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Infinite order differential equations in the space of entire functions of normal type with respect to a proximate order

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Abstract

In this paper, we consider the space of entire functions with normal type growth for a given proximate order and a continuous linear operator from such space into itself which is defined by a partial differential operator of infinite order. We will study corresponding partial differential equations with variable coefficients in the cases of regular singular type and of Korobeĭnik type.

Keywords: entire functions in several variables, partial differential operator of infinite order.

MSC: 30D15, 35R50, 32A15, 32W50.

1 Introduction

A linear operator acting continuously and locally on the holomorphic functions is represented as a partial differential operator of infinite order, more precisely a *continuous* morphism of the sheaf of holomorphic functions on a complex domain or complex manifold is characterized by a local partial differential operator of infinite order with the symbol being of minimal type with respect to the differential variable[3], so in the complex framework, it is quite natural to consider the infinite order PDE. Commenced first by Bernard[4] and then by André[5], many people studied the infinite order PDE with constant coefficients and its generalization, the convolution

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[3]Ishimura, 1978, "Homomorphismes du faisceau des germes de fonctions holomorphes dans lui-même et opérateurs différentiels";

Ishimura, 1980, "Homomorphismes du faisceau des germes de fonctions holomorphes dans lui-même et opérateurs différentiels II".

[4]Malgrange, 1956, "Existence et approximation des solutions des équations aux dérivées partielles et des équations de convolution".

[5]Martineau, 1967, "Équations différentielles d'ordre infini".

equation in each situation (for example Sébbar 1971, Morzhakov 1974, Berenstein and Struppa 1987, Ishimura and Okada 1994, Abanin, Ishimura, and Le Hai Khoi 2012). Nevertheless there are not so many results concerning the variable coefficient PDE of infinite order (see Korobeinik 1959, Korobeinik 1962, Steen 1971, Ishimura 1985, Aoki 1988).

In the preceding paper Ishimura (2007), we proved that any continuous linear operator from the space of entire functions of normal type with respect to a proximate order in itself is represented by a partial differential operator of infinite order. Recently Aoki, T., Ishimura, R., Okada, Y., Uchida, S., and Struppa, D. C. gave the complete characterization of such operator by partial differential operator as a growth condition of its symbol (arXiv:1805.00663). There we characterized also the continuous linear operator of the space of minimal type entire functions in itself as partial differential operator of infinite order.

In the present paper, we will consider infinite order partial differential equations in the space of entire functions of normal type with respect to a proximate order. We will give solvability conditions for the cases of differential equations of regular singular type and for the higher order case, of Korobeĭnik type.

2 Notations and recall

In this paper, we employ same notations as the preceding paper Ishimura (2007): for a point $z = (z_1, z_2, \dots, z_n) \in \mathbb{C}^n$ and setting $\mathbb{N} := \{0, 1, 2, 3, \dots\}$, for multi-indices $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ and $\beta = (\beta_1, \beta_2, \dots, \beta_n) \in \mathbb{N}^n$, we set:

$$|z| := \sqrt{|z_1|^2 + |z_2|^2 + \cdots + |z_n|^2}, \qquad |\alpha| := \alpha_1 + \alpha_2 + \cdots + \alpha_n,$$
$$|\vec{z}| := \big(|z_1|, |z_2|, \dots, |z_n|\big), \qquad \alpha! := \alpha_1!\alpha_2!\cdots\alpha_n!,$$
$$D^\alpha = D_z^\alpha := \frac{\partial^{|\alpha|}}{\partial z_1^{\alpha_1} \partial z_2^{\alpha_1} \cdots \partial z_n^{\alpha_n}}, \qquad \binom{\alpha}{\beta} := \frac{\alpha!}{(\alpha-\beta)!\beta!}.$$

In the sequel, we will sometimes denote $|\vec{z}| \leqslant r$ instead of $|\vec{z}| \leqslant (r, \dots, r)$.

We recall several notions and properties concerning the spaces of entire functions with growth order: principally, we will follow Lelong and Gruman[6]. Let $w(z)$ be any real-valued continuous function defined on $\mathbb{C}^n$ and consider the space of entire functions with growth with respect to w:

$$B_w := \left\{ f \in \mathcal{O}(\mathbb{C}^n) \,\middle|\, \|f\|_w := \sup_{z\in\mathbb{C}} \left| f(z) e^{-w(z)} \right| < +\infty \right\}.$$

[6] Lelong and Gruman, 1986, *Entire functions of several complex variables.*

This is a Banach space with the norm $\|\cdot\|_w$. In this article, we will make use of the notion of proximate order introduced by Valiron (1923): a differentiable function $\rho(r) \geqslant 0$ defined for $r \geqslant 0$ is said to be a *proximate order* if it satisfies

(i) $\lim_{r\to+\infty} \rho(r) = \rho$,

(ii) $\lim_{r\to+\infty} \rho'(r) r \ln r = 0$.

It is well-known that in the case where $\rho > 0$, there exists $r_0 > 1$ such that for $r > r_0$, the function $r^{\rho(r)}$ is strictly increasing (tending to $+\infty$), and if $\rho > 1$, $r^{\rho(r)-1}$ is also strictly increasing (tending to $+\infty$). We have

$$\frac{d}{dr} r^{\rho(r)} = r^{\rho(r)-1}\Big(\rho'(r) r \ln r + \rho(r)\Big).$$

We fix a differentiable function $r = \varphi(t)$ defined for $t \geqslant 0$ being the inverse function of $t = r^{\rho(r)}$ for large t and for any $q \in \mathbb{N}$, we set

$$A_q := A_{\rho,q} := \left(\frac{\varphi(q)^\rho}{\mathrm{e}\rho}\right)^{\frac{q}{\rho}}. \tag{1}$$

As in Ishimura and Miyake (2007), setting $w_\sigma(z) := \sigma |z|^{\rho(|z|)}$ for $\sigma \geqslant 0$, we define the locally convex space of entire functions of type at most $\sigma \geqslant 0$ with respect to a proximate order $\rho(r)$

$$E_\sigma^{\rho(r)} := \varprojlim_{\varepsilon>0} B_{w_{\sigma+\varepsilon}}$$

which is (by taking a decreasing sequence (ε_j) tending to 0 instead of all $\varepsilon > 0$) a (FS)-space as the following asserts:

Lemma 1 – *If $s_2 > s_1 > 0$, the inclusion map $B_{w_{s_1}} \hookrightarrow B_{w_{s_2}}$ is compact.*

Lemma is proven for example, as same as Martineau (1967, Lemma 1).

3 The space $\mathcal{E}^{\rho(r)}$

In this article, we consider the space of entire functions of normal type with respect to the proximate order $\rho(r)$:

Definition 1 –

$$\mathcal{E}^{\rho(r)} := \varinjlim_{\sigma>0} B_{w_\sigma}$$

which is a (DFS)-space.

For any function $f = \sum_{\alpha\in\mathbb{N}^n} f_\alpha z^\alpha \in \mathcal{O}(\mathbb{C}^n)$, we set

$$H_q(f) = H_q := \sup_{|z|\leqslant 1}\left|P_q(z)\right| \quad \text{with} \quad P_q(z) := \sum_{|\alpha|=q} f_\alpha z^\alpha.$$

We recall Lelong and Gruman (1986, Theorem 1.23):

Theorem 1 – *Let f be an entire function of finite order $\rho > 0$ and of proximate order $\rho(r)$. Then its type σ with respect to $\rho(r)$ is given by*

$$\frac{1}{\rho}\ln\sigma = \limsup_{q\to+\infty}\left(\frac{1}{q}\ln H_q + \ln\varphi(q)\right) - \frac{1}{\rho} - \frac{\ln\rho}{\rho}.$$

Using the Theorem 1, we have:

Lemma 2 – *An entire function $f(z) = \sum_\alpha f_\alpha z^\alpha$ belongs to $E_\sigma^{\rho(r)}$ if and only if*

$$\limsup_{q\to\infty}\left(H_q A_{\rho,q}\right)^{\frac{\rho}{q}} \leqslant \sigma. \tag{2}$$

Proof. In fact, $f \in E_\sigma^{\rho(r)}$ is equivalent to saying for any $\varepsilon > 0$, there exists $D_\varepsilon > 0$ such that we have

$$|f(z)| \leqslant D_\varepsilon \cdot \mathrm{e}^{w_{\sigma+\varepsilon}(z)} = D_\varepsilon \cdot \mathrm{e}^{(\sigma+\varepsilon)|z|^{\rho(|z|)}}$$

for all $z \in \mathbb{C}^n$, that is, setting $M_f(r) := \sup_{|z|\leqslant r}\ln|f(z)|$, we have

$$\limsup_{r\to+\infty}\frac{M_f(r)}{r^{\rho(r)}} \leqslant \limsup_{r\to+\infty}\frac{\ln D_\varepsilon + (\sigma+\varepsilon)r^{\rho(r)}}{r^{\rho(r)}} = \sigma.$$

In view of Theorem 1, this is also equivalent to say that we have

$$\limsup_{q\to+\infty}\left(\frac{\rho}{q}\ln H_q + \rho\ln\varphi(q)\right) - \ln(\mathrm{e}\rho) \leqslant \ln\sigma :$$

this means (2). □

Corollary 1 – *In the hypothesis of Lemma 2, we have:*

$$\limsup_{q\to\infty}\left(\max_{|\alpha|=q}|f_\alpha|A_{\rho,q}\right)^{\frac{\rho}{q}} \leqslant \sqrt{n}^{\rho}\sigma \tag{3}$$

Conversely if we assume this inequality, we have $f(z) \in E^{\rho(r)}_{\sqrt{n}^\rho\sigma}$.

Proof. By using Cauchy's inequality, we have

$$|f_\alpha| = \frac{1}{\alpha!}\left|\partial^\alpha P_q(0)\right| \leqslant r^{-q}\sup_{|\vec{z}|\leqslant r}\left|P_q(z)\right| = \sup_{|\vec{z}|\leqslant 1}\left|P_q(z)\right|,$$

here we denoted $(r,\ldots,r)$ simply by r. As $|\vec{z}| \leqslant r$ implies $|z| \leqslant \sqrt{n}\, r$ for any $r > 0$, setting $v := z/\sqrt{n}$, we can continue

$$\sup_{|\vec{z}|\leqslant 1}\left|P_q(z)\right| \leqslant \sup_{|z|\leqslant\sqrt{n}}\left|P_q(z)\right| = \sup_{|v|\leqslant 1}\left|P_q\left(\sqrt{n}v\right)\right| = \sqrt{n}^q \sup_{|z|\leqslant 1}\left|P_q(z)\right| = \sqrt{n}^q H_q.$$

By the Lemma 2 on the preceding page, for any $\varepsilon > 0$, if $q = |\alpha| > 0$ is large enough, we have

$$\left(|f_\alpha| A_{\rho,q}\right)^{\frac{\rho}{q}} \leqslant \sqrt{n}^\rho\left(H_q A_{\rho,q}\right)^{\frac{\rho}{q}} < \sqrt{n}^\rho(\sigma+\varepsilon)$$

so we have (3).

Conversely if we have (3), for any $\varepsilon > 0$, there exists $C > 0$ such that for $|\alpha| = q \gg 1$, taking into account the fact

$$\mathrm{H}^n_q := \binom{n+q-1}{q} = \left(1+\frac{q}{n-1}\right)\left(1+\frac{q}{n-2}\right)\cdots\left(1+\frac{q}{1}\right) \leqslant (q+1)^{n-1},$$

we have

$$\begin{aligned} H_q = \sup_{|z|\leqslant 1}\left|P_q(z)\right| &\leqslant \sup_{|z|\leqslant 1}\sum_{|\alpha|=q}|f_\alpha||z|^q \\ &\leqslant \left(\sqrt{n}^\rho\sigma+\varepsilon\right)^{\frac{q}{\rho}}\cdot\frac{\mathrm{H}^n_q}{A_{\rho,q}} \leqslant \left(\sqrt{n}^\rho\sigma+\varepsilon\right)^{\frac{q}{\rho}}\cdot\frac{(q+1)^{n-1}}{A_{\rho,q}} \end{aligned}$$

i.e. $\limsup_{q\to\infty}\left(H_q A_{\rho,q}\right)^{\frac{\rho}{q}} \leqslant n^{\frac{\rho}{2}}\sigma$ as $\limsup_{q\to\infty}((q+1)^{n-1})^{\frac{\rho}{q}} = 1$ and so we have $f(z) \in E^{\rho(r)}_{\sqrt{n}^\rho\sigma}$. □

By this Corollary, we obtain the following result:

Proposition 1 – *An entire function $f = \sum_\alpha f_\alpha z^\alpha$ belongs to $\mathcal{E}^{\rho(r)}$ if and only if we have*

$$\limsup_{q\to+\infty}\left(\max_{|\alpha|=q}|f_\alpha| A_{\rho,q}\right)^{\frac{1}{q}} < +\infty. \tag{4}$$

As for the norm of B_{w_σ}, we have

Proposition 2 – *For $f = \sum_\alpha f_\alpha z^\alpha \in B_{w_\sigma}$, there exist $C, D > 0$ such that we have*

$$\|f\|_{w_\sigma} \leqslant C\sum_{q\in\mathbb{N}}\left(\sum_{|\alpha|=q}|f_\alpha| D^q A_q\right).$$

and for any $\alpha \in \mathbb{N}^n$,

$$C^{-1} D^{-|\alpha|} A_{|\alpha|} |f_\alpha| \leqslant \|f\|_{w_\sigma}. \tag{5}$$

Proof. Ishimura and Miyake (2007, Lemma 1) assures that there exist $C, D > 0$ such that for any α, we have

$$\|z^\alpha\|_{w_\sigma} \leqslant C D^q A_q \quad (|\alpha| = q)$$

and so

$$\|f\|_{w_\sigma} \leqslant \sum_{\alpha \in \mathbb{N}^n} |f_\alpha| \|z^\alpha\|_{w_\sigma} \leqslant C \sum_{q \in \mathbb{N}} \Big(\sum_{|\alpha|=q} |f_\alpha| D^q A_q \Big).$$

As in the proof of Lelong and Gruman (1986, Lemma 9.2), by Cauchy's formula in one variable $\lambda \in \mathbb{C}$, for any q, we have

$$P_q(z) = \frac{1}{2\pi i} \oint_{|\lambda|=1} \frac{f(\lambda z)}{\lambda^{q+1}} d\lambda :$$

so for any $r > 0$, we have

$$\sup_{|z|=r} \frac{|P_q(z)|}{|z|^q} \leqslant \sup_{|z|=r} \left(\left| f(z) e^{-w_\sigma(z)} \right| \frac{e^{\sigma r^{\rho(r)}}}{r^q} \right)$$

and then for any $r > 0$, we have

$$\sup_{|z|=r} \frac{|P_q(z)|}{|z|^q} \leqslant \|f\|_{w_\sigma} \frac{e^{\sigma r^{\rho(r)}}}{r^q} :$$

as in the proof of Lelong and Gruman (1986, Lemma 9.2), the function $\frac{e^{\sigma r^{\rho(r)}}}{r^q}$ of r attains its minimum which, by choosing $C, D > 0$ large enough, is estimated by CD^q/A_q: therefore we have the inequality

$$C^{-1} D^{-q} A_q \sup_z \frac{|P_q(z)|}{|z|^q} \leqslant \|f\|_{w_\sigma}.$$

As by the proof of Corollary 1 on p. 72, setting $q = |\alpha|$, we have

$$|f_\alpha| \leqslant \sqrt{n}^q \sup_{|z|=1} |P_q(z)| = \sqrt{n}^q \sup_z \frac{|P_q(z)|}{|z|^q},$$

taking $C, D > 0$ large enough, we have (5). □

4 Duality between the spaces $\mathcal{E}^{\rho(r)}$ and $E_0^{\rho^*(s)}$

In this section, we will study the dual space of $\mathcal{E}^{\rho(r)}$ (c.f. Lelong and Gruman 1986, Theorem 9.5 and Theorem 9.16). In the rest of this section, we will suppose $\rho > 1$: in this case, the equation $s = r^{\rho(r)-1}$ has the unique solution $r = \lambda(s)$ for any s large enough. A proximate order $\rho^*(s)$ is said to be a *conjugate proximate order* of $\rho(r)$ if it satisfied for large s (so for large r)

$$\rho^*(s) := \frac{\rho(r)}{\rho(r)-1} \quad \text{i.e.} \quad \frac{1}{\rho(r)} + \frac{1}{\rho^*(s)} = 1. \tag{6}$$

It is easy to see that any differentiable function $\rho^*(s)$ satisfying (6) for large s is in fact a proximate order (see for example Lelong and Gruman 1986, Prop. 9.4). In this section, we will take a conjugate proximate order $\rho^*(s)$. Set $\rho^* := \lim_{s\to\infty} \rho^*(s) > 1$.

As the case of $r = \varphi(t)$, let $s = \varphi^*(u)$ be a differentiable function being the inverse function of $u = s^{\rho^*(s)}$ when u is large enough. We set also

$$A_q^* = A_{\rho^*,q} := \left(\frac{\varphi^*(q)^{\rho^*}}{\mathrm{e}\rho^*}\right)^{\frac{q}{\rho^*}}. \tag{7}$$

Now consider the space $E_0^{\rho^*(s)}$ of entire functions of minimal type with respect to $\rho^*(s)$, which is an (FS)-space: we will show that it is the dual space of $\mathcal{E}^{\rho(r)}$.

The function $z \mapsto \mathrm{e}^{z\cdot\zeta}$ is in $\mathcal{E}^{\rho(r)}$ if $\rho \geqslant 1$: for any $T \in (\mathcal{E}^{\rho(r)})'$, we denote its Fourier-Borel transform by

$$\hat{T}(\zeta) = T_z(\mathrm{e}^{z\cdot\zeta}).$$

Proposition 3 – *Suppose $\rho > 1$. The map*

$$T \mapsto \widehat{T} : (\mathcal{E}^{\rho(r)})' \xrightarrow{\sim} E_0^{\rho^*(s)} \tag{8}$$

is a continuous bijection of Fréchet spaces so it is an isomorphism.

Proof. For $r > 0$ large enough, we will find the upper bound of the function

$$h(r) = h(r,\zeta) := \sup_{|z|=r}\Big(\operatorname{Re} z\cdot\zeta - w_\sigma(z)\Big) = r|\zeta| - \sigma r^{\rho(r)}$$

For this, we remark that for large $|\zeta|$, the equation

$$h'(r) = |\zeta| - \sigma r^{\rho(r)-1}\Big(\rho'(r) r \ln r + \rho(r)\Big) = 0 \tag{9}$$

has unique solution $r_0 = r(\zeta)$ ($\gg 1$): in general, it is not possible to know its exact value, but for any $\delta > 0$, if $|\zeta|$ is large enough, we have

$$\left|\Big(\rho'(r_0) r_0 \ln r_0 + \rho(r_0)\Big) - \rho\right| < \delta$$

so by the (9), we have

$$\frac{|\zeta|}{\sigma(\rho+\delta)} < {r_0}^{\rho(r_0)-1} < \frac{|\zeta|}{\sigma(\rho-\delta)}.$$

Therefore for some $c > 0$, we have

$$\begin{aligned}
h(r,\zeta) \leqslant h(r_0,\zeta) &= r_0|\zeta| - \sigma {r_0}^{\rho(r_0)} \\
&< \left(\frac{|\zeta|}{\sigma(\rho-\delta)}\right)^{\frac{1}{\rho(r_0)-1}} |\zeta| - \left(\frac{|\zeta|}{\sigma(\rho+\delta)}\right)^{\frac{\rho(r_0)}{\rho(r_0)-1}} \\
&= \left(\left(\frac{1}{\sigma(\rho-\delta)}\right)^{\frac{1}{\rho(r_0)-1}} - \left(\frac{1}{\sigma(\rho+\delta)}\right)^{\frac{\rho(r_0)}{\rho(r_0)-1}}\right) |\zeta|^{\frac{\rho(r_0)}{\rho(r_0)-1}} \\
&< \left(\left(\frac{1}{\sigma\rho}\right)^{\frac{1}{\rho(r_0)-1}} - \left(\frac{1}{\sigma\rho}\right)^{\frac{\rho(r_0)}{\rho(r_0)-1}} + c\delta\right) |\zeta|^{\frac{\rho(r_0)}{\rho(r_0)-1}} \\
&= \left((\sigma\rho - 1)\left(\frac{1}{\sigma\rho}\right)^{\frac{\rho(r_0)}{\rho(r_0)-1}} + c\delta\right) |\zeta|^{\frac{\rho(r_0)}{\rho(r_0)-1}}
\end{aligned}$$

as

$$\left(\frac{1}{\sigma(\rho\pm\delta)}\right)^{\frac{1}{\rho(r_0)-1}} = \left(\frac{1}{\sigma\rho}\right)^{\frac{1}{\rho(r_0)-1}} + O(\delta):$$

we may replace δ by $c^{-1}(\sigma\rho-1)\left(\frac{1}{\sigma\rho}\right)^{\frac{\rho(r_0)}{\rho(r_0)-1}}\delta$ and then we continue

$$< (1+\delta)(\sigma\rho-1)\left(\frac{1}{\sigma\rho}\right)^{\frac{\rho(r_0)}{\rho(r_0)-1}} |\zeta|^{\frac{\rho(r_0)}{\rho(r_0)-1}} = (1+\delta)(\sigma\rho-1)\left(\frac{|\zeta|}{\sigma\rho}\right)^{\frac{\rho(r_0)}{\rho(r_0)-1}}.$$

As for large $|\zeta|$ (so $r_0 \gg 1$), setting $s_0 := {r_0}^{\rho(r_0)-1}$, we have

$$0 = h'(r_0) = |\zeta| - \sigma {r_0}^{\rho(r_0)-1}\Big(\rho'(r_0) r_0 \ln r_0 + \rho(r_0)\Big) < |\zeta| - \sigma s_0(\rho - \delta)$$

and

$$0 = h'(r_0) = |\zeta| - \sigma {r_0}^{\rho(r_0)-1}\Big(\rho'(r_0) r_0 \ln r_0 + \rho(r_0)\Big) > |\zeta| - \sigma s_0(\rho + \delta),$$

we have

$$\frac{|\zeta|}{\sigma(\rho+\delta)} < s_0 < \frac{|\zeta|}{\sigma(\rho-\delta)} \quad \text{i.e.} \quad \left|\frac{|\zeta|}{\sigma\rho} - s_0\right| < \frac{s_0\delta}{\rho}.$$

As $\rho^*(s)$ is a proximate order, with some θ such that $0 < \theta < 1$, we have

$$\begin{aligned}
\rho^*(s_0) &= \rho^*\left(\frac{|\zeta|}{\sigma\rho}\right) + \rho^*\left(s_0 + \theta\cdot\left(\frac{|\zeta|}{\sigma\rho} - s_0\right)\right)' \cdot \left(\frac{|\zeta|}{\sigma\rho} - s_0\right) \\
&< \rho^*\left(\frac{|\zeta|}{\sigma\rho}\right) + \frac{1}{\left(s_0 - \frac{s_0\delta}{\rho}\right)\ln\left(s_0 - \frac{s_0\delta}{\rho}\right)} \cdot \frac{s_0\delta}{\rho} \\
&= \rho^*\left(\frac{|\zeta|}{\sigma\rho}\right) + \frac{\delta}{(\rho-\delta)\ln\left(s_0\left(1-\frac{\delta}{\rho}\right)\right)} < \rho^*\left(\frac{|\zeta|}{\sigma\rho}\right) + \frac{\delta}{\ln\left(\frac{|\zeta|}{\sigma(\rho+\delta)}\left(1-\frac{\delta}{\rho}\right)\right)} \\
&= \rho^*\left(\frac{|\zeta|}{\sigma\rho}\right) + \frac{\delta}{\ln\left(\frac{|\zeta|}{\sigma\rho}\frac{\rho-\delta}{\rho+\delta}\right)} < \rho^*\left(\frac{|\zeta|}{\sigma\rho}\right) + \frac{\delta}{\ln\left(\frac{|\zeta|}{\sigma\rho}(1-2\delta)\right)} \\
&< \rho^*\left(\frac{|\zeta|}{\sigma\rho}\right) + \frac{\delta}{\ln\frac{|\zeta|}{\sigma\rho} - 3\delta}
\end{aligned}$$

if $|\zeta|$ large enough and if $\rho - \delta > 1$ and $\ln(1-2\delta) > -3\delta$: so we have

$$\begin{aligned}
&\sup_{|z|=r}\left(\operatorname{Re} z\cdot\zeta - w_\sigma(z)\right) = h(r,\zeta) < (1+\delta)(\sigma\rho-1)\left(\frac{|\zeta|}{\sigma\rho}\right)^{\frac{\rho(r_0)}{\rho(r_0)-1}} \\
&= (1+\delta)(\sigma\rho-1)\left(\frac{|\zeta|}{\sigma\rho}\right)^{\rho^*(s_0)} < (1+\delta)(\sigma\rho-1)\left(\frac{|\zeta|}{\sigma\rho}\right)^{\rho^*\left(\frac{|\zeta|}{\sigma\rho}\right)+\frac{\delta}{\ln\frac{|\zeta|}{\sigma\rho}-3\delta}}. \qquad (10)
\end{aligned}$$

Setting $t := \frac{|\zeta|}{\sigma\rho}$, for $0 < \delta \ll 1$, if $t > 0$ is large enough, we have

$$\begin{aligned}
\ln\frac{h(r,\zeta)}{(1+\delta)(\sigma\rho-1)} &< \ln t^{\rho^*(t)+\frac{2\delta}{\ln t - 3\delta}} = \left(\rho^*(t) + \frac{2\delta}{\ln t - 3\delta}\right)\ln t \\
&< \rho^*(t)\ln t + 3\delta = \ln t^{\rho^*(t)} + 3\delta < \ln t^{\rho^*(\sigma\rho t)} + 4\delta
\end{aligned}$$

as by, for example, Lelong and Gruman (1986, Theorem 1.18), we have $\frac{t^{\rho^*(t)}}{t^{\rho^*((\sigma\rho)t)}} < \mathrm{e}^{\delta}$: thus we have

$$\frac{h(r,\zeta)}{(1+\delta)(\sigma\rho-1)} < \mathrm{e}^{4\delta} t^{\rho^*((\sigma\rho)t)}.$$

Therefore for large $|\zeta|$, continuing (10), we have

$$\sup_z \Big(\operatorname{Re} z\cdot\zeta - w_\sigma(z)\Big) = h(r,\zeta) < (1+\delta)(\sigma\rho - 1)\mathrm{e}^{4\delta}\left(\frac{|\zeta|}{\sigma\rho}\right)^{\rho^*(|\zeta|)}$$
$$= (1+\delta)\mathrm{e}^{4\delta}\sigma^*|\zeta|^{\rho^*(|\zeta|)} = (1+\delta)\mathrm{e}^{4\delta}\sigma^* w^*(\zeta) \tag{11}$$

here we set $\sigma^* := \frac{\sigma\rho-1}{(\sigma\rho)^{\rho^*}}$ (which is decreasing as a function of $\sigma \geqslant 1$ and tending to 0 if σ tends to ∞). Thus for any $\varepsilon > 0$, there exists a constant $C = C_\varepsilon > 0$ such that we have

$$\Big|\widehat{T}(\zeta)\Big| = \Big|T_z(\mathrm{e}^{z\cdot\zeta})\Big| \leqslant \|T\|_{(B_{w_\sigma})'}\cdot\Big\|\mathrm{e}^{z\cdot\zeta}\Big\|_{w_\sigma} = \|T\|_{(B_{w_\sigma})'}\cdot\sup_z \exp\Big(\operatorname{Re} z\cdot\zeta - w_\sigma(z)\Big)$$
$$\leqslant C\|T\|_{(B_{w_\sigma})'}\cdot\exp\Big((\sigma^*+\varepsilon)w^*(\zeta)\Big)$$

where $\|T\|_{(B_{w_\sigma})'} := \sup_{\|f\|_{w_\sigma}=1}|T(f)|$ is the operator norm of T in $(B_{w_\sigma})'$: so $T \mapsto \hat{T}$ is a continuous map $(\mathcal{E}^{\rho(r)})' \to E_0^{\rho^*(s)}$ which is evidently injective.

Conversely taking into account of Corollary 1 on p. 72 for ρ^* instead of ρ and in the case $\sigma = 0$, an entire function $F(\zeta) = \sum F_\alpha \zeta^\alpha$ belongs to $E_0^{\rho^*(s)}$ if and only if we have

$$\limsup_{q\to\infty}\left(\max_{|\alpha|=q}|F_\alpha|A_{\rho^*,q}\right)^{\frac{1}{q}} = 0:$$

by (7), for any $\varepsilon > 0$, there exists $C > 0$ such that for any $q \geqslant 0$, we have

$$\max_{|\alpha|=q}|F_\alpha| \leqslant C\frac{\varepsilon^q}{\varphi^*(q)^q}.$$

Now take any $F(\zeta) = \sum F_\alpha\zeta^\alpha \in E_0^{\rho^*(s)}$ and set for any $f(z) = \sum f_\alpha z^\alpha \in \mathcal{E}^{\rho(r)}$,

$$Tf := \sum_{\alpha\in\mathbb{N}^n}\alpha! f_\alpha F_\alpha : \tag{12}$$

we prove that this is in fact convergent and so well-defined. By Proposition 1 on p. 73, taking into account (1), there exist $C' > 0$ and $D > 0$ such that for any $q \geqslant 0$, we have

$$\max_{|\alpha|=q}|f_\alpha| \leqslant C'\frac{D^q}{\varphi(q)^q}.$$

Set $r = r_q = \varphi(q)$ and $s = s_q = \varphi^*(q)$ i.e. $q^{\frac{1}{\rho(r)}} = r, q^{\frac{1}{\rho^*(s)}} = s$, then for any α with $|\alpha| = q$, we have

$$\frac{\alpha!}{\varphi(q)^q\varphi^*(q)^q} \leqslant \frac{q^q}{(rs)^q} = 1$$

and thus (12) is convergent, that is T is in fact, an element of $(\mathcal{E}^{\rho(r)})'$ and we have

$$\hat{T}(\zeta) = T_z(\mathrm{e}^{z\cdot\zeta}) = T\left(\sum_{\alpha\in\mathbb{N}^n} \frac{1}{\alpha!}\zeta^\alpha z^\alpha\right) = \sum_{\alpha\in\mathbb{N}^n} \zeta^\alpha F_\alpha = F(\zeta)$$

and thus $T \mapsto \hat{T} : (\mathcal{E}^{\rho(r)})' \to E_0^{\rho^*(s)}$ is a continuous bijection and so by a Theorem of Banach, it is in fact an isomorphism of Fréchet spaces. □

5 Partial differential equations of infinite order in $\mathcal{E}^{\rho(r)}$

We proved that any continuous linear operator from $\mathcal{E}^{\rho(r)}$ into itself is represented by an infinite order partial differential operator of the form

$$P = P(z, D_z) = \sum_{\alpha\in\mathbb{N}^n} a_\alpha(z) D_z^\alpha$$

such as with a $\sigma > 1$, $\sum_{\alpha\in\mathbb{N}^n} \|a_\alpha\|_{w_\sigma} \alpha!^{1-\frac{1}{\rho}} \zeta^\alpha$ is holomorphic at 0.[7] We call

$$P(z, \zeta) := \sum_{|\alpha|=0}^{\infty} a_\alpha(z)\zeta^\alpha$$

the *symbol* of P: Expanding each coefficient

$$a_\alpha(z) := \sum_{\beta\in\mathbb{N}^n} a_\alpha^\beta z^\beta,$$

we define at least formally the *transpose* of $P = P(\zeta, D_\zeta)$:

$${}^tP := {}^tP(\zeta, D_\zeta) := \sum_{\beta\in\mathbb{N}^n}\left(\sum_{\alpha\in\mathbb{N}^n} a_\alpha^\beta \zeta^\alpha\right) D_\zeta^\beta.$$

For a formal power series $f(z) := \sum_\nu f_\nu z^\nu \in \mathbb{C}[[z]]$, we see formally

$$P(z, D_z) f(z) = \sum_{\mu\in\mathbb{N}^n}\left(\sum_{\nu\in\mathbb{N}^n}\left(\sum_{\lambda\leqslant\mu,\nu} \frac{\nu!}{\lambda!} a_{\nu-\lambda}^{\mu-\lambda}\right) f_\nu\right) z^\mu :$$

[7] Ishimura and Miyake, 2007, "Endomorphisms of the space of entire functions with proximate order and infinite order differential operators".

thus by the identification $f \mapsto (f_\gamma)\colon \mathbb{C}[[z]] \xrightarrow{\sim} \mathbb{C}^{\mathbb{N}^n}$, the operator $P\colon \mathbb{C}[[z]] \to \mathbb{C}[[z]]$ is identified with the infinite matrix which we call its *characteristic matrix*

$$C_P := \left(c_\nu^\mu\right)_{\mu,\nu} := \left(\sum_{\lambda\leqslant\mu,\nu} \frac{\nu!}{\lambda!} a_{\nu-\lambda}^{\mu-\lambda}\right)_{\mu,\nu} : \mathbb{C}^{\mathbb{N}^n} \to \mathbb{C}^{\mathbb{N}^n}. \tag{13}$$

Similarly for $F(\zeta) = \sum_\nu F_\nu \zeta^\nu \in \mathbb{C}[[\zeta]]$, we have

$${}^tP(\zeta, D_\zeta)F(\zeta) = \sum_{\mu\in\mathbb{N}^n}\left(\sum_{\nu\in\mathbb{N}^n}\left(\sum_{\lambda\leqslant\mu,\nu} \frac{\nu!}{\lambda!} a_{\mu-\lambda}^{\nu-\lambda}\right)F_\nu\right)\zeta^\mu :$$

so we have

$$C_{{}^tP} = \left(\sum_{\lambda\leqslant\mu,\nu} \frac{\nu!}{\lambda!} a_{\mu-\lambda}^{\nu-\lambda}\right)_{\mu,\nu} = \left(\frac{\nu!}{\mu!} c_\mu^\nu\right)_{\mu,\nu}. \tag{14}$$

Now we will consider the partial differential equation in $\mathcal{E}^{\rho(r)}$:

$$P(z, D_z)f = \sum_{|\alpha|=0}^{\infty} a_\alpha(z) D_z^\alpha f(z) = g.$$

By (13), expanding $f = \sum_{\nu\in\mathbb{N}^n} f_\nu z^\nu, g = \sum_{\mu\in\mathbb{N}^n} g_\mu z^\mu$, this means formally

$$\sum_{\nu\in\mathbb{N}^n} c_\nu^\mu f_\nu \equiv \sum_{\nu\in\mathbb{N}^n}\left(\sum_{\substack{\lambda\leqslant\nu\\ \lambda\leqslant\mu}} \frac{\nu!}{\lambda!} a_{\nu-\lambda}^{\mu-\lambda}\right) f_\nu = g_\mu \quad \text{(for any} \quad \mu \in \mathbb{N}^n). \tag{15}$$

In this article, as in Ishimura (2007), we will study the following two cases: partial differential operator of infinite order $P = \sum_{|\alpha|=0}^{\infty} a_\alpha(z) D^\alpha$ of

(I) *regular singular type*, that is, each $a_\alpha(z)$ is divisible by z^α

$$a_\alpha(z) = \sum_{\alpha\leqslant\beta} a_\alpha^\beta z^\beta :$$

(II) *Korobeĭnik type*, namely, each $a_\alpha(z)$ is a polynomial of order $\leqslant \alpha$

$$a_\alpha(z) = \sum_{\beta\leqslant\alpha} a_\alpha^\beta z^\beta.$$

This second type operator was first studied by Korobeinik (1959) in one variable case; we remark also Korobeinik (1962) and van der Steen (1971) studied the operators of first type, always in one variable case.

(O) We will call P an operator of *Euler type* if it is regular singular type and Korobeĭnik type at the same time i.e. with a constant $a_\alpha \in \mathbb{C}$, P has the form

$$P = \sum_{|\alpha|=0}^{\infty} a_\alpha z^\alpha D_z^\alpha.$$

Theorem 2 – *Let a continuous linear operator $P : \mathcal{E}^{\rho(r)} \to \mathcal{E}^{\rho(r)}$ be of regular singular type as differential operator. Suppose the following conditions hold: there exist $C, R, \kappa > 0$ such that*

1. *for all $\mu \in \mathbb{N}^n$, we have*

$$C^{-1}\kappa^{|\mu|} < \left|\sum_{\lambda \leqslant \mu} \frac{\mu!}{\lambda!} a^{\mu-\lambda}_{\mu-\lambda}\right|,$$

2. *whenever $\nu < \mu$, we have*

$$\frac{\left|\sum_{\lambda\leqslant\nu} \frac{\nu!}{\lambda!} a^{\mu-\lambda}_{\nu-\lambda}\right|}{\left|\sum_{\lambda\leqslant\mu} \frac{\mu!}{\lambda!} a^{\mu-\lambda}_{\mu-\lambda}\right|} \leqslant C \cdot \frac{A_{|\nu|}}{A_{|\mu|}} R^{|\mu-\nu|}.$$

Then $P\colon \mathcal{E}^{\rho(r)} \to \mathcal{E}^{\rho(r)}$ is surjective; so is an isomorphism of Fréchet spaces.

Proof. Assume that $Pf = g \in \mathcal{E}^{\rho(r)}$ and expand $f = \sum_\nu f_\nu z^\nu, g = \sum_\mu g_\mu z^\mu$: taking into account that in this case, (15) is just $\sum_{\nu\in\mathbb{N}^n}\left(\sum_{\lambda\leqslant\nu} \frac{\nu!}{\lambda!} a^{\mu-\lambda}_{\nu-\lambda}\right) f_\nu = \sum_{\nu\leqslant\mu} c^\mu_\nu f_\nu = g_\mu$, the condition (1) assures that the coefficients (f_μ) are determined uniquely from (g_μ) and so, in view of Proposition 1 on p. 73, it suffices to show that there exist $D, K > 0$ such that for all $\mu \in \mathbb{N}^n$, we have

$$\left|f_\mu\right| \leqslant D \cdot \frac{K^{|\mu|}}{A_{|\mu|}} : \tag{16}$$

we argue by induction on $q := |\mu|$. First we remark, by the condition (1) and Proposition 1 on p. 73, there exist $D_0, L > 0$ such that for all $\mu \in \mathbb{N}^n$, we have

$$\frac{\left|g_\mu\right|}{\left|c^\mu_\mu\right|} \leqslant D_0 \frac{L^{|\mu|}}{A_{|\mu|}}. \tag{17}$$

Assuming the condition (2), take any $K > \max(L, R, C)$ and $D > D_0$ so that for all $\mu \in \mathbb{N}^n$, we have

$$\frac{D_0}{D}\left(\frac{L}{K}\right)^{|\mu|} + C \cdot \left(\frac{K^n}{(K-R)^n} - 1\right) < 1.$$

For $q = |\mu|$, we have

$$\frac{1}{\left|c_\mu^\mu\right|}\sum_{\nu<\mu}\left|c_\nu^\mu f_\nu\right| \leqslant \sum_{\nu<\mu} C\cdot\frac{A_{|\nu|}}{A_q}R^{|\mu-\nu|}\cdot D\cdot\frac{K^{|\nu|}}{A_{|\nu|}} = \frac{CD}{A_q}\left(R^q\sum_{\nu\leqslant\mu}\left(\frac{K}{R}\right)^{|\nu|} - K^q\right)$$

$$= \frac{CD}{A_q}\left(R^q\sum_{\nu_1\leqslant\mu_1}\left(\frac{K}{R}\right)^{\nu_1}\cdots\sum_{\nu_n\leqslant\mu_n}\left(\frac{K}{R}\right)^{\nu_n} - K^q\right)$$

$$= \frac{CD}{A_q}\left(R^q\cdot\frac{\left(\frac{K}{R}\right)^{\mu_1+1}-1}{\frac{K}{R}-1}\cdots\frac{\left(\frac{K}{R}\right)^{\mu_n+1}-1}{\frac{K}{R}-1} - K^q\right)$$

$$\leqslant \frac{CD}{A_q}\left(R^q\cdot\frac{\left(\frac{K}{R}\right)^{q+n}}{\left(\frac{K}{R}-1\right)^n} - K^q\right)$$

$$= \frac{CD}{A_q}\left(\frac{K^{q+n}}{(K-R)^n} - K^q\right) = CD\cdot\frac{K^q}{A_q}\left(\frac{K^n}{(K-R)^n}-1\right).$$

Therefore, by (17), we have

$$\left|f_\mu\right| \leqslant \frac{\left|g_\mu\right|}{\left|c_\mu^\mu\right|} + \frac{1}{\left|c_\mu^\mu\right|}\sum_{\nu<\mu}\left|c_\nu^\mu f_\nu\right|$$

$$\leqslant D_0\cdot\frac{L^q}{A_q} + CD\cdot\frac{K^q}{A_q}\left(\frac{K^n}{(K-R)^n}-1\right) < D\cdot\frac{K^q}{A_q},$$

that is (16) as desired. □

In particular, when P is of Euler type, we have:

Corollary 2 – *If P is of Euler type, then $P\colon \mathcal{E}^{\rho(r)} \to \mathcal{E}^{\rho(r)}$ is an isomorphism if and only if there exist $C, \kappa > 0$ such that for any $\mu \in \mathbb{N}^n$, we have*

$$C^{-1}\kappa^{|\mu|} < \left|\sum_{\lambda\leqslant\mu}\frac{\mu!}{\lambda!}a_{\mu-\lambda}^{\mu-\lambda}\right|.$$

Proof. We need only to prove the "only if" part: in fact, taking $g = \sum_\mu \frac{1}{A_{|\mu|}}$ and $f := P^{-1}g$, by Proposition 1 on p. 73, with some $C, \kappa > 0$, we have

$$\frac{1}{\left|c_\mu^\mu\right|} = \left|f_\mu\right|A_{|\mu|} \leqslant C\kappa^{-q}.$$

□

Example 1 – Suppose $n=1$, $\rho>1$. In the following cases, $P=\sum a_\alpha(z)D^\alpha$ satisfies the condition of Theorem 2 on p. 81:

(i) with $k>0$, set

$$a_\alpha^\beta := \begin{cases} \dfrac{k^\alpha}{\alpha!} & \beta=\alpha \\ \dfrac{(-k)^\alpha}{\alpha!} & \beta=\alpha+1 \\ 0 & \text{otherwise} \end{cases}.$$

(ii) with $k\neq -1$, set

$$a_\alpha^\beta := \begin{cases} \dfrac{k^\alpha}{\alpha!} & \beta=\alpha \\ \dfrac{(-1)^\alpha}{\alpha!(\beta-\alpha)!} & \beta>\alpha \\ 0 & \text{otherwise} \end{cases}.$$

Theorem 3 – *Suppose $\rho>1$. Let a continuous linear operatorm $P:\mathcal{E}^{\rho(r)}\to\mathcal{E}^{\rho(r)}$ be of Korobeĭnik type as partial differential operator. Suppose the following conditions hold:*

1. *there exist $C,\kappa>0$ such that for all $\mu\in\mathbb{N}^n$, we have*

$$C^{-1}\kappa^{|\mu|} < \left|\sum_{\lambda\leqslant\mu}\frac{\mu!}{\lambda!}a_{\mu-\lambda}^{\mu-\lambda}\right|,$$

2. *for any $\delta>0$, there exists $M=M_\delta>0$ such that whenever $|\mu|\geqslant M$ and $\nu<\mu$, we have*

$$\frac{\left|\sum_{\lambda\leqslant\nu}\frac{1}{\lambda!}a_{\mu-\lambda}^{\nu-\lambda}\right|}{\left|\sum_{\lambda\leqslant\mu}\frac{1}{\lambda!}a_{\mu-\lambda}^{\mu-\lambda}\right|} \leqslant \frac{A_{|\nu|}^*}{A_{|\mu|}^*}\delta^{|\mu-\nu|}.$$

Then $P\colon\mathcal{E}^{\rho(r)}\to\mathcal{E}^{\rho(r)}$ is surjective; so is an epimorphism of Fréchet spaces.

Proof. In the case, the characteristic matrix $C_{{}^tP}=\left(\frac{\nu!}{\mu!}c_\mu^\nu\right)_{\mu,\nu}$ (see (14)) being "lower diagonal", ${}^tP$ is injective: so by virtue of the closed range theorem, we only need to show that ${}^tP\left(E_0^{\rho^*(s)}\right)$ is closed in $E_0^{\rho^*(s)}$: let $\widehat{T}^j\in E_0^{\rho^*(s)}$ be a sequence so that the sequence $\widehat{S}^j := {}^tP(\widehat{T}^j)$ is a Cauchy sequence in $E_0^{\rho^*(s)}$. We will show that the sequence $(\widehat{T}^j)$ is a Cauchy sequence in $E_0^{\rho^*(s)}$ i.e. for any $\sigma>0$ and any $\varepsilon>0$, there exists $N>0$ such that for any $i,j>N$, we have $\left\|\widehat{T}^i-\widehat{T}^j\right\|_{w_\sigma}\leqslant\varepsilon$. Now expanding $\widehat{T}^j(\zeta)=\sum_{\mu\in\mathbb{N}^n}\widehat{T}_\mu^j\zeta^\mu$, and $\widehat{S}^j(\zeta)=\sum_{\mu\in\mathbb{N}^n}\widehat{S}_\mu^j\zeta^\mu$, taking into account of Proposition 2 on p. 73, we will show by induction, that for any $\varepsilon > 0$, there exists $N > 0$ such that for any $i, j > N$ and any $\mu \in \mathbb{N}^n$, we have

$$\left|\widehat{T}^i_\mu - \widehat{T}^j_\mu\right| \leqslant \frac{\varepsilon^{|\mu|+1}}{A^*_{|\mu|}}. \tag{18}$$

Remark for any $\mu \in \mathbb{N}^n$, inductively, we have

$$\widehat{T}^i_\mu - \widehat{T}^j_\mu = \frac{1}{c^\mu_\mu}\left(\left(\widehat{S}^i_\mu - \widehat{S}^j_\mu\right) - \sum_{\nu<\mu} \frac{\nu!}{\mu!} c^\nu_\mu \left(\widehat{T}^i_\nu - \widehat{T}^j_\nu\right)\right):$$

by this, we first remark that for $M = M_\delta$ being fixed, there exists $C_M > 1$ such that whenever $|\mu| < M$, we have

$$\left|\widehat{T}^i_\mu - \widehat{T}^j_\mu\right| \leqslant C_M \varepsilon^{|\mu|} \max_{\nu \leqslant \mu} \frac{A^*_{|\nu|}}{A^*_{|\mu|}} \left|\widehat{S}^i_\nu - \widehat{S}^j_\nu\right|.$$

The sequence $(\widehat{S^j})$ being a Cauchy sequence, we may assume that taking $N > 0$ large enough, for any $i, j > N$ and $\mu \in \mathbb{N}^n$, we have

$$\left|\widehat{S}^i_\mu - \widehat{S}^j_\mu\right| \leqslant \frac{\varepsilon}{2C_M} \cdot \frac{(\varepsilon\kappa)^{|\mu|}}{A^*_{|\mu|}}, \tag{19}$$

so by (1), for all $i, j > N$ and $\mu \in \mathbb{N}^n$, we have then

$$\frac{1}{\left|c^\mu_\mu\right|}\left|\widehat{S}^i_\mu - \widehat{S}^j_\mu\right| \leqslant \frac{\varepsilon^{|\mu|+1}}{2C_M A^*_{|\mu|}} < \frac{\varepsilon^{|\mu|+1}}{2A^*_{|\mu|}}. \tag{20}$$

In the condition (1), we may of course assume $0 < \kappa < 1$ and so from (19), if $|\mu| < M$, we have

$$\left|\widehat{T}^i_\mu - \widehat{T}^j_\mu\right| \leqslant \frac{\varepsilon^{|\mu|+1}}{2A^*_{|\mu|}} < \frac{\varepsilon^{|\mu|+1}}{A^*_{|\mu|}} \tag{21}$$

thus we have (18) in the case $|\mu| < M$. For the case $|\mu| \geqslant M$, remark the following

Lemma 3 – *If $\mu \in \mathbb{N}^n$, then*

$$\sum_{\nu<\mu} \frac{\nu!}{\mu!} \leqslant 2^n.$$

For the proof, we only remark that in the case where $n = 1$, if $\mu = 0, 1$, it is evident and if $\mu \geqslant 2$, we have

$$\sum_{\nu=0}^{\mu} \frac{\nu!}{\mu!} \leqslant 1 + \frac{1}{\mu} + (\mu - 1)\frac{1}{\mu(\mu-1)} = 1 + \frac{2}{\mu} \leqslant 2.$$

In the hypothesis (2), take $\delta > 0$ so that $\delta < \varepsilon/2^{n+1}$ and apply the Lemma: then for $q = |\mu| \geqslant M$, we have

$$\left|\frac{1}{c_\mu^\mu}\sum_{\nu<\mu}\frac{\nu!}{\mu!}c_\mu^\nu\left(\widehat{T}_\nu^i - \widehat{T}_\nu^j\right)\right| \leqslant \sum_{|\nu|=0}^{q-1}\frac{\nu!}{\mu!}\cdot\delta^{q-|\nu|}\cdot\frac{A^*_{|\nu|}}{A^*_q}\cdot\frac{\varepsilon^{|\nu|+1}}{A^*_{|\nu|}}$$
$$= \frac{\varepsilon^{q+1}}{A^*_q}\sum_{|\nu|=0}^{q-1}\frac{\nu!}{\mu!}\left(\frac{\delta}{\varepsilon}\right)^{q-|\nu|} \leqslant \frac{\varepsilon^{q+1}}{A^*_q}\cdot\frac{\delta}{\varepsilon}\sum_{|\nu|=0}^{q-1}\frac{\nu!}{\mu!} < \frac{\varepsilon^{q+1}}{2A^*_q},$$

this and (21) for the case $|\mu| < M$, combining with (20), imply the desired estimate (18). □

Example 2 – Let $n = 1$, $\rho > 1$. In the following cases, $P = \sum a_\alpha(z)D^\alpha$ satisfies the condition of Theorem 3 on p. 83:

(i) with $k > 0$, define

$$a_\alpha^\beta := \begin{cases} \dfrac{k^\alpha}{\alpha!} & \beta = \alpha \\ \dfrac{(-k)^{\alpha-1}}{(\alpha-1)!} & \beta = \alpha - 1 \\ 0 & \text{otherwise} \end{cases}.$$

(ii) with $k \neq -1$, define

$$a_\alpha^\beta := \begin{cases} \dfrac{k^\alpha}{\alpha!} & \beta = \alpha \\ \dfrac{(-1)^\beta}{\beta!}\left(\dfrac{1}{(\alpha-\beta)!}\right)^2 & \beta < \alpha \\ 0 & \text{otherwise} \end{cases}.$$

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