$ is countable.*

Proof. Suppose that (n_k) is Jamison. By Theorem 8(5) there is some ϵ such that $\Lambda_\epsilon^{(n_k)} = \{1\}$. Conversely, suppose that there is some $\delta > 0$ such that $\Lambda_\delta^{(n_k)}$ is countable and let $T \in \mathcal{L}(X)$ be such that $\sup_k \|T^{n_k}\| < \infty$. Then, the set E in Lemma 3 (for instance for $\epsilon = \frac{\delta}{2}$) is countable and thus, $\sigma_p(T) \cap S^1$ is countable, from where it follows that (n_k) is a Jamison sequence. □

It follows from Theorem 9 that we have a relatively practical way to check if a sequence is Jamison or not. The following result will be heavily used in the rest of this paper. It is a slight reformulation of some remarks that appear in Badea and Grivaux (2007), after Corollary 2.11. We simply present it here in a more convenient way and include a proof for the sake of completeness.

Lemma 5 – *Let (n_k) be any increasing sequence of positive integers. Then, (n_k) is a non-Jamison sequence if and only if for each $\epsilon > 0$ one has that $\Lambda_\epsilon^{(n_k)} \neq \{1\}$.*

Proof. If there is some $\epsilon > 0$ such that $\Lambda_\epsilon^{(n_k)} = \{1\}$, then it follows immediately from Theorem 9 that (n_k) is a Jamison sequence. Conversely, suppose that each $\Lambda_\epsilon^{(n_k)}$ has at least two elements. If there is any $\delta > 0$ such that $\Lambda_\delta^{(n_k)}$ is countable, by Theorem 9 we have that (n_k) is a Jamison sequence. But this is impossible, since by Theorem 8(5) there is some $\epsilon > 0$ such that $\Lambda_\epsilon^{(n_k)}$ has only one element. □

In order to illustrate that there is a strong relation between the growth of a sequence (n_k) and whether or not (n_k) is Jamison, we provide some examples. Henceforth, (n_k) is an increasing sequence of positive integers:

(1) If $\sup_k\left(\frac{n_{k+1}}{n_k}\right) < \infty$, then (n_k) is Jamison. In particular, $(n_k) = (k)$ is a Jamison sequence. This was firstly proven in Ransford and Roginskaya (2006) (Theorem 1.5) and can be seen as a consequence of Theorem 8 (see example 2.3 in Badea and Grivaux 2007).

(2) If $\lim_n \frac{n_{k+1}}{n_k} = \infty$, then (n_k) is non Jamison (see example 2.13 in Badea and Grivaux 2007).

(3) If (n_k) contains blocks of arbitrary length, then (n_k) is Jamison (see example 2.5 in Badea and Grivaux 2007).

(4) If (n_k) is a set of positive upper density, then (n_k) is Jamison (see example 2.6 in Badea and Grivaux 2007). However, if (n_k) is of zero upper density, then (n_k) can be either Jamison ($(n_k) = (k^2)$, using (1)) or non Jamison ($(n_k) = (k!)$, using (2)).

(5) By Szemeredi's Theorem, any set of positive upper density contains arbitrarily long arithmetic progressions. Thus, in view of example (3) it is natural to ask whether a sequence (n_k) containing arbitrarily long arithmetic progressions is

Jamison or not. However, example 2.14 in Badea and Grivaux (2007) provides a counter-example.

(6) If (n_k) is a sequence such that $n_k|n_{k+1}$, then (n_k) is a Jamison sequence if and only if $\sup_k\left(\frac{n_{k+1}}{n_k}\right) < \infty$. This is Corollary 2.16 in Badea and Grivaux (2007).

4 Main Results

We start this section with the following observation:

Lemma 6 – *If* $\lambda \in \overline{\Lambda_\epsilon^{(n_k)}}$, *then* $\sup_k |\lambda^{n_k} - 1| \leq \epsilon$.

Furthermore, we have the following:

Lemma 7 – *Suppose* (n_k) *is a non Jamison sequence and let* $\epsilon > 0$. *Then,* $1 \in \overline{\Lambda_\epsilon^{(n_k)} \setminus \{1\}}$.

Proof. Since (n_k) is non Jamison, it follows by Lemma 5 that each $\Lambda_\epsilon^{(n_k)} \setminus \{1\}$ is non empty. Moreover, since $\Lambda_\epsilon^{(n_k)} \subseteq \Lambda_\delta^{(n_k)}$ for $\epsilon < \delta$, one has that $\{\overline{\Lambda_\epsilon^{(n_k)} \setminus \{1\}}\}_{\epsilon>0}$ is a family of closed non empty subsets of S^1 with the finite intersection property. It follows by compactness of S^1 that there is some $z \in \bigcap_{\epsilon>0} \overline{\Lambda_\epsilon^{(n_k)} \setminus \{1\}}$. Thus, by Lemma 6 one has that for each $\epsilon > 0$, $|z-1| \leq \epsilon$. Hence, $z = 1 \in \bigcap_{\epsilon>0} \overline{\Lambda_\epsilon^{(n_k)} \setminus \{1\}}$. □

Theorem 10 – *Let* (n_k) *be a non Jamison sequence and suppose that* (t_k) *is an increasing sequence of positive integers such that* $\sup_k |t_k - n_k| < \infty$. *Then,* (t_k) *is non Jamison.*

Proof. Let $\sup_k |t_k - n_k| = M < \infty$ and fix some small $\epsilon > 0$. By Lemma 5 it is enough to prove that $\Lambda_\epsilon^{(t_k)} \neq \{1\}$. Since (n_k) is non Jamison, using Lemma 7 we can pick some $z = e^{i\theta}$ such that $z \neq 1$, $z \in \Lambda_{\frac{\epsilon}{3}}^{(n_k)}$ and with θ small enough so that $\frac{M\theta}{2\pi} < \frac{\epsilon}{3}$. Then, for any k:

$$|z^{t_k} - 1| \leq |z^{t_k} - z^{n_k}| + |z^{n_k} - 1| < \frac{2\epsilon}{3} < \epsilon$$

Hence, $\Lambda_\epsilon^{(t_k)} \neq \{1\}$ and thus, (t_k) is non Jamison. □

Corollary 2 – *Let* (n_k) *be a (non) Jamison sequence and suppose that* (t_k) *is an increasing sequence of positive integers such that* $\sup_k |t_k - n_k| < \infty$. *Then,* (t_k) *is (non) Jamison.*

Let $\mathcal{I} \subset \mathbb{N}^{\mathbb{N}}$ be the set of increasing sequences of positive integers. Let $\mathcal{G}$ be the topology on $\mathcal{I}$ generated by the subbasis consisting of subsets of the form $\mathcal{U}(x) = \{y \in \mathcal{I} : \sup_k |x_k - y_k| \leq N\}$, for some $x \in \mathcal{I}$ and $N \in \mathbb{N}$. It follows from Corollary 2 that the subsets of Jamison and non Jamison sequences are open. Moreover, we have seen that these subsets are non empty and thus we can conclude the following:

Corollary 3 – $(\mathcal{I}, \mathcal{G})$ *is a disconnected topological space.*

In what follows, it is useful to note the following remarks:

Remark 3 – Suppose that (n_k) is non Jamison and that (t_k) is any subsequence. Then, (t_k) is also an increasing sequence of positive integers and clearly, $\Lambda_\epsilon^{(n_k)} \subseteq \Lambda_\epsilon^{(t_k)}$. Hence, by Lemma 5 it follows that (t_k) is also non Jamison.

Remark 4 – Let $\lambda \in S^1$. Then:

$$|\lambda^n - 1| = |\lambda(\lambda^{n-1} - 1) + \lambda - 1| \le |\lambda^{n-1} - 1| + |\lambda - 1|$$

It follows by induction that $|\lambda^n - 1| \le n|\lambda - 1|$.

Remark 5 – Let (n_k) be a (non) Jamison sequence. If (t_k) is an increasing sequence of positive integers such that $\{k : t_k \ne n_k\}$ is finite, then (t_k) is also (non) Jamison.

Lemma 8 – *Let c be any positive integer. Then, (cn_k) is non Jamison if and only if (n_k) is non Jamison.*

Proof. Suppose that (cn_k) is non Jamison and fix some $\epsilon > 0$. We prove that $\Lambda_\epsilon^{(n_k)} \ne \{1\}$, from where it follows that (n_k) is non Jamison by Lemma 5. By Lemma 7, there is some sequence $(\lambda_n) \subseteq \Lambda_\epsilon^{(cn_k)} \setminus \{1\}$ such that $\lambda \to 1$. We pick any element from this sequence, say $\lambda_k := \lambda$. Then, $\lambda \ne 1$ and $\sup_k |\lambda^{cn_k} - 1| < \epsilon$. If $\lambda^c \ne 1$, then $\lambda^c \in \Lambda_\epsilon^{(n_k)} \setminus \{1\}$ and we are done. Otherwise, suppose that $\lambda^c = 1$ and note that any other cth-root of unity μ is such that $|\lambda - \mu| \ge \frac{2\pi}{c}$. Since (λ_n) is a Cauchy sequence, let $\mu = \lambda_m$ be such that $|\mu - \lambda| < \frac{2\pi}{c}$. It is clear that $\mu^c \ne 1$ and since $\mu \in \Lambda_\epsilon^{(cn_k)}$, it follows that $\mu^c \in \Lambda_\epsilon^{(n_k)} \setminus \{1\}$.

Conversely, let (cn_k) be a Jamison sequence and suppose that for some operator $T \in \mathcal{L}(X)$ one has that $\sup_k \|T^{n_k}\| = M < \infty$. For any k:

$$\|T^{cn_k}\| \le c\|T^{n_k}\| \le cM < \infty$$

Hence, $\sup_k \|T^{cn_k}\| < \infty$. By assumption, (cn_k) is Jamison and thus $\sigma_p(T) \cap S^1$ is countable. □

The next result can be seen as a generalization of Lemma 8. We consider two increasing sequences of positive integers, (n_k) and (t_k) such that there is a sequence of positive integers (a_k) such that for each k, $t_k = a_k n_k$. Lemma 8 was simply the case with (a_k) a constant sequence.

Lemma 9 – *Let (t_k) and (n_k) be increasing sequences of positive integers such that $(t_k) = (a_k n_k)$, for some sequence (a_k) of positive integers such that $\sup_k(a_k) = A < \infty$. Then, (t_k) is non Jamison if and only if (n_k) is non Jamison.*

Proof. Suppose that (t_k) is non Jamison. Since $\sup_k(a_k) = A$ one has that $a_k \in \{1,\dots,A\}$. Without loss of generality, and appealing to Remark 5 we can assume that for each $j \leq A$ the subset $I_j := \{k \in \mathbb{N} : a_k = j\}$ is either infinite or empty. Moreover, we will see that it is enough to consider the case when each $I_j \neq \emptyset$. Fix any $\epsilon > 0$ and let $P := \prod_{i=1}^{A} i < \infty$.

Since (t_k) is non Jamison, there is some $\lambda \in \Lambda^{(t_k)}_{\frac{\epsilon}{P}}$ such that $\lambda \neq 1$. Note that by Remark 3, for each $j \leq A$ one has that $\sup_{k \in I_j} |\lambda^{j n_k} - 1| < \frac{\epsilon}{P}$. Our strategy is to use Lemma 5 to prove that $(P n_k)$ is non Jamison and then, by Lemma 8 it follows that (n_k) is also non Jamison. We can assume that for any k there is some $j \leq A$ such that $k \in I_j$ and using Remark 4.2 it follows that:

$$|\lambda^{P n_k} - 1| \leq \left(\prod_{i \neq j}^{A} i\right) |\lambda^{j n_k} - 1| < \left(\prod_{i \neq j}^{A} i\right) \frac{\epsilon}{P} = \frac{\epsilon}{j} < \epsilon$$

Conversely, suppose that (t_k) is Jamison. Suppose that $T \in \mathcal{L}(X)$ is such that $\sup_k \|T^{n_k}\| = M < \infty$. Then:

$$\|T^{t_k}\| = \|T^{a_k n_k}\| \leq a_k \|T^{n_k}\| \leq AM < \infty$$

By assmption, (t_k) is Jamison and thus, $\sigma_p(T) \cap S^1$ is countable. □

Before proving the main result, it is convenient to recall the notation that was previously introduced: Given two increasing sequences of positive integers (t_k) and (n_k), we define another sequence $(r_k)_{\frac{t_k}{n_k}}$ to be such that:

$$r_k := \left| t_k - \left[\frac{t_k}{n_k}\right] n_k \right|$$

Here, $[\cdot]$ denotes the closest integer function.

Theorem 11 – *Let (t_k) be non Jamison and suppose that (n_k) is an increasing sequence of positive integers. Then, if one of the following conditions hold, (n_k) is also non Jamison:*

1. $\sup_k\left(\frac{t_k}{n_k}\right) < \infty$ *and* $\sup_k (r_k)_{\frac{t_k}{n_k}} < \infty$
2. $\sup_k\left(\frac{n_k}{t_k}\right) < \infty$ *and* $\sup_k (r_k)_{\frac{n_k}{t_k}} < \infty$

Proof. Suppose that 1. holds. Since $\sup_k(r_k) < \infty$, it follows by Theorem 10 that $[\frac{t_k}{n_k}]n_k$ is non Jamison. Furthermore, since $\sup_k\left(\frac{t_k}{n_k}\right) < \infty$ it follows by Lemma 9 that (n_k) is non Jamison.

Now, suppose that 2. holds. Note that $n_k = \left[\frac{n_k}{t_k}\right] t_k \pm r_k$ for all k. Since $\sup_k\left(\frac{n_k}{t_k}\right) < \infty$, it follows by Lemma 9 that $\left(\left[\frac{n_k}{t_k}\right] t_k\right)$ is non Jamison and since $\sup_k(r_k) < \infty$, it follows by Theorem 10 that (n_k) is non Jamison. □

Corollary 4 – *Let (t_k) be (non) Jamison and suppose that (n_k) is an increasing sequence of positive integers. Then, if one of the following conditions hold, (n_k) is also (non) Jamison:*

1. $\sup_k\left(\frac{t_k}{n_k}\right) < \infty$ *and* $\sup_k (r_k)_{\frac{t_k}{n_k}} < \infty$
2. $\sup_k\left(\frac{n_k}{t_k}\right) < \infty$ *and* $\sup_k (r_k)_{\frac{n_k}{t_k}} < \infty$

References

Badea, C. and S. Grivaux (2007). "Size of the peripheral point spectrum under power or resolvent growth conditions". *J. Funct. Anal. 246*, pp. 302–329 (cit. on pp. 88, 92–94).

Jamison, B. (1965). "Eigenvalues of modulus 1". *Proc. Amer. Math. Soc. 16*, pp. 375–377 (cit. on p. 88).

Pontrjagin, L. (1939). *Topological Groups* (cit. on p. 89).

Ransford, T. (2005). "Eigenvalues and power growth". *Israel J. Math. 146*, pp. 93–110 (cit. on p. 89).

Ransford, T. and M. Roginskaya (2006). "Point spectra of partially power-bounded operators". *J. Funct. Anal. 230*, pp. 432–445 (cit. on pp. 90, 93).

Contents