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Functors between Reedy model categories of diagrams

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Abstract

If $\mathcal{D}$ is a Reedy category and $\mathcal{M}$ is a model category, the category $\mathcal{M}^{\mathcal{D}}$ of $\mathcal{D}$-diagrams in $\mathcal{M}$ is a model category under the Reedy model category structure. If $\mathcal{C} \to \mathcal{D}$ is a Reedy functor between Reedy categories, then there is an induced functor of diagram categories $\mathcal{M}^{\mathcal{D}} \to \mathcal{M}^{\mathcal{C}}$. Our main result is a characterization of the Reedy functors $\mathcal{C} \to \mathcal{D}$ that induce right or left Quillen functors $\mathcal{M}^{\mathcal{D}} \to \mathcal{M}^{\mathcal{C}}$ for every model category $\mathcal{M}$. We apply these results to various situations, and in particular show that certain important subdiagrams of a fibrant multicosimplicial object are fibrant.

Keywords: Reedy category, Quillen functor, diagram category.

MSC: Primary 55U35, 18G55.

1 Introduction

The interesting functors between model categories are the *left Quillen functors* and *right Quillen functors* (see Hirschhorn 2003, Def. 8.5.2). In this paper, we study Quillen functors between diagram categories with the Reedy model category structure (see Theorem 3 on p. 25).

In more detail, if $\mathcal{C}$ is a Reedy category (see Definition 1 on p. 23) and $\mathcal{M}$ is a model category, then there is a *Reedy model category structure* on the category $\mathcal{M}^{\mathcal{C}}$ of $\mathcal{C}$-diagrams in $\mathcal{M}$ (see Definition 2 on p. 23 and Theorem 3 on p. 25). The original (and most well known) examples of Reedy model category structures are the model categories of *cosimplicial objects in a model category* and of *simplicial objects in a model category* (see Section 3.5).

Any functor $G\colon \mathcal{C} \to \mathcal{D}$ between Reedy categories induces a functor

$$G^*\colon \mathcal{M}^{\mathcal{D}} \longrightarrow \mathcal{M}^{\mathcal{C}}$$

of diagram categories (see Definition 6 on p. 26), and it is important to know when such a functor G^* is a left or a right Quillen functor, since, for example, a right

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Quillen functor takes fibrant objects to fibrant objects, and takes weak equivalences between fibrant objects to weak equivalences (see Proposition 11 on p. 35). The results in this paper provide a complete characterization of the Reedy functors (functors between Reedy categories that preserve the structure; see Definition 5 on p. 26) between diagram categories for which this is the case for all model categories $\mathcal{M}$.

To be clear, we point out that for any Reedy functor $G\colon \mathcal{C} \to \mathcal{D}$ there exist model categories $\mathcal{M}$ such that the induced functor $G^*\colon \mathcal{M}^{\mathcal{D}} \to \mathcal{M}^{\mathcal{C}}$ is a (right or left) Quillen functor. For example, if $\mathcal{M}$ is a model category in which the weak equivalences are the isomorphisms of $\mathcal{M}$ and all maps of $\mathcal{M}$ are both cofibrations and fibrations, then *every* Reedy functor $G\colon \mathcal{C} \to \mathcal{D}$ induces a right Quillen functor $G^*\colon \mathcal{M}^{\mathcal{D}} \to \mathcal{M}^{\mathcal{C}}$ (which is also a left Quillen functor). In this paper, we characterize those Reedy functors that induce right Quillen functors for *all* model categories $\mathcal{M}$. More precisely, we have:

Theorem 1 – *If $G\colon \mathcal{C} \to \mathcal{D}$ is a Reedy functor (see Definition 5 on p. 26), then the induced functor of diagram categories $G^*\colon \mathcal{M}^{\mathcal{D}} \to \mathcal{M}^{\mathcal{C}}$ is a right Quillen functor for every model category $\mathcal{M}$ if and only if G is a fibering Reedy functor (see Definition 8 on p. 30).*

In fact, we show that if $G\colon \mathcal{C} \to \mathcal{D}$ is a Reedy functor that is not fibering, then the induced functor of diagram categories $G^*\colon \mathcal{M}^{\mathcal{D}} \to \mathcal{M}^{\mathcal{C}}$ fails to be a right Quillen functor when $\mathcal{M}$ is the standard model category of topological spaces (see Theorem 13 on p. 45).

We also have a dual result:

Theorem 2 – *If $G\colon \mathcal{C} \to \mathcal{D}$ is a Reedy functor, then the induced functor of diagram categories $G^*\colon \mathcal{M}^{\mathcal{D}} \to \mathcal{M}^{\mathcal{C}}$ is a left Quillen functor for every model category $\mathcal{M}$ if and only if G is a cofibering Reedy functor (see Definition 8 on p. 30).*

In an attempt to make these results accessible to a more general audience, we've included a description of some background material that is well known to the experts. The structure of the paper is as follows: We provide some background on Reedy categories and functors in Section 2, including discussions of filtrations, opposites, Quillen functors, and cofinality. The only new content for this part is in Section 2.3, where we define inverse and direct $\mathcal{C}$-factorizations and (co)fibering Reedy functors, and prove some results about them. We then discuss several examples and applications of Theorem 1 and Theorem 2 in Section 3. More precisely, we look at the subdiagrams given by truncations, diagrams defined as skeleta, and three kinds of subdiagrams determined by (co)simplicial and multi(co)simplicial diagrams: restricted (co)simplicial objects, diagonals of multi(co)simplicial objects, and slices of multi(co)simplicial objects. We then finally present the proofs of Theorem 1 and Theorem 2 in Section 4. Theorem 1 will follow immediately from Theorem 11 on p. 44, which is its slight elaboration. Theorem 2 can be proved by dualizing the proof of Theorem 1 on the preceding page, but we will instead derive it in Section 4.5 from Theorem 1 on the preceding page and a careful discussion of opposite categories.

Lastly, it should be noted that, upon completing this paper, the authors learned that Theorem 1 also appears as Barwick (2010, Theorem 3.22). However, the methods presented in this paper are different, and the proof that appears here is more elementary. This paper additionally provides examples that make the material digestible for the reader, as well as a number of applications. In particular, we study how the main result applies to the various subdiagrams of multicosimplicial objects (restricted multicosimplicial objects, diagonals of multicosimplicial objects, and slices of multicosimplicial objects) that figure heavily in recent applications of functor calculus to the study of links.

2 Reedy model category structures

In this section, we give the definitions and results needed for the statements and proofs of our theorems. We assume the reader is familiar with the basic language of model categories. The material here is standard, with the exception of Section 2.3 where the key notions for characterizing Quillen functors between Reedy model categories are introduced (Definition 7 on p. 26 and Definition 8 on p. 30).

2.1 Reedy categories and their diagram categories

Definition 1 – A *Reedy category* is a small category $\mathcal{C}$ together with two subcategories $\overrightarrow{\mathcal{C}}$ (the *direct subcategory*) and $\overleftarrow{\mathcal{C}}$ (the *inverse subcategory*), both of which contain all the objects of $\mathcal{C}$, in which every object can be assigned a nonnegative integer (called its *degree*) such that

1. Every non-identity map of $\overrightarrow{\mathcal{C}}$ raises degree.
2. Every non-identity map of $\overleftarrow{\mathcal{C}}$ lowers degree.
3. Every map g in $\mathcal{C}$ has a unique factorization $g = \overrightarrow{g}\overleftarrow{g}$ where $\overrightarrow{g}$ is in $\overrightarrow{\mathcal{C}}$ and $\overleftarrow{g}$ is in $\overleftarrow{\mathcal{C}}$.

Remark 1 – The function that assigns to every object of a Reedy category its degree is not a part of the structure, but we will generally assume that such a degree function has been chosen.

Definition 2 – Let $\mathcal{C}$ be a Reedy category and let $\mathcal{M}$ be a model category.

1. A $\mathcal{C}$*-diagram in* $\mathcal{M}$ is a functor from $\mathcal{C}$ to $\mathcal{M}$.
2. The category $\mathcal{M}^{\mathcal{C}}$ of $\mathcal{C}$-diagrams in $\mathcal{M}$ is the category with objects the functors from $\mathcal{C}$ to $\mathcal{M}$ and with morphisms the natural transformations of such functors.

In order to describe the *Reedy model category structure* on the diagram category $\mathcal{M}^{\mathcal{C}}$ in Theorem 3 on the next page, we first define the *latching maps* and *matching maps* of a $\mathcal{C}$-diagram in $\mathcal{M}$ as follows.

Definition 3 – Let $\mathcal{C}$ be a Reedy category, let $\mathcal{M}$ be a model category, let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be $\mathcal{C}$-diagrams in $\mathcal{M}$, let $f\colon \boldsymbol{X} \to \boldsymbol{Y}$ be a map of diagrams, and let α be an object of $\mathcal{C}$.

1. The *latching category* $\partial(\overrightarrow{\mathcal{C}}\downarrow\alpha)$ of $\mathcal{C}$ at α is the full subcategory of $(\overrightarrow{\mathcal{C}}\downarrow\alpha)$ (the category of objects of $\overrightarrow{\mathcal{C}}$ over α; see Hirschhorn 2003, Def. 11.8.1) containing all of the objects except the identity map of α.

2. The *latching object* of $\boldsymbol{X}$ at α is

   $$\mathrm{L}_\alpha \boldsymbol{X} = \operatorname*{colim}_{\partial(\overrightarrow{\mathcal{C}}\downarrow\alpha)} \boldsymbol{X}$$

   and the *latching map* of $\boldsymbol{X}$ at α is the natural map

   $$\mathrm{L}_\alpha \boldsymbol{X} \longrightarrow \boldsymbol{X}_\alpha.$$

   We will use $\mathrm{L}^{\mathcal{C}}_\alpha \boldsymbol{X}$ to denote the latching object if the indexing category is not obvious.

3. The *relative latching map* of $f\colon \boldsymbol{X} \to \boldsymbol{Y}$ at α is the natural map

   $$\boldsymbol{X}_\alpha \amalg_{\mathrm{L}_\alpha \boldsymbol{X}} \mathrm{L}_\alpha \boldsymbol{Y} \longrightarrow \boldsymbol{Y}_\alpha.$$

4. The *matching category* $\partial(\alpha\downarrow\overleftarrow{\mathcal{C}})$ of $\mathcal{C}$ at α is the full subcategory of $(\alpha\downarrow\overleftarrow{\mathcal{C}})$ (the category of objects of $\overleftarrow{\mathcal{C}}$ under α; see Hirschhorn 2003, Def. 11.8.3) containing all of the objects except the identity map of α.

5. The *matching object* of $\boldsymbol{X}$ at α is

   $$\mathrm{M}_\alpha \boldsymbol{X} = \lim_{\partial(\alpha\downarrow\overleftarrow{\mathcal{C}})} \boldsymbol{X}$$

   and the *matching map* of $\boldsymbol{X}$ at α is the natural map

   $$\boldsymbol{X}_\alpha \longrightarrow \mathrm{M}_\alpha \boldsymbol{X}.$$

   We will use $\mathrm{M}^{\mathcal{C}}_\alpha \boldsymbol{X}$ to denote the matching object if the indexing category is not obvious.

6. The *relative matching map* of $f\colon \boldsymbol{X} \to \boldsymbol{Y}$ at α is the map

   $$\boldsymbol{X}_\alpha \longrightarrow \boldsymbol{Y}_\alpha \times_{\mathrm{M}_\alpha \boldsymbol{Y}} \mathrm{M}_\alpha \boldsymbol{X}.$$

Theorem 3 – *Let $\mathcal{C}$ be a Reedy category and let $\mathcal{M}$ be a model category. There is a model category structure on the category $\mathcal{M}^{\mathcal{C}}$ of $\mathcal{C}$-diagrams in $\mathcal{M}$, called the* Reedy model category structure, *in which a map $f\colon \boldsymbol{X} \to \boldsymbol{Y}$ of $\mathcal{C}$-diagrams in $\mathcal{M}$ is*

- *a* weak equivalence *if for every object α of $\mathcal{C}$ the map $f_\alpha\colon \boldsymbol{X}_\alpha \to \boldsymbol{Y}_\alpha$ is a weak equivalence in $\mathcal{M}$,*
- *a* cofibration *if for every object α of $\mathcal{C}$ the relative latching map $\boldsymbol{X}_\alpha \amalg_{\mathrm{L}_\alpha \boldsymbol{X}} \mathrm{L}_\alpha \boldsymbol{Y} \to \boldsymbol{Y}_\alpha$ (see Definition 3 on the preceding page) is a cofibration in $\mathcal{M}$, and*
- *a* fibration *if for every object α of $\mathcal{C}$ the relative matching map $\boldsymbol{X}_\alpha \to \boldsymbol{Y}_\alpha \times_{\mathrm{M}_\alpha \boldsymbol{Y}} \mathrm{M}_\alpha \boldsymbol{X}$ (see Definition 3 on the preceding page) is a fibration in $\mathcal{M}$.*

Proof. See Hirschhorn (2003, Def. 15.3.3 and Thm. 15.3.4). □

We also record the following standard result, which can also be obtained from the Yoneda lemma (see MacLane 1971, p. 61); we will have use for it in the proof of Proposition 22 on p. 55.

Proposition 1 – *If $\mathcal{M}$ is a category and $f\colon X \to Y$ is a map in $\mathcal{M}$, then f is an isomorphism if and only if it induces an isomorphism of the sets of maps $f_*\colon \mathcal{M}(W,X) \to \mathcal{M}(W,Y)$ for every object W of $\mathcal{M}$.*

Proof. If $g\colon Y \to X$ is an inverse for f, then $g_*\colon \mathcal{M}(W,Y) \to \mathcal{M}(W,X)$ is an inverse for f_*.

Conversely, if $f_*\colon \mathcal{M}(W,X) \to \mathcal{M}(W,Y)$ is an isomorphism for every object W of $\mathcal{M}$, then $f_*\colon \mathcal{M}(Y,X) \to \mathcal{M}(Y,Y)$ is an epimorphism, and so there is a map $g\colon Y \to X$ such that $fg = 1_Y$. We then have two maps $gf, 1_X\colon X \to X$, and

$$f_*(gf) = fgf = 1_Y f = f = f_*(1_X).$$

Since $f_*\colon \mathcal{M}(X,X) \to \mathcal{M}(X,Y)$ is a monomorphism, this implies that $gf = 1_X$. □

2.2 Filtrations of Reedy categories

The notion of a filtration of a Reedy category will be used in the proof of Theorem 13 on p. 45.

Definition 4 – If $\mathcal{C}$ is a Reedy category (with a chosen degree function) and n is a nonnegative integer, the *n'th filtration* $\mathrm{F}^n\mathcal{C}$ of $\mathcal{C}$ (also called the *n'th truncation* $\mathcal{C}^{\le n}$ of $\mathcal{C}$) is the full subcategory of $\mathcal{C}$ with objects the objects of $\mathcal{C}$ of degree at most n.

The following is a direct consequence of the definitions.

Proposition 2 – *If $\mathcal{C}$ is a Reedy category then each of its filtrations $\mathrm{F}^n\mathcal{C}$ is a Reedy category with $\overrightarrow{\mathrm{F}^n\mathcal{C}} = \overrightarrow{\mathcal{C}} \cap \mathrm{F}^n\mathcal{C}$ and $\overleftarrow{\mathrm{F}^n\mathcal{C}} = \overleftarrow{\mathcal{C}} \cap \mathrm{F}^n\mathcal{C}$, and $\mathcal{C}$ equals the union of the increasing sequence of subcategories $\mathrm{F}^0\mathcal{C} \subset \mathrm{F}^1\mathcal{C} \subset \mathrm{F}^2\mathcal{C} \subset \cdots$.*

The following will be used in the proof of Theorem 13 on p. 45 (which is one direction of Theorem 1 on p. 22).

Proposition 3 – *For $n > 0$, extending a diagram $\boldsymbol{X}$ on $\mathrm{F}^{n-1}\mathcal{D}$ to one on $\mathrm{F}^{n}\mathcal{D}$ consists of choosing, for every object γ of degree n, an object $\boldsymbol{X}_\gamma$ and a factorization $\mathrm{L}_\gamma\boldsymbol{X} \to \boldsymbol{X}_\gamma \to \mathrm{M}_\gamma\boldsymbol{X}$ of the natural map $\mathrm{L}_\gamma\boldsymbol{X} \to \mathrm{M}_\gamma\boldsymbol{X}$ from the latching object of $\boldsymbol{X}$ at γ to the matching object of $\boldsymbol{X}$ at γ.*

Proof. See Hirschhorn (2003, Thm. 15.2.1 and Cor. 15.2.9). □

2.3 Reedy functors

In Definition 5 we introduce the notion of a *Reedy functor* between Reedy categories; this is a functor that preserves the Reedy structure.

Definition 5 – If $\mathcal{C}$ and $\mathcal{D}$ are Reedy categories, then a *Reedy functor* $G\colon \mathcal{C} \to \mathcal{D}$ is a functor that takes $\overrightarrow{\mathcal{C}}$ into $\overrightarrow{\mathcal{D}}$ and takes $\overleftarrow{\mathcal{C}}$ into $\overleftarrow{\mathcal{D}}$. If $\mathcal{D}$ is a Reedy category, then a *Reedy subcategory* of $\mathcal{D}$ is a subcategory $\mathcal{C}$ of $\mathcal{D}$ that is a Reedy category for which the inclusion functor $\mathcal{C} \to \mathcal{D}$ is a Reedy functor.

Note that a Reedy functor is *not* required to respect the filtrations on the Reedy categories $\mathcal{C}$ and $\mathcal{D}$ (see Definition 4 on the previous page). Thus, a Reedy functor might take non-identity maps to identity maps (see, e.g., Proposition 22 on p. 55).

Definition 6 – If $G\colon \mathcal{C} \to \mathcal{D}$ is a Reedy functor between Reedy categories and $\mathcal{M}$ is a model category, then G induces a functor of diagram categories $G^*\colon \mathcal{M}^{\mathcal{D}} \to \mathcal{M}^{\mathcal{C}}$ under which

- a functor $\boldsymbol{X}\colon \mathcal{D} \to \mathcal{M}$ goes to the functor $G^*\boldsymbol{X}\colon \mathcal{C} \to \mathcal{M}$ that is the composition $\mathcal{C} \xrightarrow{G} \mathcal{D} \xrightarrow{\boldsymbol{X}} \mathcal{M}$ (so that for an object α of $\mathcal{C}$ we have $(G^*\boldsymbol{X})_\alpha = \boldsymbol{X}_{G\alpha}$) and
- a natural transformation of $\mathcal{D}$-diagrams $f\colon \boldsymbol{X} \to \boldsymbol{Y}$ goes to the natural transformation of $\mathcal{C}$-diagrams G^*f that on an object α of $\mathcal{C}$ is the map $f_{G\alpha}\colon \boldsymbol{X}_{G\alpha} \to \boldsymbol{Y}_{G\alpha}$ in $\mathcal{M}$.

The main results of this paper (Theorem 1 on p. 22 and Theorem 2 on p. 22) determine when the functor $G^*\colon \mathcal{M}^{\mathcal{D}} \to \mathcal{M}^{\mathcal{C}}$ is either a left Quillen functor or a right Quillen functor for all model categories $\mathcal{M}$. The characterizations will depend on the notions of the *category of inverse $\mathcal{C}$-factorizations* of a map in $\overleftarrow{\mathcal{D}}$ and the *category of direct $\mathcal{C}$-factorizations* of a map in $\overrightarrow{\mathcal{D}}$.

Definition 7 – Let $G\colon \mathcal{C} \to \mathcal{D}$ be a Reedy functor between Reedy categories, let α be an object of $\mathcal{C}$, and let β be an object of $\mathcal{D}$.

1. If $\sigma\colon G\alpha \to \beta$ is a map in $\overleftarrow{\mathcal{D}}$, then the *category of inverse $\mathcal{C}$-factorizations* of (α, σ) is the category $\mathrm{Fact}_{\overleftarrow{\mathcal{C}}}(\alpha, \sigma)$ in which

- an object is a pair

  $$\big((\nu\colon \alpha \to \gamma), (\mu\colon G\gamma \to \beta)\big)$$

  consisting of a non-identity map $\nu\colon \alpha \to \gamma$ in $\overleftarrow{\mathcal{C}}$ and a map $\mu\colon G\gamma \to \beta$ in $\overleftarrow{\mathcal{D}}$ such that the diagram

  $$\begin{array}{ccc} G\alpha & \xrightarrow{G\nu} & G\gamma \\ & {\scriptstyle\sigma}\searrow \quad \swarrow{\scriptstyle\mu} & \\ & \beta & \end{array}$$

  commutes, and

- a map from $\big((\nu\colon \alpha \to \gamma), (\mu\colon G\gamma \to \beta)\big)$ to $\big((\nu'\colon \alpha \to \gamma'), (\mu'\colon G\gamma' \to \beta)\big)$ is a map $\tau\colon \gamma \to \gamma'$ in $\overleftarrow{\mathcal{C}}$ such that the triangles

  $$\begin{array}{ccc} & \alpha & \\ & {\scriptstyle\nu}\swarrow \quad \searrow{\scriptstyle\nu'} & \\ \gamma & \xrightarrow[\tau]{} & \gamma' \end{array} \qquad \text{and} \qquad \begin{array}{ccc} G\gamma & \xrightarrow{G\tau} & G\gamma' \\ & {\scriptstyle\mu'}\searrow \quad \swarrow{\scriptstyle\mu'} & \\ & \beta & \end{array}$$

  commute.

We will often refer just to the map σ when the object α is obvious. In particular, when $G\colon \mathcal{C} \to \mathcal{D}$ is the inclusion of a subcategory the object α is determined by the morphism σ, and we will often refer to the *category of inverse $\mathcal{C}$-factorizations of σ*.

2. If $\sigma\colon \beta \to G\alpha$ is a map in $\overrightarrow{\mathcal{D}}$, then the *category of direct $\mathcal{C}$-factorizations* of (α, σ) is the category $\mathrm{Fact}_{\overrightarrow{\mathcal{C}}}(\alpha, \sigma)$ in which

   - an object is a pair

     $$\big((\nu\colon \gamma \to \alpha), (\mu\colon \beta \to G\gamma)\big)$$

     consisting of a non-identity map $\nu\colon \gamma \to \alpha$ in $\overrightarrow{\mathcal{C}}$ and a map $\mu\colon \beta \to G\gamma$ in $\overrightarrow{\mathcal{D}}$ such that the diagram

     $$\begin{array}{ccc} \beta & \xrightarrow{\mu} & G\gamma \\ & {\scriptstyle\sigma}\searrow \quad \swarrow{\scriptstyle G\nu} & \\ & G\alpha & \end{array}$$

     commutes, and

- a map from $\big((\nu\colon \gamma \to \alpha), (\mu\colon \beta \to G\gamma)\big)$ to $\big((\nu'\colon \gamma' \to \alpha), (\mu'\colon \beta \to G\gamma')\big)$ is a map $\tau\colon \gamma \to \gamma'$ in $\overrightarrow{\mathcal{C}}$ such that the triangles

$$\begin{array}{ccc} & \alpha & \\ {}^{\nu}\nearrow & & \nwarrow^{\gamma'} \\ \gamma & \xrightarrow[\tau]{} & \gamma' \end{array} \qquad \text{and} \qquad \begin{array}{ccc} G\gamma & \xrightarrow{G\tau} & G\gamma' \\ {}_{\mu}\nwarrow & & \nearrow_{\mu'} \\ & \beta & \end{array}$$

commute.

We will often refer just to the map σ when the object α is obvious. In particular, when $G\colon \mathcal{C} \to \mathcal{D}$ is the inclusion of a subcategory the object α is determined by the morphism σ, and we will often refer to the *category of direct $\mathcal{C}$-factorizations of σ*.

Proposition 4 – *Let $G\colon \mathcal{C} \to \mathcal{D}$ be a Reedy functor between Reedy categories, let α be an object of $\mathcal{C}$, and let β be an object of $\mathcal{D}$.*

1. *If $\sigma\colon G\alpha \to \beta$ is a map in $\overleftarrow{\mathcal{D}}$, then we have an induced functor*

$$G_*\colon \partial(\alpha \downarrow \overleftarrow{\mathcal{C}}) \longrightarrow (G\alpha \downarrow \overleftarrow{\mathcal{D}})$$

from the matching category of $\mathcal{C}$ at α to the category of objects of $\overleftarrow{\mathcal{D}}$ under $G\alpha$ that takes the object $\alpha \to \gamma$ of $\partial(\alpha \downarrow \overleftarrow{\mathcal{C}})$ to the object $G\alpha \to G\gamma$ of $(G\alpha \downarrow \overleftarrow{\mathcal{D}})$, and the category $\mathrm{Fact}_{\overleftarrow{\mathcal{C}}}(\alpha, \sigma)$ of inverse $\mathcal{C}$-factorizations of (α, σ) (see Definition 7 on p. 26) is the category $(G_* \downarrow \sigma)$ of objects of $\partial(\alpha \downarrow \overleftarrow{\mathcal{C}})$ over σ.

2. *If $\sigma\colon \beta \to G\alpha$ is a map in $\overrightarrow{\mathcal{D}}$, then we have an induced functor*

$$G_*\colon \partial(\overrightarrow{\mathcal{C}} \downarrow \alpha) \longrightarrow (\overrightarrow{\mathcal{D}} \downarrow G\alpha)$$

from the latching category of $\mathcal{C}$ at α to the category of objects of $\overrightarrow{\mathcal{D}}$ over $G\alpha$ that takes the object $\gamma \to \alpha$ of $\partial(\overrightarrow{\mathcal{C}} \downarrow \alpha)$ to the object $G\gamma \to G\alpha$ of $(\overrightarrow{\mathcal{D}} \downarrow G\alpha)$, and the category $\mathrm{Fact}_{\overrightarrow{\mathcal{C}}}(\alpha, \sigma)$ of direct $\mathcal{C}$-factorizations of (α, σ) is the category $(\sigma \downarrow G_*)$ of objects of $\partial(\overrightarrow{\mathcal{C}} \downarrow \alpha)$ under σ.

Proof. We will prove part 1; the proof of part 2 is similar. An object of $(G_* \downarrow \sigma)$ is a pair $\big((\nu\colon \alpha \to \gamma), (\mu\colon G\gamma \to \beta)\big)$ where $\nu\colon \alpha \to \gamma$ is an object of $\partial(\alpha \downarrow \overleftarrow{\mathcal{C}})$ and $\mu\colon G\gamma \to \beta$ is a map in $\overleftarrow{\mathcal{D}}$ that makes the triangle

$$\begin{array}{ccc} & G\alpha & \\ {}^{G\nu}\swarrow & & \searrow^{\sigma} \\ G\gamma & \xrightarrow[\mu]{} & \beta \end{array}$$

commute. A map from $\big((\nu\colon \alpha \to \gamma), (\mu\colon G\gamma \to \beta)\big)$ to $\big((\nu'\colon \alpha \to \gamma'), (\mu'\colon G\gamma' \to \beta)\big)$ is a map $\tau\colon \gamma \to \gamma'$ in $\overleftarrow{\mathcal{C}}$ that makes the triangles

$$\begin{array}{ccc} & \alpha & \\ {\scriptstyle\nu}\swarrow & & \searrow{\scriptstyle\nu'} \\ \gamma & \xrightarrow[\tau]{} & \gamma' \end{array} \qquad \text{and} \qquad \begin{array}{ccc} G\gamma & \xrightarrow{G\tau} & G\gamma' \\ {\scriptstyle\mu}\searrow & & \swarrow{\scriptstyle\mu'} \\ & \beta & \end{array}$$

commute. This is exactly the definition of the category of inverse $\mathcal{C}$-factorizations of (α, σ). □

Proposition 5 – *Let $\mathcal{C}$ and $\mathcal{D}$ be Reedy categories, let $G\colon \mathcal{C} \to \mathcal{D}$ be a Reedy functor, and let α be an object of $\mathcal{C}$.*

1. *If G takes every non-identity map $\alpha \to \gamma$ in $\overleftarrow{\mathcal{C}}$ to a non-identity map in $\overleftarrow{\mathcal{D}}$, then there is an induced functor of matching categories*

   $$G_*\colon \partial(\alpha \downarrow \overleftarrow{\mathcal{C}}) \to \partial(G\alpha \downarrow \overleftarrow{\mathcal{D}})$$

   (see Definition 3 on p. 24) that takes the object $\eta\colon \alpha \to \gamma$ of $\partial(\alpha \downarrow \overleftarrow{\mathcal{C}})$ to the object $G\eta\colon G\alpha \to G\gamma$ of $\partial(G\alpha \downarrow \overleftarrow{\mathcal{D}})$. If β is an object of $\mathcal{D}$ and $\sigma\colon G\alpha \to \beta$ is a map in $\overleftarrow{\mathcal{D}}$, then the category $\mathrm{Fact}_{\overleftarrow{\mathcal{C}}}(\alpha, \sigma)$ of inverse $\mathcal{C}$-factorizations of (α, σ) (see Definition 7 on p. 26) is the category $(G_* \downarrow \sigma)$ of objects of $\partial(\alpha \downarrow \overleftarrow{\mathcal{C}})$ over σ.

2. *If G takes every non-identity map $\gamma \to \alpha$ in $\overrightarrow{\mathcal{C}}$ to a non-identity map in $\overrightarrow{\mathcal{D}}$, then there is an induced functor of latching categories*

   $$G_*\colon \partial(\overrightarrow{\mathcal{C}} \downarrow \alpha) \to \partial(\overrightarrow{\mathcal{D}} \downarrow G\alpha)$$

   (see Definition 3 on p. 24) that takes the object $\eta\colon \gamma \to \alpha$ of $\partial(\overrightarrow{\mathcal{C}} \downarrow \alpha)$ to the object $G\eta\colon G\gamma \to G\alpha$ of $\partial(\overrightarrow{\mathcal{D}} \downarrow G\alpha)$. If β is an object of $\mathcal{D}$ and $\sigma\colon \beta \to G\alpha$ is a map in $\overrightarrow{\mathcal{D}}$, then the category $\mathrm{Fact}_{\overrightarrow{\mathcal{C}}}(\alpha, \sigma)$ of direct $\mathcal{C}$-factorizations of (α, σ) is the category $(\sigma \downarrow G_*)$ of objects of $\partial(\overrightarrow{\mathcal{C}} \downarrow \alpha)$ under σ.

Proof. This is identical to the proof of Proposition 4 on the preceding page, except that the requirement that certain non-identity maps go to non-identity maps ensures (in part 1) that the functor $G_*\colon \partial(\alpha \downarrow \overleftarrow{\mathcal{C}}) \to (G\alpha \downarrow \overleftarrow{\mathcal{D}})$ factors through the subcategory $\partial(G\alpha \downarrow \overleftarrow{\mathcal{D}})$ of $(G\alpha \downarrow \overleftarrow{\mathcal{D}})$ and (in part 2) that the functor $G_*\colon \partial(\overrightarrow{\mathcal{C}} \downarrow \alpha) \to (\overrightarrow{\mathcal{D}} \downarrow G\alpha)$ factors through the subcategory $\partial(\overrightarrow{\mathcal{D}} \downarrow G\alpha)$ of $(\overrightarrow{\mathcal{D}} \downarrow G\alpha)$. □

The following is the main definition of this section; it is used in the statements of our main theorems (Theorem 1 on p. 22 and Theorem 2 on p. 22).

Definition 8 – Let $G\colon \mathcal{C} \to \mathcal{D}$ be a Reedy functor between Reedy categories.

1. The Reedy functor G is a *fibering Reedy functor* if for every object α in $\mathcal{C}$, every object β in $\mathcal{D}$, and every map $\sigma\colon G\alpha \to \beta$ in $\overleftarrow{\mathcal{D}}$, the nerve of $\mathrm{Fact}_{\overleftarrow{\mathcal{C}}}(\alpha,\sigma)$, the category of inverse $\mathcal{C}$-factorizations of (α,σ), (see Definition 7 on p. 26) is either empty or connected.

   If $\mathcal{C}$ is a Reedy subcategory of $\mathcal{D}$ and if the inclusion is a fibering Reedy functor, then we will call $\mathcal{C}$ a *fibering Reedy subcategory* of $\mathcal{D}$.

2. The Reedy functor G is a *cofibering Reedy functor* if for every object α in $\mathcal{C}$, every object β in $\mathcal{D}$, and every map $\sigma\colon \beta \to G\alpha$ in $\overrightarrow{\mathcal{D}}$, the nerve of $\mathrm{Fact}_{\overrightarrow{\mathcal{C}}}(\alpha,\sigma)$, the category of direct $\mathcal{C}$-factorizations of (α,σ), (see Definition 7 on p. 26) is either empty or connected.

   If $\mathcal{C}$ is a Reedy subcategory of $\mathcal{D}$ and if the inclusion is a cofibering Reedy functor, then we will call $\mathcal{C}$ a *cofibering Reedy subcategory* of $\mathcal{D}$.

Examples of fibering Reedy functors and of cofibering Reedy functors (and of Reedy functors that are not fibering and Reedy functors that are not cofibering) are given in Section 3.

2.4 Opposites

The results in this section will be used in the proof of Theorem 2 on p. 22, which can be found in Section 4.5.

Proposition 6 – *If $\mathcal{C}$ is a Reedy category, then the opposite category $\mathcal{C}^{\mathrm{op}}$ is a Reedy category in which $\overrightarrow{\mathcal{C}^{\mathrm{op}}} = (\overleftarrow{\mathcal{C}})^{\mathrm{op}}$ and $\overleftarrow{\mathcal{C}^{\mathrm{op}}} = (\overrightarrow{\mathcal{C}})^{\mathrm{op}}$.*

Proof. A degree function for $\mathcal{C}$ will serve as a degree function for $\mathcal{C}^{\mathrm{op}}$, and factorizations $\sigma = \tau\mu$ in $\mathcal{C}$ with $\mu \in \overleftarrow{\mathcal{C}}$ and $\tau \in \overrightarrow{\mathcal{C}}$ correspond to factorizations $\sigma^{\mathrm{op}} = \mu^{\mathrm{op}}\tau^{\mathrm{op}}$ in $\mathcal{C}^{\mathrm{op}}$ with $\mu^{\mathrm{op}} \in (\overleftarrow{\mathcal{C}})^{\mathrm{op}} = \overrightarrow{\mathcal{C}^{\mathrm{op}}}$ and $\tau^{\mathrm{op}} \in (\overrightarrow{\mathcal{C}})^{\mathrm{op}} = \overleftarrow{\mathcal{C}^{\mathrm{op}}}$. □

Proposition 7 – *If $\mathcal{C}$ and $\mathcal{D}$ are Reedy categories, then a functor $G\colon \mathcal{C} \to \mathcal{D}$ is a Reedy functor if and only if its opposite $G^{\mathrm{op}}\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$ is a Reedy functor.*

Proof. This follows from Proposition 6. □

Lemma 1 – *Let $G\colon \mathcal{C} \to \mathcal{D}$ be a Reedy functor between Reedy categories, let α be an object of $\mathcal{C}$, and let β be an object of $\mathcal{D}$.*

1. *If $\sigma\colon G\alpha \to \beta$ is a map in $\overleftarrow{\mathcal{D}}$, then the opposite of the category of inverse $\mathcal{C}$-factorizations of (α,σ) is the category of direct $\mathcal{C}^{\mathrm{op}}$-factorizations of $(\alpha,\sigma^{\mathrm{op}}\colon \beta \to G\alpha)$ in $\overrightarrow{\mathcal{D}^{\mathrm{op}}}$.*

2. *If* $\sigma\colon \beta \to G\alpha$ *is a map in* $\overrightarrow{\mathcal{D}}$*, then the opposite of the category of direct* $\mathcal{C}$*-factorizations of* (α, σ) *is the category of inverse* $\mathcal{C}^{\mathrm{op}}$*-factorizations of* $(\alpha, \sigma^{\mathrm{op}}\colon G\alpha \to \beta)$ *in* $\overleftarrow{\mathcal{D}^{\mathrm{op}}}$*.*

Proof. We will prove part (1); part (2) will then follow from applying part (1) to $\sigma^{\mathrm{op}}\colon G\alpha \to \beta$ in $\mathcal{C}^{\mathrm{op}}$ and remembering that $(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}} = \mathcal{C}$ and $(\mathcal{D}^{\mathrm{op}})^{\mathrm{op}} = \mathcal{D}$.

Let $\sigma\colon G\alpha \to \beta$ be a map in $\overleftarrow{\mathcal{D}}$. Recall from Definition 7 on p. 26 that

- an object of the category of inverse $\mathcal{C}$-factorizations of $(\alpha, \sigma\colon G\alpha \to \beta)$ is a pair

$$\big((\nu\colon \alpha \to \gamma), (\mu\colon G\gamma \to \beta)\big)$$

  consisting of a non-identity map $\nu\colon \alpha \to \gamma$ in $\overleftarrow{\mathcal{C}}$ and a map $\mu\colon G\gamma \to \beta$ in $\overleftarrow{\mathcal{D}}$ such that the composition $G\alpha \xrightarrow{G\nu} G\gamma \xrightarrow{\mu} \beta$ equals σ, and

- a map from $\big((\nu\colon \alpha \to \gamma), (\mu\colon G\gamma \to \beta)\big)$ to $\big((\nu'\colon \alpha \to \gamma'), (\mu'\colon G\gamma' \to \beta)\big)$ is a map $\tau\colon \gamma \to \gamma'$ in $\overleftarrow{\mathcal{C}}$ such that the triangles

$$\begin{array}{ccc} & \alpha & \\ {}^{\nu}\swarrow & & \searrow^{\nu'} \\ \gamma & \xrightarrow[\tau]{} & \gamma' \end{array} \qquad \text{and} \qquad \begin{array}{ccc} G\gamma & \xrightarrow{G\tau} & G\gamma' \\ {}_{\mu}\searrow & & \swarrow_{\mu'} \\ & \beta & \end{array}$$

  commute.

The opposite of this category has the same objects, but

- a non-identity map $\nu\colon \alpha \to \gamma$ in $\overleftarrow{\mathcal{C}}$ is equivalently a non-identity map $\nu^{\mathrm{op}}\colon \gamma \to \alpha$ in $(\overleftarrow{\mathcal{C}})^{\mathrm{op}} = \overrightarrow{\mathcal{C}^{\mathrm{op}}}$, and

- a factorization $G\alpha \xrightarrow{G\nu} G\gamma \xrightarrow{\mu} \beta$ of σ such that $\mu \in \overleftarrow{\mathcal{D}}$ is equivalently a factorization $\beta \xrightarrow{\mu^{\mathrm{op}}} G\gamma \xrightarrow{G\nu^{\mathrm{op}}} G\alpha$ of $\sigma^{\mathrm{op}}\colon \beta \to G\alpha$ in $(\overleftarrow{\mathcal{D}})^{\mathrm{op}} = \overrightarrow{\mathcal{D}^{\mathrm{op}}}$

Thus, the opposite category can be described as the category in which

- An object is a pair

$$\big((\nu^{\mathrm{op}}\colon \gamma \to \alpha), (\mu^{\mathrm{op}}\colon \beta \to G\gamma)\big)$$

  consisting of a non-identity map $\nu^{\mathrm{op}}\colon \gamma \to \alpha$ in $(\overleftarrow{\mathcal{C}})^{\mathrm{op}} = \overrightarrow{\mathcal{C}^{\mathrm{op}}}$ and a map $\mu^{\mathrm{op}}\colon \beta \to G\gamma$ in $(\overleftarrow{\mathcal{D}})^{\mathrm{op}} = \overrightarrow{\mathcal{D}^{\mathrm{op}}}$ such that the composition $\beta \xrightarrow{\mu^{\mathrm{op}}} G\gamma \xrightarrow{G\nu^{\mathrm{op}}} G\alpha$ equals σ^{op}, and

- a map from $\big((\nu^{\mathrm{op}}\colon \gamma \to \alpha), (\mu^{\mathrm{op}}\colon \beta \to G\gamma)\big)$ to $\big(((\nu')^{\mathrm{op}}\colon \gamma' \to \alpha), ((\mu')^{\mathrm{op}}\colon \beta \to G\gamma')\big)$ is a map $\tau^{\mathrm{op}}\colon \gamma' \to \gamma$ in $(\overleftarrow{\mathcal{C}})^{\mathrm{op}} = \overrightarrow{\mathcal{C}^{\mathrm{op}}}$ such that the triangles

$$\begin{array}{ccc} & \alpha & \\ {\scriptstyle \nu^{\mathrm{op}}}\nearrow & & \nwarrow{\scriptstyle \nu'^{\mathrm{op}}} \\ \gamma & \xleftarrow[\tau^{\mathrm{op}}]{} & \gamma' \end{array} \qquad \text{and} \qquad \begin{array}{ccc} G\gamma & \xleftarrow{G\tau^{\mathrm{op}}} & G\gamma' \\ {\scriptstyle \mu^{\mathrm{op}}}\nwarrow & & \nearrow{\scriptstyle \mu'^{\mathrm{op}}} \\ & \beta & \end{array}$$

  commute.

This is exactly the category of direct $\mathcal{C}^{\mathrm{op}}$-factorizations of $(\alpha, \sigma^{\mathrm{op}}\colon \beta \to G\alpha)$ in $\overrightarrow{\mathcal{D}^{\mathrm{op}}}$. □

Proposition 8 – *If $G\colon \mathcal{C} \to \mathcal{D}$ is a Reedy functor between Reedy categories, then G is a fibering Reedy functor if and only if $G^{\mathrm{op}}\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$ is a cofibering Reedy functor.*

Proof. Since the nerve of a category is empty or connected if and only if the nerve of the opposite category is, respectively, empty or connected, this follows from Lemma 1 on p. 30. □

Lemma 2 – *Let $\boldsymbol{X}$ be a $\mathcal{C}$-diagram in $\mathcal{M}$ (which can also be viewed as a $\mathcal{C}^{\mathrm{op}}$-diagram in $\mathcal{M}^{\mathrm{op}}$), and let α be an object of $\mathcal{C}$.*

1. *The latching object $\mathrm{L}^{\mathcal{C}}_{\alpha}$ of $\boldsymbol{X}$ as a $\mathcal{C}$-diagram in $\mathcal{M}$ at α is the matching object $\mathrm{M}^{\mathcal{C}^{\mathrm{op}}}_{\alpha}$ of $\boldsymbol{X}$ as a $\mathcal{C}^{\mathrm{op}}$-diagram in $\mathcal{M}^{\mathrm{op}}$ at α, and the opposite of the latching map $\mathrm{L}^{\mathcal{C}}_{\alpha}\boldsymbol{X} \to \boldsymbol{X}$ of $\boldsymbol{X}$ as a $\mathcal{C}$-diagram in $\mathcal{M}$ at α is the matching map $\boldsymbol{X} \to \mathrm{L}^{\mathcal{C}}_{\alpha}\boldsymbol{X} = \mathrm{M}^{\mathcal{C}^{\mathrm{op}}}_{\alpha}\boldsymbol{X}$ of $\boldsymbol{X}$ as a $\mathcal{C}^{\mathrm{op}}$-diagram in $\mathcal{M}^{\mathrm{op}}$ at α.*

2. *The matching object $\mathrm{M}^{\mathcal{C}}_{\alpha}$ of $\boldsymbol{X}$ as a $\mathcal{C}$-diagram in $\mathcal{M}$ at α is the latching object $\mathrm{L}^{\mathcal{C}^{\mathrm{op}}}_{\alpha}$ of $\boldsymbol{X}$ as a $\mathcal{C}^{\mathrm{op}}$-diagram in $\mathcal{M}^{\mathrm{op}}$ at α, and the opposite of the matching map $\boldsymbol{X} \to \mathrm{M}^{\mathcal{C}}_{\alpha}\boldsymbol{X}$ of $\boldsymbol{X}$ as a $\mathcal{C}$-diagram in $\mathcal{M}$ at α is the latching map $\mathrm{L}^{\mathcal{C}^{\mathrm{op}}}_{\alpha}\boldsymbol{X} = \mathrm{M}^{\mathcal{C}}_{\alpha}\boldsymbol{X} \to \boldsymbol{X}$ of $\boldsymbol{X}$ as a $\mathcal{C}^{\mathrm{op}}$-diagram in $\mathcal{M}^{\mathrm{op}}$ at α.*

Proof. We will prove part 1; part 2 then follows by applying part 1 to the $\mathcal{C}^{\mathrm{op}}$-diagram $\boldsymbol{X}$ in $\mathcal{M}^{\mathrm{op}}$ and remembering that $(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}} = \mathcal{C}$ and $(\mathcal{M}^{\mathrm{op}})^{\mathrm{op}} = \mathcal{M}$.

The latching object $\mathrm{L}^{\mathcal{C}}_{\alpha}\boldsymbol{X}$ of $\boldsymbol{X}$ at α is the colimit of the diagram in $\mathcal{M}$ with an object $\boldsymbol{X}_{\beta}$ for every non-identity map $\sigma\colon \beta \to \alpha$ in $\overrightarrow{\mathcal{C}}$ and a map $\mu_*\colon \boldsymbol{X}_{\beta} \to \boldsymbol{X}_{\gamma}$ for every commutative triangle

$$\begin{array}{ccc} & \alpha & \\ {\scriptstyle \sigma}\nearrow & & \nwarrow{\scriptstyle \tau} \\ \beta & \xrightarrow[\mu]{} & \gamma \end{array}$$

in $\overrightarrow{\mathcal{C}}$ in which σ and τ are non-identity maps. Thus, $\mathrm{L}^{\mathcal{C}}_{\alpha}\boldsymbol{X}$ can also be described as the limit of the diagram in $\mathcal{M}^{\mathrm{op}}$ with one object $\boldsymbol{X}_{\beta}$ for every non-identity map

$\sigma^{\mathrm{op}}\colon \alpha \to \beta$ in $(\overrightarrow{\mathcal{C}})^{\mathrm{op}} = \overleftarrow{\mathcal{C}^{\mathrm{op}}}$ and a map $(\mu^{\mathrm{op}})_*\colon \boldsymbol{X}_\gamma \to \boldsymbol{X}_\beta$ for every commutative triangle

$$\begin{array}{ccc} & \alpha & \\ {\scriptstyle \sigma^{\mathrm{op}}}\swarrow & & \searrow{\scriptstyle \tau^{\mathrm{op}}} \\ \beta & \underset{\mu^{\mathrm{op}}}{\longleftarrow} & \gamma \end{array}$$

in $(\overrightarrow{\mathcal{C}})^{\mathrm{op}} = \overleftarrow{\mathcal{C}^{\mathrm{op}}}$ in which σ^{op} and τ^{op} are non-identity maps. Thus, $\mathrm{L}^{\mathcal{C}}_\alpha \boldsymbol{X} = \mathrm{M}^{\mathcal{C}^{\mathrm{op}}}_\alpha \boldsymbol{X}$.

The latching map $\mathrm{L}^{\mathcal{C}}_\alpha \boldsymbol{X} \to \boldsymbol{X}_\alpha$ is the unique map in $\mathcal{M}$ such that for every non-identity map $\sigma\colon \beta \to \alpha$ in $\overrightarrow{\mathcal{C}}$ the triangle

$$\begin{array}{ccc} & & \boldsymbol{X}_\alpha \\ & {\scriptstyle \sigma_*}\nearrow & \uparrow \\ \boldsymbol{X}_\beta & \longrightarrow & \mathrm{L}^{\mathcal{C}}_\alpha \boldsymbol{X} \end{array}$$

commutes, and so the opposite of the latching map is the unique map $\boldsymbol{X}_\alpha \to \mathrm{L}^{\mathcal{C}}_\alpha \boldsymbol{X} = \mathrm{M}^{\mathcal{C}^{\mathrm{op}}}_\alpha \boldsymbol{X}$ in $\mathcal{M}^{\mathrm{op}}$ such that for every non-identity map $\sigma^{\mathrm{op}}\colon \alpha \to \beta$ in $(\overrightarrow{\mathcal{C}})^{\mathrm{op}} = \overleftarrow{\mathcal{C}^{\mathrm{op}}}$ the triangle

$$\begin{array}{ccc} & & \boldsymbol{X}_\alpha \\ & {\scriptstyle (\sigma^{\mathrm{op}})^*}\swarrow & \downarrow \\ \boldsymbol{X}_\beta & \longleftarrow & \mathrm{M}^{\mathcal{C}^{\mathrm{op}}}_\alpha \boldsymbol{X} \end{array}$$

commutes, i.e., the opposite of the latching map of $\boldsymbol{X}$ at α in $\mathcal{C}$ is the matching map of $\boldsymbol{X}$ at α in $\mathcal{C}^{\mathrm{op}}$. □

Lemma 3 – *Let $f\colon \boldsymbol{X} \to \boldsymbol{Y}$ be a map of $\mathcal{C}$-diagrams in $\mathcal{M}$ and let α be an object of $\mathcal{C}$.*

1. *The opposite of the relative latching map (see Definition 3 on p. 24) of f at α is the relative matching map of the map $f^{\mathrm{op}}\colon \boldsymbol{Y} \to \boldsymbol{X}$ of $\mathcal{C}^{\mathrm{op}}$-diagrams in $\mathcal{M}^{\mathrm{op}}$ at α.*
2. *The opposite of the relative matching map (see Definition 3 on p. 24) of f at α is the relative latching map of the map $f^{\mathrm{op}}\colon \boldsymbol{Y} \to \boldsymbol{X}$ of $\mathcal{C}^{\mathrm{op}}$-diagrams in $\mathcal{M}^{\mathrm{op}}$ at α.*

Proof. We will prove part (1); part (2) then follows by applying part (1) to the map of $\mathcal{C}^{\mathrm{op}}$-diagrams $f^{\mathrm{op}}\colon \boldsymbol{Y} \to \boldsymbol{X}$ in $\mathcal{M}^{\mathrm{op}}$ and remembering that $(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}} = \mathcal{C}$ and $(\mathcal{M}^{\mathrm{op}})^{\mathrm{op}} = \mathcal{M}$.

If $P = \boldsymbol{X}_\alpha \amalg_{\mathrm{L}^{\mathcal{C}}_\alpha \boldsymbol{X}} \mathrm{L}^{\mathcal{C}}_\alpha \boldsymbol{Y}$, then the relative latching map is the unique map $P \to \boldsymbol{Y}_\alpha$ that makes the diagram

$$\begin{array}{ccccc} \mathrm{L}^{\mathcal{C}}_\alpha \boldsymbol{X} & & \longrightarrow & & \mathrm{L}^{\mathcal{C}}_\alpha \boldsymbol{Y} \\ \downarrow & & & \swarrow & \downarrow \\ & & P & & \\ \downarrow & \nearrow & & \searrow & \downarrow \\ \boldsymbol{X}_\alpha & & \longrightarrow & & \boldsymbol{Y}_\alpha \end{array}$$

commute. The opposite of that diagram is the diagram

$$\begin{array}{ccccc}
\mathrm{M}_\alpha^{\mathcal{C}^{\mathrm{op}}}\boldsymbol{X} & & \longleftarrow & & \mathrm{M}_\alpha^{\mathcal{C}^{\mathrm{op}}}\boldsymbol{Y} \\
\uparrow & & & \nearrow & \uparrow \\
 & & P & & \\
\uparrow & \swarrow & & \nwarrow & \uparrow \\
\boldsymbol{X}_\alpha & & \longleftarrow & & \boldsymbol{Y}_\alpha
\end{array}$$

in $\mathcal{M}^{\mathrm{op}}$ (see Lemma 2 on p. 32), in which $P = \boldsymbol{X}_\alpha \times_{\mathrm{M}_\alpha^{\mathcal{C}^{\mathrm{op}}}\boldsymbol{X}} \mathrm{M}_\alpha^{\mathcal{C}^{\mathrm{op}}}\boldsymbol{Y}$, and the opposite of the relative latching map is the unique map in $\mathcal{M}^{\mathrm{op}}$ that makes this diagram commute, i.e., it is the relative matching map. □

Proposition 9 – *If $\mathcal{M}$ is a model category and $\mathcal{C}$ is a Reedy category, then the opposite $(\mathcal{M}^{\mathcal{C}})^{\mathrm{op}}$ of the Reedy model category $\mathcal{M}^{\mathcal{C}}$ (see Definition 2 on p. 23) is naturally isomorphic as a model category to the Reedy model category $(\mathcal{M}^{\mathrm{op}})^{\mathcal{C}^{\mathrm{op}}}$.*

Proof. The opposite $(\mathcal{M}^{\mathcal{C}})^{\mathrm{op}}$ of $\mathcal{M}^{\mathcal{C}}$ is a model category in which

- the cofibrations of $(\mathcal{M}^{\mathcal{C}})^{\mathrm{op}}$ are the opposites of the fibrations of $\mathcal{M}^{\mathcal{C}}$,
- the fibrations of $(\mathcal{M}^{\mathcal{C}})^{\mathrm{op}}$ are the opposites of the cofibrations of $\mathcal{M}^{\mathcal{C}}$, and
- the weak equivalences of $(\mathcal{M}^{\mathcal{C}})^{\mathrm{op}}$ are the opposites of the weak equivalences of $\mathcal{M}^{\mathcal{C}}$.

Proposition 6 on p. 30 implies that we have a Reedy model category structure on $(\mathcal{M}^{\mathrm{op}})^{\mathcal{C}^{\mathrm{op}}}$. The objects and maps of $(\mathcal{M}^{\mathcal{C}})^{\mathrm{op}}$ coincide with those of $(\mathcal{M}^{\mathrm{op}})^{(\mathcal{C}^{\mathrm{op}})}$, and so we need only show that the model category structures coincide. This follows because the opposites of the objectwise weak equivalences of $\mathcal{M}^{\mathcal{C}}$ are the objectwise weak equivalences of $(\mathcal{M}^{\mathrm{op}})^{\mathcal{C}^{\mathrm{op}}}$, and Lemma 3 on the previous page implies that the opposites of the cofibrations of $\mathcal{M}^{\mathcal{C}}$ are the fibrations of $(\mathcal{M}^{\mathrm{op}})^{\mathcal{C}^{\mathrm{op}}}$ and that the opposites of the fibrations of $\mathcal{M}^{\mathcal{C}}$ are the cofibrations of $(\mathcal{M}^{\mathrm{op}})^{\mathcal{C}^{\mathrm{op}}}$ (see Theorem 3 on p. 25). □

2.5 Quillen functors

Definition 9 – Let $\mathcal{M}$ and $\mathcal{N}$ be model categories and let $G\colon \mathcal{M} \rightleftarrows \mathcal{N} : U$ be a pair of adjoint functors. The functor G is a *left Quillen functor* and the functor U is a *right Quillen functor* if

- the left adjoint G preserves both cofibrations and trivial cofibrations, and
- the right adjoint U preserves both fibrations and trivial fibrations.

Proposition 10 – *If $\mathcal{M}$ and $\mathcal{N}$ are model categories and $G\colon \mathcal{M} \rightleftarrows \mathcal{N} : U$ is a pair of adjoint functors, then the following are equivalent:*

1. *The left adjoint G is a left Quillen functor and the right adjoint U is a right Quillen functor.*
2. *The left adjoint G preserves both cofibrations and trivial cofibrations.*
3. *The right adjoint U preserves both fibrations and trivial fibrations.*

Proof. This is Hirschhorn (2003, Prop. 8.5.3). □

Proposition 11 – *Let $\mathcal{M}$ and $\mathcal{N}$ be model categories and let $G\colon \mathcal{M} \rightleftarrows \mathcal{N} : U$ be a pair of adjoint functors.*

1. *If G is a left Quillen functor, then G takes cofibrant objects of $\mathcal{M}$ to cofibrant objects of $\mathcal{N}$ and takes weak equivalences between cofibrant objects in $\mathcal{M}$ to weak equivalences between cofibrant objects of $\mathcal{N}$.*
2. *If U is a right Quillen functor, then U takes fibrant objects of $\mathcal{N}$ to fibrant objects of $\mathcal{M}$ and takes weak equivalences between fibrant objects in $\mathcal{N}$ to weak equivalences between fibrant objects of $\mathcal{M}$.*

Proof. Since left adjoints take initial objects to initial objects, if the left adjoint G takes cofibrations to cofibrations then it takes cofibrant objects to cofibrant objects. The statement about weak equivalences follows from Hirschhorn (2003, Cor. 7.7.2).

Dually, since right adjoints take terminal objects to terminal objects, if the right adjoint U takes fibrations to fibrations then it takes fibrant objects to fibrant objects. The statement about weak equivalences follows from Hirschhorn (2003, Cor. 7.7.2). □

Proposition 12 – *A functor between model categories $G\colon \mathcal{M} \to \mathcal{N}$ is a left Quillen functor if and only if its opposite $G^{\mathrm{op}}\colon \mathcal{M}^{\mathrm{op}} \to \mathcal{N}^{\mathrm{op}}$ is a right Quillen functor.*

Proof. This follows because the cofibrations and trivial cofibrations of $\mathcal{M}^{\mathrm{op}}$ are the opposites of the fibrations and trivial fibrations, respectively, of $\mathcal{M}$ and the fibrations and trivial fibrations of $\mathcal{M}^{\mathrm{op}}$ are the opposites of the cofibrations and trivial cofibrations, respectively, of $\mathcal{M}$ (with a similar statement for $\mathcal{N}$). □

2.6 Cofinality

Definition 10 – Let $\mathcal{A}$ and $\mathcal{B}$ be small categories and let $G\colon \mathcal{A} \to \mathcal{B}$ be a functor.

- The functor G is *left cofinal* (or *initial*) if for every object α of $\mathcal{B}$ the nerve $\mathrm{N}(G\downarrow\alpha)$ of the overcategory $(G\downarrow\alpha)$ is non-empty and connected. If in addition G is the inclusion of a subcategory, then we will say that $\mathcal{A}$ is a *left cofinal subcategory* (or *initial subcategory*) of $\mathcal{B}$.

- The functor G is *right cofinal* (or *terminal*) if for every object α of $\mathcal{B}$ the nerve $\mathrm{N}(\alpha\downarrow G)$ of the undercategory $(\alpha\downarrow G)$ is non-empty and connected. If in addition G is the inclusion of a subcategory, then we will say that $\mathcal{A}$ is a *right cofinal subcategory* (or *terminal subcategory*) of $\mathcal{B}$.

For the proof of the following, see MacLane (1971, p. IX.3) or Hirschhorn (2003, Thm. 14.2.5).

Theorem 4 – *Let $\mathcal{A}$ and $\mathcal{B}$ be small categories and let $G\colon \mathcal{A} \to \mathcal{B}$ be a functor.*

1. *The functor G is left cofinal if and only if for every complete category $\mathcal{M}$ (i.e., every category in which all small limits exist) and every diagram $\boldsymbol{X}\colon \mathcal{B} \to \mathcal{M}$ the natural map $\lim_{\mathcal{B}} \boldsymbol{X} \to \lim_{\mathcal{A}} G^{*}\boldsymbol{X}$ is an isomorphism.*

2. *The functor G is right cofinal if and only if for every cocomplete category $\mathcal{M}$ (i.e., every category in which all small colimits exist) and every diagram $\boldsymbol{X}\colon \mathcal{B} \to \mathcal{M}$ the natural map $\operatorname{colim}_{\mathcal{A}} G^{*}\boldsymbol{X} \to \operatorname{colim}_{\mathcal{B}} \boldsymbol{X}$ is an isomorphism.*

3 Examples

In this section, we present various examples to illustrate Theorem 1 on p. 22 and Theorem 2 on p. 22.

3.1 A Reedy functor that is not fibering

The following is an example of a Reedy subcategory that is not fibering.

Example 1 – Let $\mathcal{D}$ be the category

$$\begin{array}{ccc} & \alpha & \\ {}^{p}\swarrow & & \searrow^{r} \\ \gamma & & \delta \\ {}_{q}\searrow & & \swarrow_{s} \\ & \beta & \end{array} \qquad \text{in which } qp = sr.$$

- Let α be of degree 2,
- let γ and δ be of degree 1, and
- let β be of degree 0.

$\mathcal{D}$ is then a Reedy category in which $\overleftarrow{\mathcal{D}} = \mathcal{D}$ and $\overrightarrow{\mathcal{D}}$ has only identity maps.

Let $\mathcal{C}$ be the full subcategory of $\mathcal{D}$ on the objects $\{\alpha, \gamma, \delta\}$, and let $\mathcal{C}$ have the structure of a Reedy category that makes it a Reedy subcategory of $\mathcal{D}$. Although $\mathcal{C}$ is a Reedy subcategory of $\mathcal{D}$, it is not a fibering Reedy subcategory because the map $qp\colon \alpha \to \beta$ in $\overleftarrow{\mathcal{D}}$ has only two factorizations in which the first map is in $\overleftarrow{\mathcal{C}}$ and is not an identity map and the second is in $\overleftarrow{\mathcal{D}}$, $q \circ p$ and $s \circ r$, and neither of those factorizations maps to the other; thus the nerve of the category of such factorizations is nonempty and not connected. Theorem 1 on p. 22 thus implies that there is a model category $\mathcal{M}$ such that the restriction functor $\mathcal{M}^{\mathcal{D}} \to \mathcal{M}^{\mathcal{C}}$ is not a right Quillen functor. This is actually proved in Theorem 13 on p. 45, which constructs a fibrant $\mathcal{D}$-diagram in the standard model category of topological spaces for which the induced $\mathcal{C}$-diagram is not fibrant. For the categories $\mathcal{C}$ and $\mathcal{D}$ of Example 1 on the preceding page, that $\mathcal{D}$-diagram is the functor that takes every object of $\mathcal{D}$ to I, the unit interval, and takes every morphism of $\mathcal{D}$ to the identity map. This is a fibrant diagram because every matching map is a homeomorphism, and is thus a fibration. The induced $\mathcal{C}$-diagram is not fibrant, though, because the matching map at α is the diagonal map $I \to I \times I$, which is not a fibration.

3.2 A Reedy functor that is not cofibering

Proposition 8 on p. 32 implies that the opposite of Example 1 on the preceding page is a Reedy subcategory that is not cofibering.

3.3 Truncations

Proposition 13 – *If $\mathcal{C}$ is a Reedy category and $n \geq 0$, then the inclusion functor $G\colon \mathcal{C}^{\leq n} \to \mathcal{C}$ (see Definition 4 on p. 25) is both a fibering Reedy functor and a cofibering Reedy functor.*

Proof. We will prove that the inclusion is a fibering Reedy functor; the proof that it is a cofibering Reedy functor is similar.

If $\mathrm{degree}(\alpha) \leq n$, then the inclusion functor $G\colon \mathcal{C}^{\leq n} \to \mathcal{C}$ induces an isomorphism of undercategories $G_*\colon (\alpha \downarrow \overleftarrow{\mathcal{C}^{\leq n}}) \to (\alpha \downarrow \overleftarrow{\mathcal{C}})$. Let $\sigma\colon \alpha \to \beta$ be a map in $\overleftarrow{\mathcal{C}}$. If σ is the identity map, then the category of inverse $\mathcal{C}$-factorizations of σ is empty; if σ is not an identity map, then the object $\big((\sigma\colon \alpha \to \beta), 1_\beta\big)$ is a terminal object of the category of inverse $\mathcal{C}$-factorizations of σ, and so the nerve of the category of inverse $\mathcal{C}$-factorizations of σ is connected. Thus, G is fibering. □

Proposition 14 – *If $\mathcal{M}$ is a model category, $\mathcal{C}$ is a Reedy category, and $n \geq 0$, then the restriction functor $\mathcal{M}^{\mathcal{C}} \to \mathcal{M}^{\mathcal{C}^{\leq n}}$ (see Definition 4 on p. 25) is both a left Quillen functor and a right Quillen functor.*

Proof. This follows from Proposition 13, Theorem 1 on p. 22, and Theorem 2 on p. 22. □

Proposition 14 extends to products of Reedy categories as follows.

Proposition 15 – *If $\mathcal{C}$ and $\mathcal{D}$ are Reedy categories, $\mathcal{M}$ is a model category, and $n \geq 0$, then the restriction functor $\mathcal{M}^{\mathcal{C}\times\mathcal{D}} \to \mathcal{M}^{(\mathcal{C}^{\leq n}\times\mathcal{D})}$ (see Definition 4 on p. 25) is both a left Quillen functor and a right Quillen functor.*

Proof. The category $\mathcal{M}^{\mathcal{C}\times\mathcal{D}}$ of $(\mathcal{C}\times\mathcal{D})$-diagrams in $\mathcal{M}$ is isomorphic as a model category to the category $(\mathcal{M}^{\mathcal{D}})^{\mathcal{C}}$ of $\mathcal{C}$-diagrams in $\mathcal{M}^{\mathcal{D}}$ (see Hirschhorn 2003, Thm. 15.5.2), and so the result follows from Proposition 14 on the previous page. □

Proposition 16 – *If $\mathcal{M}$ is a model category, m is a positive integer, and for $1 \leq i \leq m$ we have a Reedy category $\mathcal{C}_i$ and a nonnegative integer n_i, then the restriction functor*

$$\mathcal{M}^{\mathcal{C}_1\times\mathcal{C}_2\times\cdots\mathcal{C}_m} \longrightarrow \mathcal{M}^{\mathcal{C}_1^{\leq n_1}\times\mathcal{C}_2^{\leq n_2}\times\cdots\mathcal{C}_m^{\leq n_m}}$$

(see Definition 4 on p. 25) is both a left Quillen functor and a right Quillen functor.

Proof. The restriction functor is the composition of the restriction functors

$$\begin{aligned}\mathcal{M}^{\mathcal{C}_1\times\mathcal{C}_2\times\cdots\mathcal{C}_m} &\longrightarrow \mathcal{M}^{\mathcal{C}_1^{\leq n_1}\times\mathcal{C}_2\times\cdots\mathcal{C}_m}\\ &\longrightarrow \mathcal{M}^{\mathcal{C}_1^{\leq n_1}\times\mathcal{C}_2^{\leq n_2}\times\cdots\mathcal{C}_m} \longrightarrow \cdots \longrightarrow \mathcal{M}^{\mathcal{C}_1^{\leq n_1}\times\mathcal{C}_2^{\leq n_2}\times\cdots\mathcal{C}_m^{\leq n_m}}\end{aligned}$$

and so the result follows from Proposition 15. □

3.4 Skeleta

Definition 11 – Let $\mathcal{C}$ be a Reedy category, let $n \geq 0$, and let $\mathcal{M}$ be a model category.

1. Since $\mathcal{M}$ is cocomplete, the restriction functor $\mathcal{M}^{\mathcal{C}} \to \mathcal{M}^{\mathcal{C}^{\leq n}}$ has a left adjoint $\mathbf{L}\colon \mathcal{M}^{\mathcal{C}^{\leq n}} \to \mathcal{M}^{\mathcal{C}}$ (see Borceux 1994, Thm. 3.7.2), and we define the *n-skeleton functor* $\mathrm{sk}_n\colon \mathcal{M}^{\mathcal{C}} \to \mathcal{M}^{\mathcal{C}}$ to be the composition
$$\mathcal{M}^{\mathcal{C}} \xrightarrow{\text{restriction}} \mathcal{M}^{\mathcal{C}^{\leq n}} \xrightarrow{\quad\mathbf{L}\quad} \mathcal{M}^{\mathcal{C}}.$$

2. Since $\mathcal{M}$ is complete, the restriction functor $\mathcal{M}^{\mathcal{C}} \to \mathcal{M}^{\mathcal{C}^{\leq n}}$ has a right adjoint $\mathbf{R}\colon \mathcal{M}^{\mathcal{C}^{\leq n}} \to \mathcal{M}^{\mathcal{C}}$ (see Borceux 1994, Thm. 3.7.2), and we define the *n-coskeleton functor* $\mathrm{cosk}_n\colon \mathcal{M}^{\mathcal{C}} \to \mathcal{M}^{\mathcal{C}}$ to be the composition
$$\mathcal{M}^{\mathcal{C}} \xrightarrow{\text{restriction}} \mathcal{M}^{\mathcal{C}^{\leq n}} \xrightarrow{\quad\mathbf{R}\quad} \mathcal{M}^{\mathcal{C}}.$$

Proposition 17 – *If $\mathcal{C}$ is a Reedy category, $n \geq 0$, and $\mathcal{M}$ is a model category, then*

1. *the n-skeleton functor $\mathrm{sk}_n\colon \mathcal{M} \to \mathcal{M}$ is a left Quillen functor, and*

2. *the n-coskeleton functor $\mathrm{cosk}_n\colon \mathcal{M} \to \mathcal{M}$ is a right Quillen functor.*

Proof. Since the restriction functor is a right Quillen functor (see Proposition 14 on p. 37), its left adjoint is a left Quillen functor (see Proposition 10 on p. 35). Since the restriction is also a left Quillen functor (see Proposition 14 on p. 37), its composition with its left adjoint is a left Quillen functor. Similarly, the composition of restriction with its right adjoint is a right Quillen functor. □

3.5 (Multi)cosimplicial and (multi)simplicial objects

In this section we consider simplicial and cosimplicial diagrams, as well as their multidimensional versions, m-cosimplicial and m-simplicial diagrams (see Definition 12). Simplicial and cosimplicial diagrams are standard tools in homotopy theory, while m-simplicial and m-cosimplicial ones have seen an increase in usage in recent years, most notably through their appearance in the calculus of functors (see Eldred 2013, Koytcheff, Munson, and Volić 2013).

The important questions are whether the restrictions to various subdiagrams of m-simplicial and m-cosimplicial diagrams are Quillen functors (and the answer will be yes in all cases that we consider here). The subdiagrams we will look at are the restricted (co)simplicial objects, diagonals of m-(co)simplicial objects, and slices of m-(co)simplicial objects. These are considered separately below. In particular, the fibrancy of the slices of a fibrant m-dimensional cosimplicial object is needed to justify taking its totalization one dimension at a time, as is done in both Eldred (2013) and Koytcheff, Munson, and Volić (2013). This and some further results about totalizations of m-cosimplicial objects will be addressed in future work.

We begin by recalling the definitions:

Definition 12 – For every nonnegative integer n, we let $[n]$ denote the ordered set $(0,1,2,\dots,n)$.

1. The *cosimplicial indexing category* $\mathbf{\Delta}$ is the category with objects the $[n]$ for $n \geq 0$ and with $\mathbf{\Delta}\big([n],[k]\big)$ the set of weakly monotone functions $[n] \to [k]$.

2. A *cosimplicial object* in a category $\mathcal{M}$ is a functor from $\mathbf{\Delta}$ to $\mathcal{M}$.

3. If m is a positive integer, then an *m-cosimplicial object* in $\mathcal{M}$ is a functor from $\mathbf{\Delta}^m$ to $\mathcal{M}$.

4. The *simplicial indexing category* $\mathbf{\Delta}^{\mathrm{op}}$, the opposite category of $\mathbf{\Delta}$.

5. A *simplicial object* in a category $\mathcal{M}$ is a functor from $\mathbf{\Delta}^{\mathrm{op}}$ to $\mathcal{M}$.

6. If m is a positive integer, then an *m-simplicial object* in $\mathcal{M}$ is a functor from $(\mathbf{\Delta}^m)^{\mathrm{op}} = (\mathbf{\Delta}^{\mathrm{op}})^m$ to $\mathcal{M}$.

Definition 13 – The *standard Reedy category structure* on the cosimplicial indexing category $\mathbf{\Delta}$ (see Definition 12 on the previous page) is the one in which

- the direct subcategory $\overrightarrow{\mathbf{\Delta}}$ consists of the injective functions and
- the inverse subcategory $\overleftarrow{\mathbf{\Delta}}$ consists of the surjective functions,

and the *standard degree function* assigns the object $[n]$ degree n.

Restricted cosimplicial objects and restricted simplicial objects

For examples of fibering Reedy subcategories and cofibering Reedy subcategories that include all of the objects, we consider the restricted cosimplicial (or semi-cosimplicial) and restricted simplicial (or semi-simplicial) indexing categories.

Definition 14 – For n a nonnegative integer, we denote the ordered set $(0,1,2,\ldots,n)$ by $[n]$.

1. The *restricted cosimplicial indexing category* $\mathbf{\Delta}_{\mathrm{rest}}$ is the category with objects the ordered sets $[n]$ for $n \geq 0$ and with $\mathbf{\Delta}_{\mathrm{rest}}\big([n],[k]\big)$ the *injective* order preserving maps $[n] \to [k]$.

   The category $\mathbf{\Delta}_{\mathrm{rest}}$ is thus a subcategory of $\mathbf{\Delta}$, the cosimplicial indexing category (see Definition 12 on the previous page). In fact, $\mathbf{\Delta}_{\mathrm{rest}} = \overrightarrow{\mathbf{\Delta}}$, the direct subcategory of $\mathbf{\Delta}$ (see Definition 13).

2. The *restricted simplicial indexing category* $\mathbf{\Delta}^{\mathrm{op}}_{\mathrm{rest}}$ is the opposite of the restricted cosimplicial indexing category.

3. If $\mathcal{M}$ is a category, then a *restricted cosimplicial object* in $\mathcal{M}$ is a functor from $\mathbf{\Delta}_{\mathrm{rest}}$ to $\mathcal{M}$.

4. If $\mathcal{M}$ is a category, a *restricted simplicial object* in $\mathcal{M}$ is a functor from $(\mathbf{\Delta}_{\mathrm{rest}})^{\mathrm{op}}$ to $\mathcal{M}$.

If we let $G\colon \mathbf{\Delta}_{\mathrm{rest}} \to \mathbf{\Delta}$ be the inclusion, then for $\boldsymbol{X}$ a cosimplicial object in $\mathcal{M}$ the induced diagram $G^*\boldsymbol{X}$ is a restricted cosimplicial object in $\mathcal{M}$, called the *underlying restricted cosimplicial object* of $\boldsymbol{X}$; it is obtained from $\boldsymbol{X}$ by "forgetting the codegeneracy operators". Similarly, if we let $G\colon \mathbf{\Delta}^{\mathrm{op}}_{\mathrm{rest}} \to \mathbf{\Delta}^{\mathrm{op}}$ be the inclusion, then for $\boldsymbol{Y}$ a simplicial object in $\mathcal{M}$ the induced diagram $G^*\boldsymbol{Y}$ is a restricted simplicial object in $\mathcal{M}$, called the *underlying restricted simplicial object* of $\boldsymbol{Y}$, obtained from $\boldsymbol{Y}$ by "forgetting the degeneracy operators".

Proposition 18 – *Let $\mathcal{D}$ be a Reedy category and let $\mathcal{C} = \overrightarrow{\mathcal{D}}$, the direct subcategory of $\mathcal{D}$.*

1. *The inclusion $\mathcal{C} \to \mathcal{D}$ is both a fibering Reedy functor and a cofibering Reedy functor.*

2. *The inclusion $\mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$ is both a fibering Reedy functor and a cofibering Reedy functor.*

Proof. We will prove part 1; part 2 will then follow from Proposition 8 on p. 32.

We first prove that the inclusion $\mathcal{C} \to \mathcal{D}$ is the inclusion of a cofibering Reedy subcategory. Let $\sigma\colon \beta \to \alpha$ be a map in $\mathcal{D}$. If σ is an identity map, then the category of direct $\mathcal{C}$-factorizations of σ is empty. If σ is not an identity map, then $\big((\sigma\colon \beta \to \alpha), 1_\beta\big)$ is an object of the category of direct $\mathcal{C}$-factorizations of σ that maps to every other object of that category, and so the nerve of that category is connected.

We now prove that the inclusion $\mathcal{C} \to \mathcal{D}$ is the inclusion of a fibering Reedy subcategory. Let $\sigma\colon \alpha \to \beta$ be a map in $\overleftarrow{\mathcal{D}}$. Since there are no non-identity maps in $\mathcal{C}$, the category of inverse $\mathcal{C}$-factorizations of σ is empty. □

Theorem 5 –

1. *The inclusion $\mathbf{\Delta}_{\mathrm{rest}} \to \mathbf{\Delta}$ of the restricted cosimplicial indexing category into the cosimplicial indexing category is both a fibering Reedy functor and a cofibering Reedy functor.*
2. *The inclusion $\mathbf{\Delta}^{\mathrm{op}}_{\mathrm{rest}} \to \mathbf{\Delta}^{\mathrm{op}}$ of the restricted simplicial indexing category into the simplicial indexing category is both a fibering Reedy functor and a cofibering Reedy functor.*

Proof. This follows from Proposition 18 on the preceding page. □

Theorem 6 – *Let $\mathcal{M}$ be a model category.*

1. *The functor $\mathcal{M}^{\mathbf{\Delta}} \to \mathcal{M}^{\mathbf{\Delta}_{\mathrm{rest}}}$ that "forgets the codegeneracies" of a cosimplicial object is both a left Quillen functor and a right Quillen functor.*
2. *The functor $\mathcal{M}^{\mathbf{\Delta}^{\mathrm{op}}} \to \mathcal{M}^{\mathbf{\Delta}^{\mathrm{op}}_{\mathrm{rest}}}$ that "forgets the degeneracies" of a simplicial object is both a left Quillen functor and a right Quillen functor.*

Proof. This follows from Theorem 5, Theorem 1 on p. 22, and Theorem 2 on p. 22. □

Diagonals of multicosimplicial and multisimplicial objects

Definition 15 – Let m be a positive integer.

1. The *diagonal embedding* of the category $\mathbf{\Delta}$ into $\mathbf{\Delta}^m$ is the functor $D\colon \mathbf{\Delta} \to \mathbf{\Delta}^m$ that takes the object $[k]$ of $\mathbf{\Delta}$ to the object $\big(\underbrace{[k],[k],\ldots,[k]}_{m \text{ times}}\big)$ of $\mathbf{\Delta}^m$ and the morphism $\phi\colon [p] \to [q]$ of $\mathbf{\Delta}$ to the morphism (ϕ^m) of $\mathbf{\Delta}^m$.
2. If $\mathcal{M}$ is a category and $\boldsymbol{X}$ is an m-cosimplicial object in $\mathcal{M}$, then the *diagonal* $\operatorname{diag}\boldsymbol{X}$ of $\boldsymbol{X}$ is the cosimplicial object in $\mathcal{M}$ that is the composition

   $$\mathbf{\Delta} \xrightarrow{D} \mathbf{\Delta}^m \xrightarrow{\boldsymbol{X}} \mathcal{M},$$

   so that $(\operatorname{diag}\boldsymbol{X})^k = \boldsymbol{X}^{(k,k,\ldots,k)}$.

3. If $\mathcal{M}$ is a category and $\boldsymbol{X}$ is an m-simplicial object in $\mathcal{M}$, then the *diagonal* $\operatorname{diag}\boldsymbol{X}$ of $\boldsymbol{X}$ is the simplicial object in $\mathcal{M}$ that is the composition

$$\boldsymbol{\Delta}^{\mathrm{op}} \xrightarrow{D^{\mathrm{op}}} (\boldsymbol{\Delta}^m)^{\mathrm{op}} = (\boldsymbol{\Delta}^{\mathrm{op}})^m \xrightarrow{\boldsymbol{X}} \mathcal{M},$$

so that $(\operatorname{diag}\boldsymbol{X})_k = \boldsymbol{X}_{(k,k,\ldots,k)}$.

Theorem 7 – *Let m be a positive integer.*

1. *The diagonal embedding $D\colon \boldsymbol{\Delta} \to \boldsymbol{\Delta}^m$ is a fibering Reedy functor.*
2. *The diagonal embedding $D^{\mathrm{op}}\colon \boldsymbol{\Delta}^{\mathrm{op}} \to (\boldsymbol{\Delta}^m)^{\mathrm{op}} = (\boldsymbol{\Delta}^{\mathrm{op}})^m$ is a cofibering Reedy functor.*

Proof. We will prove part 1; part 2 will then follow from Proposition 8 on p. 32.

We will identify $\boldsymbol{\Delta}$ with its image in $\boldsymbol{\Delta}^m$, so that the objects of $\boldsymbol{\Delta}$ are the m-tuples $\big([k],[k],\ldots,[k]\big)$. If $(\alpha_1,\alpha_2,\ldots,\alpha_m)\colon \big([k],[k],\ldots,[k]\big) \to \big([p_1],[p_2],\ldots,[p_m]\big)$ is a map in $\overleftarrow{\boldsymbol{\Delta}^m}$, then Hirschhorn (2017, Lem. 5.1) implies that it has a terminal factorization through a diagonal object of $\boldsymbol{\Delta}^m$. If that terminal factorization is through the identity map of $\big([k],[k],\ldots,[k]\big)$, then the category of inverse $\boldsymbol{\Delta}$-factorizations of $(\alpha_1,\alpha_2,\ldots,\alpha_m)$ is empty; if that terminal factorization is not through the identity map, then it is a terminal object of the category of inverse $\boldsymbol{\Delta}$-factorizations of $(\alpha_1,\alpha_2,\ldots,\alpha_m)$, and so the nerve of that category is connected. □

Part 1 of the following corollary appears in Hirschhorn (2017).

Corollary 1 – *Let m be a positive integer and let $\mathcal{M}$ be a model category.*

1. *The functor that takes an m-cosimplicial object in $\mathcal{M}$ to its diagonal cosimplicial object is a right Quillen functor.*
2. *The functor that takes an m-simplicial object in $\mathcal{M}$ to its diagonal simplicial object is a left Quillen functor.*

Proof. This follows from Theorem 7, Theorem 1 on p. 22, and Theorem 2 on p. 22. □

Slices of multicosimplicial and multisimplicial objects

Definition 16 – Let n be a positive integer and for $1 \le i \le n$ let $\mathcal{C}_i$ be a category. If K is a subset of $\{1,2,\ldots,n\}$, then a *K-slice* of the product category $\prod_{i=1}^{n} \mathcal{C}_i$ is the category $\prod_{i\in K} \mathcal{C}_i$. (If K consists of a single integer j, then we will use the term *j-slice* to refer to the K-slice.) An *inclusion of the K-slice* is a functor $\prod_{i\in K} \mathcal{C}_i \to \prod_{i=1}^{n} \mathcal{C}_i$ defined by choosing an object α_i of $\mathcal{C}_i$ for $i \in \big(\{1,2,\ldots,n\} - K\big)$ and inserting α_i into the i'th coordinate for $i \in \big(\{1,2,\ldots,n\} - K\big)$.

Theorem 8 – *Let n be a positive integer and for $1 \le i \le n$ let $\mathcal{C}_i$ be a Reedy category. For every subset K of $\{1,2,\ldots,n\}$ both the product $\prod_{i=1}^{n} \mathcal{C}_i$ and the product $\prod_{i\in K} \mathcal{C}_i$ are Reedy categories (see Hirschhorn 2003, Prop. 15.1.6), and every inclusion of a K-slice $\prod_{i\in K} \mathcal{C}_i \to \prod_{i=1}^{n} \mathcal{C}_i$ (see Definition 16 on the preceding page) is both a fibering Reedy functor and a cofibering Reedy functor.*

Proof. We will show that every inclusion is a fibering Reedy functor; the proof that it is a cofibering Reedy functor is similar (and also follows from applying the fibering case to the inclusion $\prod_{i\in K} \mathcal{C}_i^{\mathrm{op}} \to \prod_{i=1}^{n} \mathcal{C}_i^{\mathrm{op}}$; see Proposition 8 on p. 32). We will assume that $K = \{1,2\}$; the other cases are similar.

Let $(\beta_1,\beta_2,\alpha_3,\alpha_4,\ldots,\alpha_n)$ be an object of $\prod_{i\in K} \mathcal{C}_i$ and let

$$(\sigma_1,\sigma_2,\ldots,\sigma_n)\colon (\beta_1,\beta_2,\alpha_3,\alpha_4,\ldots,\alpha_n) \longrightarrow (\gamma_1,\gamma_2,\ldots,\gamma_n)$$

be a map in $\overleftarrow{\prod_{i=1}^{n} \mathcal{C}_i}$. Since $\overleftarrow{\prod_{i=1}^{n} \mathcal{C}_i} = \prod_{i=1}^{n} \overleftarrow{\mathcal{C}_i}$, each $\sigma_i \in \overleftarrow{\mathcal{C}_i}$. If σ_1 and σ_2 are both identity maps, then the category of inverse $\prod_{i\in K} \mathcal{C}_i$-factorizations of $(\sigma_1,\sigma_2,\ldots,\sigma_n)$ is empty. Otherwise, the category of inverse $\prod_{i\in K} \mathcal{C}_i$-factorizations of $(\sigma_1,\sigma_2,\ldots,\sigma_n)$ contains the object

$$(\beta_1,\beta_2,\alpha_3,\alpha_4,\ldots,\alpha_n) \xrightarrow{(\sigma_1,\sigma_2,1_{\alpha_3},1_{\alpha_4},\ldots,1_{\alpha_n})} (\gamma_1,\gamma_2,\alpha_3,\alpha_4,\ldots,\alpha_n) \xrightarrow{(1_{\gamma_1},1_{\gamma_2},\sigma_3,\sigma_4,\ldots,\sigma_n)} (\gamma_1,\gamma_2,\ldots,\gamma_n)$$

and every other object of the category of inverse $\prod_{i\in K} \mathcal{C}_i$-factorizations of $(\sigma_1,\sigma_2,\ldots,\sigma_n)$ maps to this one. Thus the nerve of the category of inverse $\prod_{i\in K} \mathcal{C}_i$-factorizations of $(\sigma_1,\sigma_2,\ldots,\sigma_n)$ is connected. □

Theorem 9 – *If $\mathcal{M}$ is a model category, n, $\mathcal{C}_i$ for $1 \le i \le n$, and K are as in Theorem 8, and the functor $\prod_{i\in K} \mathcal{C}_i \to \prod_{i=1}^{n} \mathcal{C}_i$ is the inclusion of a K-slice, then the restriction functor*

$$\mathcal{M}^{(\prod_{i=1}^{n} \mathcal{C}_i)} \longrightarrow \mathcal{M}^{(\prod_{i\in K} \mathcal{C}_i)}$$

is both a left Quillen functor and a right Quillen functor.

Proof. This follows from Theorem 1 on p. 22, Theorem 2 on p. 22, and Theorem 8. □

Definition 17 – Let $\mathcal{M}$ be a model category and let m be a positive integer.

1. If $\boldsymbol{X}$ is an m-cosimplicial object in $\mathcal{M}$, then a *slice* of $\boldsymbol{X}$ is a cosimplicial object in $\mathcal{M}$ defined by restricting all but one factor of $\boldsymbol{\Delta}^m$.

2. If $\boldsymbol{X}$ is an m-simplicial object in $\mathcal{M}$, then a *slice* of $\boldsymbol{X}$ is a simplicial object in $\mathcal{M}$ defined by restricting all but one factor of $(\boldsymbol{\Delta}^{\mathrm{op}})^m$.

Theorem 10 – *Let $\mathcal{M}$ be a model category and let m be a positive integer.*

1. *The functor $\mathcal{M}^{\mathbf{\Delta}^m} \to \mathcal{M}^{\mathbf{\Delta}}$ that restricts a multicosimplicial object to a slice (see Definition 17 on the previous page) is a both a left Quillen functor and a right Quillen functor.*

2. *The functor $\mathcal{M}^{(\mathbf{\Delta}^{\mathrm{op}})^m} \to \mathcal{M}^{\mathbf{\Delta}^{\mathrm{op}}}$ that restricts a multisimplicial object to a slice is both a left Quillen functor and a right Quillen functor.*

Proof. This follows from Theorem 9 on the previous page. □

Corollary 2 – *Let $\mathcal{M}$ be a model category and let m be a positive integer.*

1. *If $\boldsymbol{X}$ is a cofibrant m-cosimplicial object in $\mathcal{M}$, then every slice of $\boldsymbol{X}$ is a cofibrant cosimplicial object.*

2. *If $\boldsymbol{X}$ is a fibrant m-cosimplicial object in $\mathcal{M}$, then every slice of $\boldsymbol{X}$ is a fibrant cosimplicial object.*

3. *If $\boldsymbol{X}$ is a cofibrant m-simplicial object in $\mathcal{M}$, then every slice of $\boldsymbol{X}$ is a cofibrant simplicial object.*

4. *If $\boldsymbol{X}$ is a fibrant m-simplicial object in $\mathcal{M}$, then every slice of $\boldsymbol{X}$ is a fibrant simplicial object.*

Proof. This follows from Theorem 10. □

4 Proofs of the main theorems

Our main result, Theorem 1 on p. 22, will follow immediately from Theorem 11 below (the latter is an elaboration of the former). The proof of its dual, Theorem 2 on p. 22, will use Theorem 1 and can be found in Section 4.5.

Theorem 11 – *If $G\colon \mathcal{C} \to \mathcal{D}$ is a Reedy functor between Reedy categories, then the following are equivalent:*

1. *The functor G is a fibering Reedy functor (see Definition 8 on p. 30).*

2. *For every model category $\mathcal{M}$ the induced functor of diagram categories $G^*\colon \mathcal{M}^{\mathcal{D}} \to \mathcal{M}^{\mathcal{C}}$ is a right Quillen functor.*

3. *For every model category $\mathcal{M}$ the induced functor of diagram categories $G^*\colon \mathcal{M}^{\mathcal{D}} \to \mathcal{M}^{\mathcal{C}}$ takes fibrant objects of $\mathcal{M}^{\mathcal{D}}$ to fibrant objects of $\mathcal{M}^{\mathcal{C}}$.*

Proof. The proof will be completed by Proposition 11 on p. 35 and the proofs of Theorems 12 and 13 on the next page. More precisely, we will have

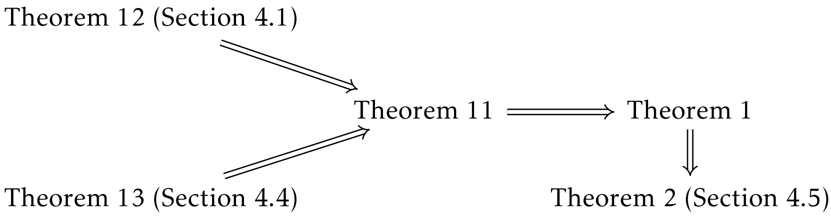
$$(1) \xRightarrow{\text{Theorem 12}} (2) \xRightarrow{\text{Proposition 11}} (3) \xRightarrow{\text{Theorem 13}} (1)$$ □

Theorem 12 – *If $G\colon \mathcal{C} \to \mathcal{D}$ is a fibering Reedy functor and $\mathcal{M}$ is a model category, then the induced functor of diagram categories $G^*\colon \mathcal{M}^{\mathcal{D}} \to \mathcal{M}^{\mathcal{C}}$ is a right Quillen functor.*

Theorem 13 – *If $G\colon \mathcal{C} \to \mathcal{D}$ is a Reedy functor that is not a fibering Reedy functor, then there is a fibrant $\mathcal{D}$-diagram of topological spaces for which the induced $\mathcal{C}$-diagram is not fibrant.*

The proof of Theorem 12 is given in Section 4.1, while the proof of Theorem 13 can be found in Section 4.4.

In summary, the proofs of our main results, Theorems 1 and 2 on p. 22, thus have the following structure:



4.1 Proof of Theorem 12

We work backward, first giving the proof of the main result. The completion of that proof will depend on two key assertions, Proposition 19 on p. 47 and Proposition 22 on p. 55, whose proofs are given in Sections 4.2 and 4.3. The assumption that we have a fibering Reedy functor is used only in the proofs of Proposition 19 on p. 47 and Proposition 20 on p. 50 (the latter is used in the proof of the former).

Proof (Proof of Theorem 12*).* Since $\mathcal{M}$ is cocomplete, the left adjoint of G^* exists (see Borceux 1994, Thm. 3.7.2 or MacLane 1971, p. 235). Thus, to show that the induced functor $\mathcal{M}^{\mathcal{D}} \to \mathcal{M}^{\mathcal{C}}$ is a right Quillen functor, we need only show that it preserves fibrations and trivial fibrations (see Proposition 10 on p. 35). Since the weak equivalences in $\mathcal{M}^{\mathcal{D}}$ and $\mathcal{M}^{\mathcal{C}}$ are the objectwise ones, any weak equivalence in $\mathcal{M}^{\mathcal{D}}$ induces a weak equivalence in $\mathcal{M}^{\mathcal{C}}$. Thus, if we show that the induced functor preserves fibrations, then we will also know that it takes maps that are both fibrations and weak equivalences to maps that are both fibrations and weak equivalences, i.e., that it also preserves trivial fibrations.

To show that the induced functor $\mathcal{M}^{\mathcal{D}} \to \mathcal{M}^{\mathcal{C}}$ preserves fibrations, let $\boldsymbol{X} \to \boldsymbol{Y}$ be a fibration of $\mathcal{D}$-diagrams in $\mathcal{M}$; we will let $G^*\boldsymbol{X}$ and $G^*\boldsymbol{Y}$ denote the induced diagrams on $\mathcal{C}$. For every object α of $\mathcal{C}$, the matching objects of $\boldsymbol{X}$ and $\boldsymbol{Y}$ at α in $\mathcal{M}^{\mathcal{C}}$ are

$$\mathrm{M}^{\mathcal{C}}_{\alpha} G^*\boldsymbol{X} = \lim_{\partial(\alpha\downarrow\overleftarrow{\mathcal{C}})} G^*\boldsymbol{X} \qquad \text{and} \qquad \mathrm{M}^{\mathcal{C}}_{\alpha} G^*\boldsymbol{Y} = \lim_{\partial(\alpha\downarrow\overleftarrow{\mathcal{C}})} G^*\boldsymbol{Y}$$

and we define $P^{\mathcal{C}}_{\alpha}$ by letting the diagram

$$\begin{array}{ccc} P^{\mathcal{C}}_{\alpha} & \dashrightarrow & (G^{*}\boldsymbol{Y})_{\alpha} \\ \downarrow & & \downarrow \\ \mathrm{M}^{\mathcal{C}}_{\alpha} G^{*}\boldsymbol{X} & \longrightarrow & \mathrm{M}^{\mathcal{C}}_{\alpha} G^{*}\boldsymbol{Y} \end{array} \tag{1}$$

be a pullback; we must show that the relative matching map $(G^{*}\boldsymbol{X})_{\alpha} \to P^{\mathcal{C}}_{\alpha}$ is a fibration (see Theorem 3 on p. 25), and there are two cases:

1. There is a non-identity map $\alpha \to \gamma$ in $\overleftarrow{\mathcal{C}}$ that G takes to the identity map of $G\alpha$.
2. G takes every non-identity map $\alpha \to \gamma$ in $\overleftarrow{\mathcal{C}}$ to a non-identity map in $\overleftarrow{\mathcal{D}}$.

In the first case, Proposition 22 on p. 55 (in Section 4.3 below) implies that the pullback Diagram 1 is isomorphic to the diagram

$$\begin{array}{ccc} P^{\mathcal{C}}_{\alpha} & \longrightarrow & (G^{*}\boldsymbol{Y})_{\alpha} \\ \downarrow & & \downarrow {\scriptstyle 1_{(G^{*}\boldsymbol{Y})_{\alpha}}} \\ (G^{*}\boldsymbol{X})_{\alpha} & \longrightarrow & (G^{*}\boldsymbol{Y})_{\alpha} \end{array}$$

in which the vertical map on the left is an isomorphism $P^{\mathcal{C}}_{\alpha} \approx (G^{*}\boldsymbol{X})_{\alpha}$. Thus, the composition of the relative matching map with that isomorphism is the identity map of $(G^{*}\boldsymbol{X})_{\alpha}$, and so the relative matching map is an isomorphism $(G^{*}\boldsymbol{X})_{\alpha} \to P^{\mathcal{C}}_{\alpha}$, and is thus a fibration.

We are left with the second case, and so we can assume that G takes every non-identity map $\alpha \to \gamma$ in $\overleftarrow{\mathcal{C}}$ to a non-identity map in $\overleftarrow{\mathcal{D}}$. In this case, G induces a functor $G_{*} \colon \partial(\alpha \downarrow \overleftarrow{\mathcal{C}}) \to \partial(G\alpha \downarrow \overleftarrow{\mathcal{D}})$ that takes the object $f \colon \alpha \to \gamma$ of $\partial(\alpha \downarrow \overleftarrow{\mathcal{C}})$ to the object $Gf \colon G\alpha \to G\gamma$ of $\partial(G\alpha \downarrow \overleftarrow{\mathcal{D}})$ (see Proposition 5 on p. 29).

The matching objects of $\boldsymbol{X}$ and $\boldsymbol{Y}$ at $G\alpha$ in $\mathcal{M}^{\mathcal{D}}$ are

$$\mathrm{M}^{\mathcal{D}}_{G\alpha}\boldsymbol{X} = \lim_{\partial(G\alpha \downarrow \overleftarrow{\mathcal{D}})} \boldsymbol{X} \qquad \text{and} \qquad \mathrm{M}^{\mathcal{D}}_{G\alpha}\boldsymbol{Y} = \lim_{\partial(G\alpha \downarrow \overleftarrow{\mathcal{D}})} \boldsymbol{Y}$$

and we define $P^{\mathcal{D}}_{G\alpha}$ by letting the diagram

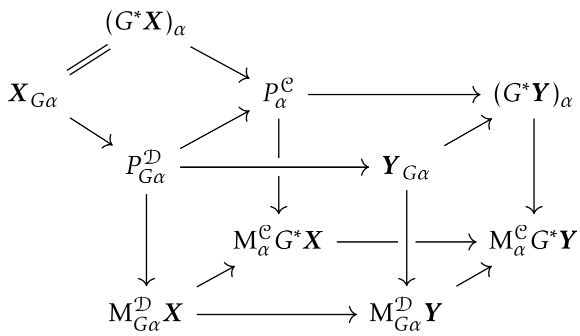
$$\begin{array}{ccc} P^{\mathcal{D}}_{G\alpha} & \dashrightarrow & \boldsymbol{Y}_{G\alpha} \\ \downarrow & & \downarrow \\ \mathrm{M}^{\mathcal{D}}_{G\alpha}\boldsymbol{X} & \longrightarrow & \mathrm{M}^{\mathcal{D}}_{G\alpha}\boldsymbol{Y} \end{array}$$

be a pullback. The functor $G_*\colon \partial(\alpha\downarrow\overleftarrow{\mathcal{C}}) \to \partial(G\alpha\downarrow\overleftarrow{\mathcal{D}})$ (see Proposition 5 on p. 29) induces natural maps

$$\mathrm{M}^{\mathcal{D}}_{G\alpha}\boldsymbol{X} = \lim_{\partial(G\alpha\downarrow\overleftarrow{\mathcal{D}})} \boldsymbol{X} \longrightarrow \lim_{\partial(\alpha\downarrow\overleftarrow{\mathcal{C}})} G^*\boldsymbol{X} = \mathrm{M}^{\mathcal{C}}_{\alpha}G^*\boldsymbol{X}$$

$$\mathrm{M}^{\mathcal{D}}_{G\alpha}\boldsymbol{Y} = \lim_{\partial(G\alpha\downarrow\overleftarrow{\mathcal{D}})} \boldsymbol{Y} \longrightarrow \lim_{\partial(\alpha\downarrow\overleftarrow{\mathcal{C}})} G^*\boldsymbol{Y} = \mathrm{M}^{\mathcal{C}}_{\alpha}G^*\boldsymbol{Y}$$

and so we have a map of pullbacks and relative matching maps



and our map $(G^*\boldsymbol{X})_\alpha \to P^{\mathcal{C}}_{\alpha}$ equals the composition

$$(G^*\boldsymbol{X})_\alpha = \boldsymbol{X}_{G\alpha} \longrightarrow P^{\mathcal{D}}_{G\alpha} \longrightarrow P^{\mathcal{C}}_{\alpha}.$$

Since the map $\boldsymbol{X} \to \boldsymbol{Y}$ is a fibration in $\mathcal{M}^{\mathcal{D}}$, the relative matching map $\boldsymbol{X}_{G\alpha} \to P^{\mathcal{D}}_{G\alpha}$ is a fibration (see Theorem 3 on p. 25), and so it is sufficient to show that the natural map

$$P^{\mathcal{D}}_{G\alpha} \longrightarrow P^{\mathcal{C}}_{\alpha} \tag{2}$$

is a fibration. That statement is the content of Proposition 19 (in Section 4.2, below) which (along with Proposition 22 on p. 55 in Section 4.3) will complete the proof of Theorem 12 on p. 45. □

4.2 Statement and proof of Proposition 19

The purpose of this section is to state and prove the following proposition, which (along with Proposition 22 in Section 4.3) will complete the proof of Theorem 12 on p. 45.

Proposition 19 – *For every object α of $\mathcal{C}$, the map*

$$P^{\mathcal{D}}_{G\alpha} \longrightarrow P^{\mathcal{C}}_{\alpha}$$

from (2) *is a fibration.*

The proof of Proposition 19 is intricate, but it does not require any new definitions. To aid the reader, here is the structure of the argument:

$$\begin{array}{ccccc} \text{Proposition 20} & \Longrightarrow & \text{Proposition 19} & & \\ & & \Uparrow & & \\ \text{Lemma 7} & \Longrightarrow & \text{Proposition 21} & \Longleftarrow & \text{Lemma 6} \\ & & & & \Uparrow \\ & & \text{Lemma 4 \& Diagram 8} & \Longrightarrow & \text{Lemma 5} \end{array} \tag{3}$$

We will start with the proof of Proposition 19 and then, as in the proof of Theorem 12 on p. 45, we will work our way backward from it.

Proof (Proof of Proposition 19*).* If the degree of α is k, we define a nested sequence of subcategories of $\partial(G\alpha \downarrow \overleftarrow{\mathcal{D}})$

$$\mathcal{A}_{-1} \subset \mathcal{A}_0 \subset \mathcal{A}_1 \subset \cdots \subset \mathcal{A}_{k-1} = \partial(G\alpha \downarrow \overleftarrow{\mathcal{D}}) \tag{4}$$

by letting $\mathcal{A}_i$ for $-1 \le i \le k-1$ be the full subcategory of $\partial(G\alpha \downarrow \overleftarrow{\mathcal{D}})$ with objects the union of

- the objects of $\partial(G\alpha \downarrow \overleftarrow{\mathcal{D}})$ whose target is of degree at most i, and
- the image under $G_*\colon \partial(\alpha \downarrow \overleftarrow{\mathcal{C}}) \to \partial(G\alpha \downarrow \overleftarrow{\mathcal{D}})$ (see Proposition 5 on p. 29) of the objects of $\partial(\alpha \downarrow \overleftarrow{\mathcal{C}})$.

The functor $G_*\colon \partial(\alpha \downarrow \overleftarrow{\mathcal{C}}) \to \partial(G\alpha \downarrow \overleftarrow{\mathcal{D}})$ factors through $\mathcal{A}_{-1}$ and, since there are no objects of negative degree, this functor, which by abuse of notation we will also call $G_*\colon \partial(\alpha \downarrow \overleftarrow{\mathcal{C}}) \to \mathcal{A}_{-1}$, maps onto the objects of $\mathcal{A}_{-1}$.

In fact, we claim that the functor $G_*\colon \partial(\alpha \downarrow \overleftarrow{\mathcal{C}}) \to \mathcal{A}_{-1}$ is left cofinal (see Definition 10 on p. 35) and thus induces isomorphisms

$$\lim_{\mathcal{A}_{-1}} \boldsymbol{X} \approx \lim_{\partial(\alpha\downarrow\overleftarrow{\mathcal{C}})} G^*\boldsymbol{X} \qquad \text{and} \qquad \lim_{\mathcal{A}_{-1}} \boldsymbol{Y} \approx \lim_{\partial(\alpha\downarrow\overleftarrow{\mathcal{C}})} G^*\boldsymbol{Y}$$

(see Theorem 4 on p. 36). To see this, note that every object of $\mathcal{A}_{-1}$ is of the form $G\sigma\colon G\alpha \to G\beta$ for some object $\sigma\colon \alpha \to \beta$ of $\partial(\alpha \downarrow \overleftarrow{\mathcal{C}})$ and Proposition 5 on p. 29 implies that the overcategory $\big(G_* \downarrow (G\sigma\colon G\alpha \to G\beta)\big)$ is exactly the category of inverse $\mathcal{C}$-factorizations of $(\alpha, G\sigma)$, and so (since G is a fibering Reedy functor) its nerve must be either empty or connected. Since it is not empty (it contains the vertex $(\alpha \xrightarrow{\sigma} \beta, 1_{G\beta})$), it is connected, and so $G_*\colon \partial(\alpha \downarrow \overleftarrow{\mathcal{C}}) \to \mathcal{A}_{-1}$ is left cofinal.

The sequence of inclusions of categories (4) thus induces sequences of maps

$$\lim_{\partial(G\alpha\downarrow\overleftarrow{\mathcal{D}})} \boldsymbol{X} = \lim_{\mathcal{A}_{k-1}} \boldsymbol{X} \to \lim_{\mathcal{A}_{k-2}} \boldsymbol{X} \to \cdots \to \lim_{\mathcal{A}_0} \boldsymbol{X} \to \lim_{\mathcal{A}_{-1}} \boldsymbol{X} \approx \lim_{\partial(\alpha\downarrow\overleftarrow{\mathcal{C}})} G^*\boldsymbol{X}$$

$$\lim_{\partial(G\alpha\downarrow\overleftarrow{\mathcal{D}})} \boldsymbol{Y} = \lim_{\mathcal{A}_{k-1}} \boldsymbol{Y} \to \lim_{\mathcal{A}_{k-2}} \boldsymbol{Y} \to \cdots \to \lim_{\mathcal{A}_0} \boldsymbol{Y} \to \lim_{\mathcal{A}_{-1}} \boldsymbol{Y} \approx \lim_{\partial(\alpha\downarrow\overleftarrow{\mathcal{C}})} G^*\boldsymbol{Y}.$$

For $-1 \le i \le k-1$ we let P_i be the pullback

$$\begin{array}{ccc} P_i & \dashrightarrow & \boldsymbol{Y}_{G\alpha} \\ \vdots & & \downarrow \\ \lim_{\mathcal{A}_i} \boldsymbol{X} & \longrightarrow & \lim_{\mathcal{A}_i} \boldsymbol{Y}. \end{array}$$

Since we have an evident map of diagrams

$$\Big(\lim_{\mathcal{A}_{i+1}} \boldsymbol{X} \to \lim_{\mathcal{A}_{i+1}} \boldsymbol{Y} \leftarrow \boldsymbol{Y}_{G\alpha}\Big) \longrightarrow \Big(\lim_{\mathcal{A}_i} \boldsymbol{X} \to \lim_{\mathcal{A}_i} \boldsymbol{Y} \leftarrow \boldsymbol{Y}_{G\alpha}\Big)$$

we also get an induced map $P_{i+1} \to P_i$ of pullbacks. We thus have a factorization of (2) as

$$P^{\mathcal{D}}_{G\alpha} = P_{k-1} \longrightarrow P_{k-2} \longrightarrow \cdots \longrightarrow P_{-1} \approx P^{\mathcal{C}}_{\alpha},$$

and we will show that the map $P_{i+1} \to P_i$ is a fibration for $-1 \le i \le k-2$.

The objects of $\mathcal{A}_{i+1}$ that are not in $\mathcal{A}_i$ are maps $G\alpha \to \beta$ where β is of degree $i+1$, and this set of maps can be divided into two subsets:

- the set S_{i+1} of maps $G\alpha \to \beta$ for which the category of inverse $\mathcal{C}$-factorizations of $(\alpha, G\alpha \to \beta)$ is nonempty, and
- the set T_{i+1} of maps for which the category of inverse $\mathcal{C}$-factorizations of $(\alpha, G\alpha \to \beta)$ is empty.

We let $\mathcal{A}'_{i+1}$ be the full subcategory of $\partial(G\alpha \downarrow \overleftarrow{\mathcal{D}})$ with objects the union of S_{i+1} with the objects of $\mathcal{A}_i$, and define P'_{i+1} as the pullback

$$\begin{array}{ccc} P'_{i+1} & \dashrightarrow & \boldsymbol{Y}_{G\alpha} \\ \vdots & & \downarrow \\ \lim_{\mathcal{A}'_{i+1}} \boldsymbol{X} & \longrightarrow & \lim_{\mathcal{A}'_{i+1}} \boldsymbol{Y}. \end{array}$$

We have inclusions of categories $\mathcal{A}_i \subset \mathcal{A}'_{i+1} \subset \mathcal{A}_{i+1}$, and the maps

$$\lim_{\mathcal{A}_{i+1}} \boldsymbol{X} \longrightarrow \lim_{\mathcal{A}_i} \boldsymbol{X} \qquad \text{and} \qquad \lim_{\mathcal{A}_{i+1}} \boldsymbol{Y} \longrightarrow \lim_{\mathcal{A}_i} \boldsymbol{Y}$$

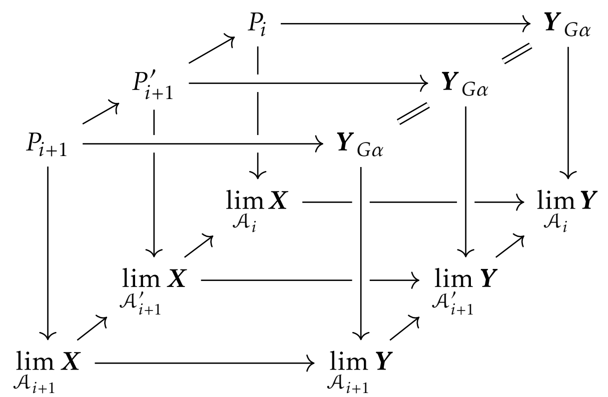
factor as

$$\lim_{\mathcal{A}_{i+1}} \boldsymbol{X} \longrightarrow \lim_{\mathcal{A}'_{i+1}} \boldsymbol{X} \longrightarrow \lim_{\mathcal{A}_i} \boldsymbol{X} \qquad \text{and} \qquad \lim_{\mathcal{A}_{i+1}} \boldsymbol{Y} \longrightarrow \lim_{\mathcal{A}'_{i+1}} \boldsymbol{Y} \longrightarrow \lim_{\mathcal{A}_i} \boldsymbol{Y}.$$

These factorizations induce a factorization

$$P_{i+1} \longrightarrow P'_{i+1} \longrightarrow P_i \tag{5}$$

of the map $P_{i+1} \to P_i$, and we have the commutative diagram



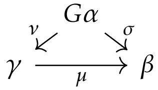
Proposition 20 below asserts that the map $P'_{i+1} \to P_i$ is an isomorphism and Proposition 21 on the next page asserts that the map $P_{i+1} \to P'_{i+1}$ is a fibration. Hence, the map $P^{\mathcal{D}}_{G\alpha} \to P^{\mathcal{C}}_{\alpha}$ is a fibration as well. □

Proposition 20 – *For $-1 \le i \le k-2$, the map $P'_{i+1} \to P_i$ in* (5) *is an isomorphism.*

Proof. We will show that for every element $\sigma\colon G\alpha \to \beta$ of $\mathcal{A}_i$ the overcategory $(\mathcal{A}_i \downarrow \sigma)$ is nonempty and connected, which will imply that the inclusion $\mathcal{A}_i \subset \mathcal{A}'_{i+1}$ is left cofinal (see Definition 10 on p. 35). This will imply that the maps $\lim_{\mathcal{A}'_{i+1}} \boldsymbol{X} \to \lim_{\mathcal{A}_i} \boldsymbol{X}$ and $\lim_{\mathcal{A}'_{i+1}} \boldsymbol{Y} \to \lim_{\mathcal{A}_i} \boldsymbol{Y}$ are isomorphisms (see Theorem 4 on p. 36), and so the induced map $P'_{i+1} \to P_i$ is an isomorphism.

If $\sigma\colon G\alpha \to \beta$ is an element of $\mathcal{A}_i$, then the overcategory $(\mathcal{A}_i \downarrow \sigma)$ has the terminal object 1_σ and is thus nonempty and connected.

Now suppose that $\sigma\colon G\alpha \to \beta$ is an object of $\mathcal{A}'_{i+1}$ that is not in $\mathcal{A}_i$. The objects of $(\mathcal{A}_i \downarrow \sigma)$ are commutative diagrams



where $\nu\colon G\alpha \to \gamma$ is in $\mathcal{A}_i$ and μ is in $\overleftarrow{\mathcal{D}}$. Since β is of degree $i+1$ and μ lowers degree (because μ cannot be an identity map, since σ isn't in $\mathcal{A}_i$), the degree of γ must be greater than $i+1$, and so the map $\nu\colon G\alpha \to \gamma$ must be of the form $G\nu'\colon G\alpha \to G\gamma'$ for some map $\nu'\colon \alpha \to \gamma'$ in $\partial(\alpha \downarrow \overleftarrow{\mathcal{C}})$. Thus, the objects of $(\mathcal{A}_i \downarrow \sigma)$ are pairs $\big((\nu'\colon \alpha \to \gamma'), (\mu\colon G\gamma' \to \beta)\big)$ where $\nu'\colon \alpha \to \gamma'$ is a non-identity map of $\overleftarrow{\mathcal{C}}$, $\mu\colon G\gamma' \to \beta$ is in $\overleftarrow{\mathcal{D}}$, and $\mu \circ G\nu' = \sigma$, and $(\mathcal{A}_i \downarrow \sigma)$ is the category of inverse $\mathcal{C}$-factorizations of (α, σ) (see Proposition 5 on p. 29). Since G is a fibering Reedy functor, the nerve of the category of inverse $\mathcal{C}$-factorizations of (α,σ) is either empty or connected. Since it is nonempty (because $\sigma\colon G\alpha \to \beta$ is an element of S_{i+1}), the nerve of the overcategory $(\mathcal{A}_i \downarrow \sigma)$ is nonempty and connected. □

Proposition 21 – *For $-1 \le i \le k-2$, the map $P_{i+1} \to P'_{i+1}$ in (5) is a fibration.*

The proof of Proposition 21 is more intricate; the reader might wish to refer to the chart (3) for its structure. Before we can present it, we will need several lemmas. For the first one, the reader should recall the definition of the sets T_i from the proof of Proposition 19 on p. 47.

Lemma 4 – *For every $\mathcal{D}$-diagram $\boldsymbol{Z}$ in $\mathcal{M}$ there is a natural pullback square*

$$\begin{array}{ccc} \lim\limits_{\mathcal{A}_{i+1}} \boldsymbol{Z} & \longrightarrow & \lim\limits_{\mathcal{A}'_{i+1}} \boldsymbol{Z} \\ \big\downarrow & & \big\downarrow \\ \prod\limits_{(G\alpha\to\beta)\in T_{i+1}} \boldsymbol{Z}_\beta & \longrightarrow & \prod\limits_{(G\alpha\to\beta)\in T_{i+1}} \lim\limits_{\partial(\beta\downarrow\overleftarrow{\mathcal{D}})} \boldsymbol{Z}. \end{array} \tag{6}$$

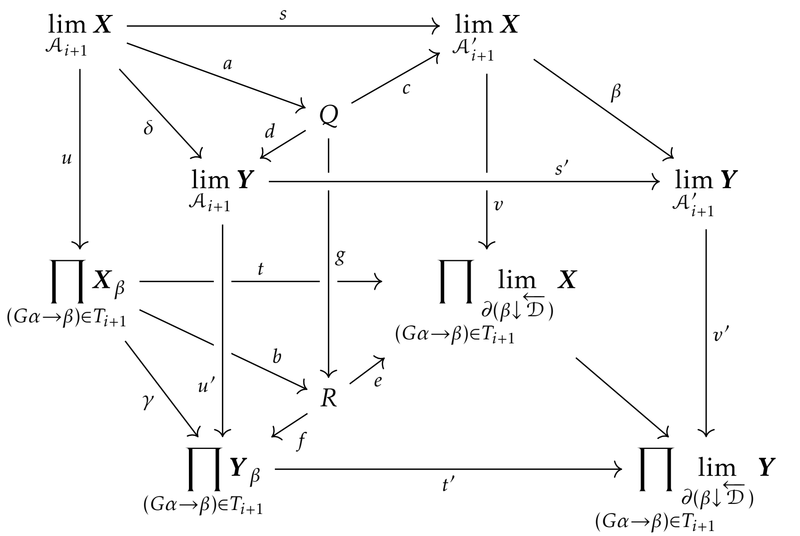
Proof. For every element $\sigma\colon G\alpha \to \beta$ of T_{i+1}, every object of the matching category $\partial(\beta \downarrow \overleftarrow{\mathcal{D}})$ is a map to an object of degree at most i, and so we have a functor $\partial(\beta \downarrow \overleftarrow{\mathcal{D}}) \to \mathcal{A}'_{i+1}$ that takes $\beta \to \gamma$ to the composition $G\alpha \xrightarrow{\sigma} \beta \to \gamma$; this induces the map $\lim_{\mathcal{A}'_{i+1}} \boldsymbol{Z} \longrightarrow \lim_{\partial(\beta\downarrow\overleftarrow{\mathcal{D}})} \boldsymbol{Z}$ that is the projection of the right hand vertical map onto the factor indexed by σ. We thus have a commutative square as in Diagram 6.

The objects of $\mathcal{A}_{i+1}$ are the objects of $\mathcal{A}'_{i+1}$ together with the elements of T_{i+1}, and so a map to $\lim_{\mathcal{A}_{i+1}} \boldsymbol{Z}$ is determined by its postcompositions with the above maps to $\lim_{\mathcal{A}'_{i+1}} \boldsymbol{Z}$ and $\prod_{(G\alpha\to\beta)\in T_{i+1}} \boldsymbol{Z}_\beta$. Since there are no non-identity maps in $\mathcal{A}_{i+1}$ with codomain an element of T_{i+1} (because $\mathrm{Fact}_{\overleftarrow{\mathcal{C}}}(\alpha, G\alpha \to \beta) = \varnothing$), and the only non-identity maps with domain an element $G\alpha \to \beta$ of T_{i+1} are the objects of the matching category $\partial(\beta \downarrow \overleftarrow{\mathcal{D}})$, maps to $\lim_{\mathcal{A}'_{i+1}} \boldsymbol{Z}$ and to $\prod_{(G\alpha\to\beta)\in T_{i+1}} X_\beta$ determine a map to $\lim_{\mathcal{A}_{i+1}} \boldsymbol{Z}$ if and only if their compositions to $\prod_{(G\alpha\to\beta)\in T_{i+1}} \lim_{\partial(\beta\downarrow\overleftarrow{\mathcal{D}})} \boldsymbol{Z}$ agree. Thus, the diagram is a pullback square. □

Now define Q and R by letting the squares

$$
\begin{array}{ccc}
Q & \dashrightarrow & \lim_{\mathcal{A}'_{i+1}} \boldsymbol{X} \\
\vdots & & \downarrow \\
\lim_{\mathcal{A}_{i+1}} \boldsymbol{Y} & \longrightarrow & \lim_{\mathcal{A}'_{i+1}} \boldsymbol{Y}
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
R & \dashrightarrow & \prod_{(G\alpha\to\beta)\in T_{i+1}} \lim_{\partial(\beta\downarrow\overleftarrow{\mathcal{D}})} \boldsymbol{X} \\
\vdots & & \downarrow \\
\prod_{(G\alpha\to\beta)\in T_{i+1}} \boldsymbol{Y}_\beta & \longrightarrow & \prod_{(G\alpha\to\beta)\in T_{i+1}} \lim_{\partial(\beta\downarrow\overleftarrow{\mathcal{D}})} \boldsymbol{Y}
\end{array}
\qquad (7)
$$

be pullbacks, and consider the commutative diagram



(8)

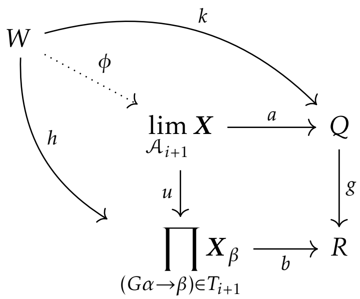
Lemma 4 on the previous page implies that the front and back rectangles are pullbacks.

Lemma 5 – *The square*

$$
\begin{array}{ccc}
\lim_{\mathcal{A}_{i+1}} \boldsymbol{X} & \xrightarrow{a} & Q \\
{\scriptstyle u}\downarrow & & \downarrow{\scriptstyle g} \\
\prod_{(G\alpha\to\beta)\in T_{i+1}} \boldsymbol{X}_\beta & \xrightarrow[b]{} & R
\end{array}
\qquad (9)
$$

is a pullback.

Proof. Let W be an object of $\mathcal{M}$ and let $h\colon W \to \prod_{(G\alpha\to\beta)\in T_{i+1}} \boldsymbol{X}_\beta$ and $k\colon W \to Q$ be maps such that $gk = bh$; we will show that there is a unique map $\phi\colon W \to \lim_{\mathcal{A}_{i+1}} \boldsymbol{X}$ such that $a\phi = k$ and $u\phi = h$.



The map $ck\colon W \to \lim_{\mathcal{A}'_{i+1}} \boldsymbol{X}$ has the property that $v(ck) = egk = ebh = th$, and since the back rectangle of Diagram 8 is a pullback, the maps ck and h induce a map $\phi\colon W \to \lim_{\mathcal{A}_{i+1}} \boldsymbol{X}$ such that $u\phi = h$ and $s\phi = ck$. We must show that $a\phi = k$, and since Q is a pullback as in Diagram 7, this is equivalent to showing that $ca\phi = ck$ and $da\phi = dk$.

Since $ck = s\phi = ca\phi$, we need only show that $da\phi = dk$. Since the front rectangle of Diagram 8 is a pullback, it is sufficient to show that $s'da\phi = s'dk$ and $u'da\phi = u'dk$. For the first of those, we have

$$s'da\phi = s'\delta\phi = \beta s\phi = \beta ck = s'dk$$

and for the second, we have

$$u'da\phi = u'\delta\phi = \gamma u\phi = fbu\phi = fbh = fgk = u'dk.$$

Thus, the map ϕ satisfies $a\phi = k$.

To see that ϕ is the unique such map, let $\psi\colon W \to \lim_{\mathcal{A}_{i+1}} \boldsymbol{X}$ be another map such that $a\psi = k$ and $u\psi = h$. We will show that $s\psi = s\phi$ and $u\psi = u\phi$; since the back rectangle of Diagram 8 is a pullback, this will imply that $\psi = \phi$.

Since $u\psi = h = u\phi$, we need only show that $s\psi = s\phi$, which follows because $s\psi = ca\psi = ck = s\phi$. □

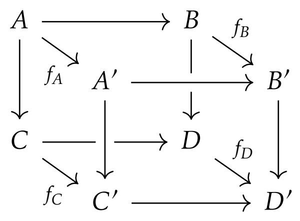
Lemma 6 – *If $\boldsymbol{X} \to \boldsymbol{Y}$ is a fibration of $\mathcal{D}$-diagrams, then the natural map*

$$\lim_{\mathcal{A}_{i+1}} \boldsymbol{X} \longrightarrow Q = \lim_{\mathcal{A}'_{i+1}} \boldsymbol{X} \times_{(\lim_{\mathcal{A}'_{i+1}} \boldsymbol{Y})} \lim_{\mathcal{A}_{i+1}} \boldsymbol{Y}$$

is a fibration.

Proof. Lemma 5 on the preceding page gives us the pullback square in Diagram 9 where Q and R are defined by the pullbacks in Diagram 7. Since $\boldsymbol{X} \to \boldsymbol{Y}$ is a fibration of $\mathcal{D}$-diagrams, the map $\prod_{(G\alpha \to \beta) \in T_{i+1}} \boldsymbol{X}_\beta \to R$ is a product of fibrations (see Definition 3 on p. 24 (6)) and is thus a fibration, and so the map $\lim_{\mathcal{A}_{i+1}} \boldsymbol{X} \to Q = \lim_{\mathcal{A}'_{i+1}} \boldsymbol{X} \times_{(\lim_{\mathcal{A}'_{i+1}} \boldsymbol{Y})} \lim_{\mathcal{A}_{i+1}} \boldsymbol{Y}$ is a pullback of a fibration and is thus a fibration. □

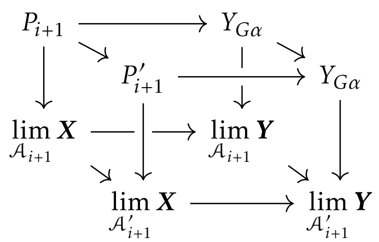
Lemma 7 (Reedy) – *If both the front and back squares in the diagram*



are pullbacks and both $f_B\colon B \to B'$ and $C \to C' \times_{D'} D$ are fibrations, then $f_A\colon A \to A'$ is a fibration.

Proof. This is the dual of a lemma of Reedy (see Hirschhorn 2003, Lem. 7.2.15 and Rem. 7.1.10). □

Proof (Proof of Proposition 21*).* We have a commutative diagram



in which the front and back squares are pullbacks (by definition), and so Lemma 7 implies that it is sufficient to show that the map

$$\lim_{\mathcal{A}_{i+1}} \boldsymbol{X} \longrightarrow \lim_{\mathcal{A}'_{i+1}} \boldsymbol{X} \times_{(\lim_{\mathcal{A}'_{i+1}} \boldsymbol{Y})} \lim_{\mathcal{A}_{i+1}} \boldsymbol{Y}$$

is a fibration; that is the statement of Lemma 6 on the previous page. □

4.3 Statement and proof of Proposition 22

The purpose of this section is to state and prove the following proposition, which (along with Proposition 19 on p. 47 in Section 4.2) completes the proof of Theorem 12 on p. 45.

Proposition 22 – *Let $G\colon \mathcal{C} \to \mathcal{D}$ be a fibering Reedy functor and let $\boldsymbol{X}$ be a $\mathcal{D}$-diagram in a model category $\mathcal{M}$. If α is an object of $\mathcal{C}$ for which there is an object $\alpha \to \gamma$ of $\partial(\alpha \downarrow \overleftarrow{\mathcal{C}})$ (i.e., a non-identity map $\alpha \to \gamma$ in $\overleftarrow{\mathcal{C}}$) that G takes to an identity map in $\overleftarrow{\mathcal{D}}$, then the matching map $(G^*\boldsymbol{X})_\alpha \to \mathrm{M}^{\mathcal{C}}_\alpha(G^*\boldsymbol{X})$ of $G^*\boldsymbol{X}$ (see Definition 6 on p. 26) at α is an isomorphism.*

The proof will require several preliminary definitions and results.

Definition 18 – The *G-kernel at α* is the full subcategory of the matching category $\partial(\alpha \downarrow \overleftarrow{\mathcal{C}})$ with objects the non-identity maps $\alpha \to \gamma$ in $\overleftarrow{\mathcal{C}}$ that G takes to the identity map of $G\alpha$.

If $\alpha \to \gamma$ is an object of the G-kernel at α, then the map $(G^*\boldsymbol{X})_\alpha \to (G^*\boldsymbol{X})_\gamma$ is the identity map. Note that the G-kernel at α is not usually left cofinal in $\partial(\mathcal{C} \downarrow \overleftarrow{\alpha})$.

Lemma 8 – *Under the hypotheses of Proposition 22, the nerve of the G-kernel at α is connected.*

Proof. Since G is a fibering Reedy functor, the nerve of the category $\mathrm{Fact}_{\overleftarrow{\mathcal{C}}}(\alpha, 1_{G\alpha})$ of inverse $\mathcal{C}$-factorizations of $(\alpha, 1_{G\alpha})$ is connected, and there is an isomorphism from the G-kernel at α to $\mathrm{Fact}_{\overleftarrow{\mathcal{C}}}(\alpha, 1_{G\alpha})$ that takes the object $\alpha \to \gamma$ to the object $\big((\alpha \to \gamma), (1_{G\alpha})\big)$. □

The matching object $\mathrm{M}^{\mathcal{C}}_\alpha(G^*\boldsymbol{X})$ is the limit of a $\partial(\alpha \downarrow \overleftarrow{\mathcal{C}})$-diagram (which we will also denote by $G^*\boldsymbol{X}$); we will refer to that diagram as the *matching diagram*. The restriction of the matching diagram to the G-kernel at α is a diagram in which every object goes to $\boldsymbol{X}_{G\alpha} = (G^*\boldsymbol{X})_\alpha$ and every map goes to the identity map of $\boldsymbol{X}_{G\alpha}$, because if there is a commutative triangle

$$\begin{array}{ccc} & \alpha & \\ {}^{f}\swarrow & & \searrow^{f'} \\ \gamma & \xrightarrow[\tau]{} & \gamma' \end{array}$$

in $\overleftarrow{\mathcal{C}}$ in which $Gf = Gf' = 1_{G\alpha}$, then $G\tau \circ 1_{G\alpha} = 1_{G\alpha}$, and so $G\tau = 1_{G\alpha}$. Together with Lemma 8, this implies the following.

Lemma 9 – *Under the hypotheses of Proposition 22, the restriction of the matching diagram to the G-kernel at α is a connected diagram in which every object goes to $\boldsymbol{X}_{G\alpha}$ and every map goes to the identity map of $\boldsymbol{X}_{G\alpha}$.*

We will prove Proposition 22 by showing that for every object W of $\mathcal{M}$ the matching map induces an isomorphism of sets of maps

$$\mathcal{M}\big(W, (G^*\boldsymbol{X})_\alpha\big) \longrightarrow \mathcal{M}\big(W, \mathrm{M}^{\mathcal{C}}_\alpha(G^*\boldsymbol{X})\big) \tag{10}$$

(see Proposition 1 on p. 25). The matching object $\mathrm{M}^{\mathcal{C}}_{\alpha}(G^*\boldsymbol{X})$ is the limit of the matching diagram, and so maps from W to $\mathrm{M}^{\mathcal{C}}_{\alpha}(G^*\boldsymbol{X})$ correspond to maps from W to the matching diagram. Lemma 9 on the previous page implies that if we restrict the matching diagram to the G-kernel at α, then maps from W to the restriction of that diagram to the G-kernel at α correspond to maps from W to $(G^*\boldsymbol{X})_{\alpha}$, and that fact allows us to define a potential inverse to (10). All that remains is to show that our potential inverse is actually an inverse.

If $\alpha \to \beta$ and $\alpha \to \gamma$ are objects of the matching category and there is a map $\tau\colon (\alpha \to \beta) \to (\alpha \to \gamma)$ in the matching category, i.e., a commutative diagram

$$\begin{array}{ccc} & \alpha & \\ \swarrow & & \searrow \\ \beta & \xrightarrow[\tau]{} & \gamma, \end{array}$$

then for every object W of $\mathcal{M}$ and map from W to the matching diagram, the projection of that map onto $(\alpha \to \gamma)$ is entirely determined by its projection onto $(\alpha \to \beta)$; we will describe this by saying that the object $(\alpha \to \gamma)$ is *controlled* by the object $(\alpha \to \beta)$. Similarly, if there is a commutative triangle

$$\begin{array}{ccc} & \alpha & \\ \swarrow & & \searrow \\ \gamma & \xrightarrow{\tau} & \gamma' \end{array}$$

in the matching category such that $G\tau$ is an identity map, then we will say that the object $(\alpha \to \gamma)$ is *controlled* by the object $(\alpha \to \gamma')$ *and* that the object $(\alpha \to \gamma')$ is *controlled* by the object $(\alpha \to \gamma)$. We will show by a downward induction on degree that all objects of the matching category are controlled by objects of the G-kernel at α (see Definition 20 and Proposition 23 on p. 58).

Definition 19 – We define an equivalence relation on the set of objects of $\partial(\alpha \downarrow \overleftarrow{\mathcal{C}})$, called *$G$-equivalence at α*, as the equivalence relation generated by the relation under which $f\colon \alpha \to \gamma$ is equivalent to $f'\colon \alpha \to \gamma'$ if there is a commutative triangle

$$\begin{array}{ccc} & \alpha & \\ {\scriptstyle f}\swarrow & & \searrow{\scriptstyle f'} \\ \gamma & \xrightarrow[\tau]{} & \gamma' \end{array}$$

with $G\tau$ an identity map.

If f and f' are G-equivalent at α, then $Gf = Gf'$, and there is a zig-zag of identity maps connecting $\boldsymbol{X}_f$ and $\boldsymbol{X}_{f'}$ in the matching diagram.

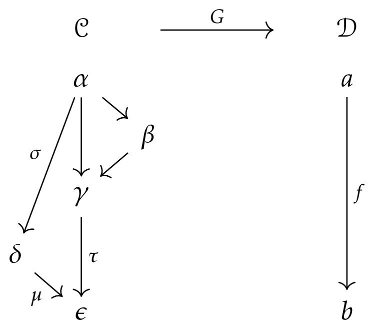
Definition 20 – We define the set of *controlled objects* $\{\alpha \to \gamma\}$ of the matching category $\partial(\alpha \downarrow \overleftarrow{\mathcal{C}})$ by a decreasing induction on degree$(G\gamma)$:

1. If $\alpha \to \gamma$ is an object of $\partial(\alpha \downarrow \overleftarrow{\mathcal{C}})$ such that degree$(G\gamma)$ = degree$(G\alpha)$ (i.e., if $G(\alpha \to \gamma) = 1_{G\alpha}$), then $\alpha \to \gamma$ is controlled. (That is, all objects of the G-kernel

at α are controlled.) Note that this initial step is non-empty, since we have assumed that the G-kernel at α is non-empty.

2. If $0 \le n < \text{degree}(G\alpha)$ and we have defined the controlled objects $\alpha \to \delta$ for $n < \text{degree}(\delta) \le \text{degree}(G\alpha)$, then we define an object $\alpha \to \gamma$ with $\text{degree}(G\gamma) = n$ to be controlled if it is G-equivalent at α to an object $\alpha \to \gamma'$ that has a factorization $\alpha \to \delta \to \gamma'$ in $\overleftarrow{\mathcal{C}}$ such that $\alpha \to \delta$ is an object of $\partial(\alpha \downarrow \overleftarrow{\mathcal{C}})$ that is controlled.

Example 2 – Let $G\colon \mathcal{C} \to \mathcal{D}$ be the fibering Reedy functor between Reedy categories as in the following diagram:



where

- $\mathcal{C}$ has five objects, α, β, γ, δ, and ϵ of degrees 4, 3, 2, 1, and 0, respectively, and the diagram commutes;
- $\mathcal{D}$ has two objects, a and b of degrees 1 and 0, respectively;
- $G\alpha = G\beta = G\gamma = a$ and G takes the maps between them to 1_a;
- $G\delta = G\epsilon = b$ and $G\mu = 1_b$; and
- $G\sigma = G\tau = f$.

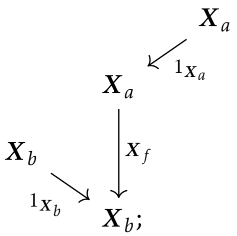
Every object of $\partial(\alpha \downarrow \overleftarrow{\mathcal{C}})$ is controlled:

- The objects $\alpha \to \beta$ and $\alpha \to \gamma$ are controlled because of the first part of Definition 20 on the preceding page.
- The object $\alpha \to \epsilon$ is controlled because it is G-equivalent at α to itself and it factors as $\alpha \to \gamma \to \epsilon$ with the object $\alpha \to \gamma$ controlled.
- The object σ is controlled because it is G-equivalent at α to $\alpha \to \epsilon$ and the latter map factors as $\alpha \to \gamma \to \epsilon$ where the object $\alpha \to \gamma$ is controlled.

If $\boldsymbol{X}$ is a $\mathcal{D}$-diagram in a model category $\mathcal{M}$, then the induced $\mathcal{C}$-diagram $G^*\boldsymbol{X}$ has

$$(G^*\boldsymbol{X})_\alpha = (G^*\boldsymbol{X})_\beta = (G^*\boldsymbol{X})_\gamma = \boldsymbol{X}_a \qquad \text{and} \qquad (G^*\boldsymbol{X})_\delta = (G^*\boldsymbol{X})_\epsilon = \boldsymbol{X}_b,$$

and the matching object of $(G^*\boldsymbol{X})$ at α is the limit of the diagram



that limit is isomorphic to $\boldsymbol{X}_a$, as guaranteed by Proposition 22 on p. 55.

The set of controlled objects has the following property.

Lemma 10 – *Under the hypotheses of Proposition 22 on p. 55, if W is an object of $\mathcal{M}$ and $h,k\colon W \to \mathrm{M}_\alpha^{\mathcal{C}}(G^*\boldsymbol{X})$ are two maps to the matching object of $G^*\boldsymbol{X}$ at α whose projections onto at least one object of the G-kernel at α agree, then their projections onto every controlled object agree.*

Proof. This follows by a decreasing induction as in Definition 20 on p. 56, using Lemma 9 on p. 55 and Definition 20 on p. 56. □

That every object in the example above was controlled was not an accident, as shown by the following result.

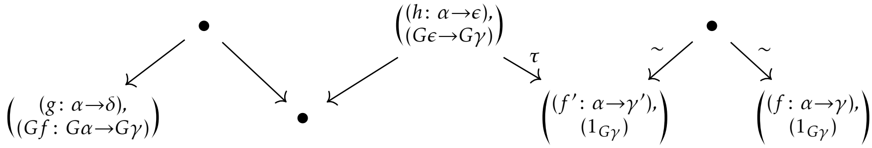
Proposition 23 – *Under the hypotheses of Proposition 22 on p. 55, every object $f\colon \alpha \to \gamma$ of $\partial(\alpha \downarrow \overleftarrow{\mathcal{C}})$ is controlled.*

Proof. We will show this by a decreasing induction on the degree of $G\gamma$ in $\mathcal{D}$, beginning with degree$(G\alpha)$. The induction is begun because the objects $f\colon \alpha \to \gamma$ in $\partial(\alpha \downarrow \overleftarrow{\mathcal{C}})$ with degree$(G\gamma) =$ degree$(G\alpha)$ are exactly the objects of the G-kernel at α, since a map in $\overleftarrow{\mathcal{D}}$ that does not lower degree must be an identity map.

Suppose now that $0 \le n <$ degree$(G\alpha)$, that every object $\alpha \to \delta$ in $\partial(\alpha \downarrow \overleftarrow{\mathcal{C}})$ with degree$(G\delta) > n$ is controlled, and that $f\colon \alpha \to \gamma$ is an object of $\partial(\alpha \downarrow \overleftarrow{\mathcal{C}})$ with degree$(G\gamma) = n$. We will show that there is a map $\tau\colon \epsilon \to \gamma'$ in $\mathrm{Fact}_{\overleftarrow{\mathcal{C}}}(\alpha, Gf)$ from an object $\big((h\colon \alpha \to \epsilon), (G\epsilon \to G\gamma)\big)$ with degree$(\epsilon) >$ degree(γ) to an object $\big((f'\colon \alpha \to \gamma'), (1\colon G\gamma' \to G\gamma' = G\gamma)\big)$ that is G-equivalent to f. The induction hypothesis will then imply that $h\colon \alpha \to \epsilon$ is controlled, and since the composition $\alpha \xrightarrow{h} \epsilon \xrightarrow{\tau} \gamma'$ equals $f'\colon \alpha \to \gamma'$, this will imply that $f\colon \alpha \to \gamma$ is controlled.

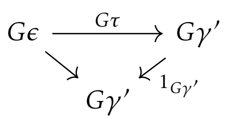
Consider the category $\mathrm{Fact}_{\overleftarrow{\mathcal{C}}}(\alpha, Gf)$ of inverse $\mathcal{C}$-factorizations of $(\alpha, Gf : G\alpha \to G\gamma)$. We first show that if $\big((f'\colon \alpha \to \gamma'), (1_{G\gamma})\big)$ is an object of $\mathrm{Fact}_{\overleftarrow{\mathcal{C}}}(\alpha, Gf)$ such that f' is G-equivalent at α to f, and if that object is the domain of a map to an object $\big((h\colon \alpha \to \epsilon), (G\epsilon \to G\gamma)\big)$, then we must have $\mathrm{degree}(G\epsilon) = \mathrm{degree}(G\gamma)$ and that target object must actually be of the form $\big((f''\colon \alpha \to \gamma''), (1_{G\gamma})\big)$ where f'' is also G-equivalent at α to f. This is because if $\tau\colon \gamma' \to \epsilon$ is a map in $\overleftarrow{\mathcal{C}}$ such that $G\tau$ is *not* an identity map, then $\mathrm{degree}(G\epsilon) < \mathrm{degree}(G\gamma') = \mathrm{degree}(G\gamma)$, which is not possible because an identity map in a Reedy category cannot factor through a degree-lowering map.

The category $\mathrm{Fact}_{\overleftarrow{\mathcal{C}}}(\alpha, Gf)$ contains the object $\big((f\colon \alpha \to \gamma), (1_{G\gamma})\big)$ and, if $g\colon \alpha \to \delta$ is an object of the G-kernel at α, then it also contains the object $\big((g\colon \alpha \to \delta), (Gf\colon G\alpha \to G\gamma)\big)$. Since G is a fibering Reedy functor, the nerve of the category $\mathrm{Fact}_{\overleftarrow{\mathcal{C}}}(\alpha, Gf)$ is connected, and so there must be a zig-zag of maps in $\mathrm{Fact}_{\overleftarrow{\mathcal{C}}}(\alpha, Gf)$ connecting those two objects. Since every map in $\mathrm{Fact}_{\overleftarrow{\mathcal{C}}}(\alpha, Gf)$ with domain an object $\big((f'\colon \alpha \to \gamma'), (1_{G\gamma})\big)$ (where $f'\colon \alpha \to \gamma'$ is G-equivalent at α to $f\colon \alpha \to \gamma$) can have as a target only another such object, and the object $\big((g\colon \alpha \to \delta), (Gf\colon G\alpha \to G\gamma)\big)$ (with $g\colon \alpha \to \delta$ an object of the G-kernel at α) is at the left end of the zig-zag, the zig-zag must look like the following:



That is, the rightmost few maps in the zig-zag can be maps that G takes to $1_{G\gamma}$ (labelled with "$\sim$" in the diagram), but at some point in the zig-zag there must be a map going to the right, from an object $\big((h\colon \alpha \to \epsilon), (G\epsilon \to G\gamma)\big)$ of $\mathrm{Fact}_{\overleftarrow{\mathcal{C}}}(\alpha, Gf)$ with h *not* G-equivalent at α to f and a map $\tau\colon \epsilon \to \gamma'$ from that object to an object $\big((f'\colon \alpha \to \gamma'), (1_{G\gamma})\big)$ where $f'\colon \alpha \to \gamma'$ is G-equivalent at α to f.

If we had $\mathrm{degree}(G\epsilon) = \mathrm{degree}(G\gamma)$, then $G\tau$ would be an identity map (and so h would be G-equivalent to f) because there would be a commutative triangle



in which the map $G\epsilon \to G\gamma'$ is a map of $\overleftarrow{\mathcal{D}}$ that does not lower degree and is thus an identity map. Thus, the only way an object $\big((f'\colon \alpha \to \gamma'), (1_{G\gamma})\big)$ with f' being G-

equivalent to f can connect via a zig-zag to an object $\big((h\colon \alpha \to \epsilon), (G\epsilon \to G\gamma)\big)$ with h not G-equivalent to f is by way of a map $\tau\colon \epsilon \to \gamma'$ from an object $\big((h\colon \alpha \to \epsilon), (G\epsilon \to G\gamma)\big)$ with degree$(G\epsilon) >$ degree$(G\gamma)$, which (by the induction hypothesis) implies that $h\colon \alpha \to \epsilon$ is controlled. In this case, the composition $\alpha \xrightarrow{h} \epsilon \xrightarrow{\tau} \gamma'$ equals $f'\colon \alpha \to \gamma'$, and so $f\colon \alpha \to \gamma$ is controlled. This completes the induction. □

Proof (Proof of Proposition 22*).* Proposition 1 on p. 25 implies that it is sufficient to show that for every object W of $\mathcal{M}$ the matching map $(G^*\boldsymbol{X})_\alpha \to \mathrm{M}^{\mathcal{C}}_\alpha(G^*\boldsymbol{X})$ induces an isomorphism of the sets of maps

$$\mathcal{M}\big(W, (G^*\boldsymbol{X})_\alpha\big) \xrightarrow{\approx} \mathcal{M}\big(W, \mathrm{M}^{\mathcal{C}}_\alpha(G^*\boldsymbol{X})\big). \tag{11}$$

Let W be an object of $\mathcal{M}$ and let $h\colon W \to \mathrm{M}^{\mathcal{C}}_\alpha(G^*\boldsymbol{X})$ be a map. If $\alpha \to \gamma$ is an object of $\partial(\alpha\downarrow \overleftarrow{\mathcal{C}})$ that is in the G-kernel at α, then $(G^*\boldsymbol{X})_{(\alpha\to\gamma)} = (G^*\boldsymbol{X})_\gamma = (G^*\boldsymbol{X})_\alpha$, and so the projection of h onto $(G^*\boldsymbol{X})_{(\alpha\to\gamma)}$ defines a map $\hat{h}\colon W \to (G^*\boldsymbol{X})_\alpha$. Lemma 9 on p. 55 implies that the map $\hat{h}$ is independent of the choice of object of the G-kernel at α.

The composition

$$W \xrightarrow{\hat{h}} (G^*\boldsymbol{X})_\alpha \longrightarrow \mathrm{M}^{\mathcal{C}}_\alpha(G^*\boldsymbol{X})$$

has the same projection onto $(G^*\boldsymbol{X})_{(\alpha\to\gamma)}$ as the map $h\colon W \to \mathrm{M}^{\mathcal{C}}_\alpha(G^*\boldsymbol{X})$; since every object of $\partial(\alpha\downarrow \overleftarrow{\mathcal{C}})$ is controlled (see Proposition 23 on p. 58), these two maps agree on every projection of $\mathrm{M}^{\mathcal{C}}_\alpha(G^*\boldsymbol{X})$ (see Lemma 10 on p. 58), and so they are equal; thus, the map (11) is a surjection. Since the composition of the matching map with the projection $\mathrm{M}^{\mathcal{C}}_\alpha(G^*\boldsymbol{X}) \to (G^*\boldsymbol{X})_{(\alpha\to\gamma)}$ is $\boldsymbol{X}\circ G$ applied to $\alpha \to \gamma$, which is the identity map, $\hat{h}$ is the only possible lift to $(G^*\boldsymbol{X})_\alpha$ of h, and so the map (11) is also an injection, and so it is an isomorphism. □

4.4 Proof of Theorem 13

We will begin by constructing the $\mathcal{D}$-diagram $\boldsymbol{X}$ whose existence is asserted in Theorem 13 on p. 45. The construction is by induction on the filtrations $\mathrm{F}^n\mathcal{D}$ of $\mathcal{D}$ (see Definition 4 on p. 25), and it will follow immediately that $\boldsymbol{X}$ is a fibrant $\mathcal{D}$-diagram (see Proposition 24 on p. 62). Proposition 25 on p. 62 will then describe the diagram $\boldsymbol{X}$ in more detail.

Proposition 26 on p. 64 describes the matching object $\mathrm{M}^{\mathcal{C}}_\alpha(G^*\boldsymbol{X})$ of the induced $\mathcal{C}$-diagram $G^*\boldsymbol{X}$ at an object α of $\mathcal{C}$, and then Proposition 27 on p. 65 shows that the matching map $(G^*\boldsymbol{X})_\alpha \to \mathrm{M}^{\mathcal{C}}_\alpha(G^*\boldsymbol{X})$ is not a fibration, which implies that $G^*\boldsymbol{X}$ is not fibrant. This plan is illustrated in the following diagram:

$$\begin{array}{ccc} & & \text{Theorem 13} \\ & \nearrow\!\!\!\!\nearrow & \Uparrow \\ \text{Proposition 24 on the next page} & & \text{Proposition 27 on p. 65} \\ & \nearrow\!\!\!\!\nearrow & \Uparrow \\ \text{Proposition 25 on the next page} & \Longrightarrow & \text{Proposition 26 on p. 64} \end{array} \tag{12}$$

Our $\mathcal{D}$-diagram $\boldsymbol{X}$ will be a diagram in the standard model category of topological spaces. Throughout its construction, the reader should keep the square diagram from Example 1 on p. 36 in mind. In that example, the diagram $\boldsymbol{X}$ that we construct here is the functor that sends each object in that square to the unit interval I with all the maps going to the identity map, and $G\colon \mathcal{C} \to \mathcal{D}$ is the inclusion of the diagram obtained by removing the degree zero object β from the square.

To construct the $\mathcal{D}$-diagram $\boldsymbol{X}$ we set the object $\boldsymbol{X}_\beta$ (for a particular object β of $\mathcal{D}$) equal to the unit interval I (see the construction below), and then Proposition 25 on the next page shows that for every object γ of $\mathcal{D}$ the space $\boldsymbol{X}_\gamma$ is a product of copies of I. We remark that there is nothing essential about the choice of the space I; it could be replaced by any space Y that is path connected and has more then one point (see the proof of Proposition 27 on p. 65).

We will define the diagram $\boldsymbol{X}$ inductively over the filtrations $\mathrm{F}^n\mathcal{D}$ of $\mathcal{D}$ (see Definition 4 on p. 25 and Proposition 3 on p. 26). To start this inductive construction, since $G\colon \mathcal{C} \to \mathcal{D}$ is not a fibering Reedy functor, there are objects $\alpha \in \mathrm{Ob}(\mathcal{C})$ and $\beta \in \mathrm{Ob}(\mathcal{D})$ and a map $\sigma\colon G\alpha \to \beta$ in $\overleftarrow{\mathcal{D}}$ such that the nerve of the category of inverse $\mathcal{C}$-factorizations of (α, σ) (see Definition 7 on p. 26) is nonempty and not connected. Let n_β be the degree of β. We have two cases:

- If $n_\beta = 0$, we begin by letting $\boldsymbol{X}\colon \mathrm{F}^0\mathcal{D} \to \mathrm{Top}$ take β to the unit interval I and all other objects of $\mathrm{F}^0\mathcal{D}$ to $*$ (the one-point space).
- If $n_\beta > 0$, we begin by letting $\boldsymbol{X}\colon \mathrm{F}^{(n_\beta)-1}\mathcal{D} \to \mathrm{Top}$ be the constant functor at $*$ (the one-point space). Then, to extend $\boldsymbol{X}$ from $\mathrm{F}^{(n_\beta)-1}\boldsymbol{D}$ to $\mathrm{F}^{n_\beta}\boldsymbol{D}$, we let $\boldsymbol{X}_\beta = I$, the unit interval. We factor $\mathrm{L}_\beta\boldsymbol{X} \to \mathrm{M}_\beta\boldsymbol{X}$ as

  $$\mathrm{L}_\beta\boldsymbol{X} \longrightarrow I \longrightarrow \mathrm{M}_\beta\boldsymbol{X}$$

  where the first map is the constant map at $0 \in I$ and the second map is the unique map $I \to *$ (since $\boldsymbol{X}_\gamma = *$ is the terminal object of Top for all objects γ of degree less than n_β, that matching object is $*$). If γ is any other object of $\mathcal{D}$ of degree n_β, we let $\boldsymbol{X}_\gamma = \mathrm{M}_\gamma\boldsymbol{X}$ and let $\mathrm{L}_\gamma\boldsymbol{X} \to \boldsymbol{X}_\gamma \to \mathrm{M}_\gamma\boldsymbol{X}$ be the natural map followed by the identity map.

We now define $X\colon F^n\mathcal{D} \to \mathrm{Top}$ for $n > n_\beta$ inductively on n by letting $X_\gamma = \mathrm{M}_\gamma X$ for every object γ of degree n and letting the factorization $\mathrm{L}_\gamma X \to X_\gamma \to \mathrm{M}_\gamma X$ be the natural map followed by the identity map.

Proposition 24 – *The $\mathcal{D}$-diagram of topological spaces X is fibrant.*

Proof. The matching map at the object β of $\mathcal{D}$ is the map $I \to *$, which is a fibration, and the matching map at every other object of $\mathcal{D}$ is an identity map, which is also a fibration. □

We now give a more detailed description of the diagram X.

Proposition 25 –

1. *For every object γ in $\mathcal{D}$ the space X_γ is homeomorphic to a product of unit intervals, one for each map $\gamma \to \beta$ in $\overleftarrow{\mathcal{D}}$ (and so, for objects γ for which there are no maps $\gamma \to \beta$ in $\overleftarrow{\mathcal{D}}$, the space X_γ is the empty product, and is thus equal to the terminal object, the one-point space $*$).*

2. *Under the isomorphisms of part 1, if $\tau\colon \gamma \to \delta$ is a map in $\overleftarrow{\mathcal{D}}$, then the projection of $X_\tau\colon X_\gamma \to X_\delta$ onto the factor I of X_δ indexed by a map $\mu\colon \delta \to \beta$ in $\overleftarrow{\mathcal{D}}$ is the projection of X_γ onto the factor I of X_γ indexed by $\mu\tau\colon \gamma \to \beta$.*

Proof. We will use an induction on n to prove both parts of the proposition simultaneously for the restriction of X to each filtration $\mathrm{F}^n\mathcal{D}$ of $\mathcal{D}$. The induction is begun at $n = n_\beta$ because the only map in $\mathrm{F}^{n_\beta}\overleftarrow{\mathcal{D}}$ to β is the identity map of β, the only object of $\mathrm{F}^{n_\beta}\overleftarrow{\mathcal{D}}$ at which X is not a single point is β, and $X_\beta = I$.

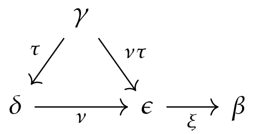
Suppose now that $n > n_\beta$, the statement is true for the restriction of X to $\mathrm{F}^{n-1}\mathcal{D}$, and that γ is an object of degree n. The space X_γ is defined to be the matching object $\mathrm{M}_\gamma X = \lim_{\partial(\gamma\downarrow\overleftarrow{\mathcal{D}})} X$. There is a discrete subcategory $\mathcal{E}_\gamma$ of the matching category $\partial(\gamma\downarrow\overleftarrow{\mathcal{D}})$ consisting of the maps $\gamma \to \beta$ in $\overleftarrow{\mathcal{D}}$, and so there is a projection map

$$\mathrm{M}_\gamma X = \lim_{\partial(\gamma\downarrow\overleftarrow{\mathcal{D}})} X \longrightarrow \lim_{\mathcal{E}_\gamma} X = \prod_{(\gamma\to\beta)\in\overleftarrow{\mathcal{D}}} X_\beta = \prod_{(\gamma\to\beta)\in\overleftarrow{\mathcal{D}}} I.$$

We will show that that projection map $p\colon \lim_{\partial(\gamma\downarrow\overleftarrow{\mathcal{D}})} X \to \prod_{(\gamma\to\beta)\in\overleftarrow{\mathcal{D}}} I$ is a homeomorphism by defining an inverse homeomorphism

$$q\colon \prod_{(\gamma\to\beta)\in\overleftarrow{\mathcal{D}}} I \longrightarrow \lim_{\partial(\gamma\downarrow\overleftarrow{\mathcal{D}})} X.$$

We define the map q by defining its projection onto $X_{(\tau\colon \gamma\to\delta)} = X_\delta$ for each object $(\tau\colon \gamma \to \delta)$ of $\partial(\gamma\downarrow\overleftarrow{\mathcal{D}})$. The induction hypothesis implies that $X_\tau = X_\delta$ is isomorphic to $\prod_{(\delta\to\beta)\in\overleftarrow{\mathcal{D}}} I$, and we let the projection onto the factor indexed by $\mu\colon \delta \to \beta$ be the projection of $\prod_{(\gamma\to\beta)\in\overleftarrow{\mathcal{D}}} I$ onto the factor indexed by $\mu\tau\colon \gamma \to \beta$. To see that this defines a map to $\lim_{\partial(\gamma\downarrow\overleftarrow{\mathcal{D}})} \boldsymbol{X}$, let $\nu\colon \delta \to \epsilon$ be a map from $\tau\colon \gamma \to \delta$ to $\nu\tau\colon \gamma \to \epsilon$ in $\partial(\gamma\downarrow\overleftarrow{\mathcal{D}})$ (see Diagram 13). The induction hypothesis implies that the projection of the map $\boldsymbol{X}_\nu\colon \boldsymbol{X}_\tau = \boldsymbol{X}_\delta \to \boldsymbol{X}_{\nu\tau} = \boldsymbol{X}_\epsilon$ onto the factor of $\boldsymbol{X}_\epsilon$ indexed by $\xi\colon \epsilon \to \beta$ in $\overleftarrow{\mathcal{D}}$ is the projection of $\boldsymbol{X}_\tau = \boldsymbol{X}_\delta$ onto the factor indexed by $\xi\nu\colon \delta \to \beta$.



(13)

Thus, the projection of the composition $\prod_{(\gamma\to\beta)\in\overleftarrow{\mathcal{D}}} I \to \boldsymbol{X}_\tau = \boldsymbol{X}_\delta \xrightarrow{\boldsymbol{X}_\nu} \boldsymbol{X}_{\nu\tau} = \boldsymbol{X}_\epsilon$ onto the factor indexed by $\xi\colon \epsilon \to \beta$ equals the projection of $\prod_{(\gamma\to\beta)\in\overleftarrow{\mathcal{D}}} I$ onto the factor indexed by $\xi\nu\tau\colon \gamma \to \beta$, which equals that same projection of the map $\prod_{(\gamma\to\beta)\in\overleftarrow{\mathcal{D}}} I \to \boldsymbol{X}_{\nu\tau:\gamma\to\epsilon} = \boldsymbol{X}_\epsilon$. Thus, we have defined the map q.

It is immediate from the definitions that pq is the identity map of $\prod_{(\gamma\to\beta)\in\overleftarrow{\mathcal{D}}} I$. To see that qp is the identity map of $\lim_{\partial(\gamma\downarrow\overleftarrow{\mathcal{D}})} \boldsymbol{X}$, we first note that the definitions immediately imply that the projection of qp onto each $\boldsymbol{X}_{(\gamma\to\beta)} = \boldsymbol{X}_\beta$ equals the corresponding projection of the identity map of $\lim_{\partial(\gamma\downarrow\overleftarrow{\mathcal{D}})} \boldsymbol{X}$. If $\tau\colon \gamma \to \delta$ is any other object of $\partial(\gamma\downarrow\overleftarrow{\mathcal{D}})$, then the induction hypothesis implies that $\boldsymbol{X}_\tau = \boldsymbol{X}_\delta$ is homeomorphic to the product $\prod_{(\delta\to\beta)\in\overleftarrow{\mathcal{D}}} I$. Every $\mu\colon \delta \to \beta$ in $\overleftarrow{\mathcal{D}}$ defines a map $\mu_*\colon (\tau\colon \gamma \to \delta) \to (\mu\tau\colon \gamma \to \beta)$ in $\partial(\gamma\downarrow\overleftarrow{\mathcal{D}})$, and the induction hypothesis implies that the map $\boldsymbol{X}_\mu\colon \boldsymbol{X}_\tau = \boldsymbol{X}_\delta \to \boldsymbol{X}_{\mu\tau} = \boldsymbol{X}_\beta = I$ is projection onto the factor indexed by μ. Thus, for any map to $\lim_{\partial(\gamma\downarrow\overleftarrow{\mathcal{D}})} \boldsymbol{X}$, its projection onto $\boldsymbol{X}_\tau = \boldsymbol{X}_\delta$ is determined by its projections onto the $\boldsymbol{X}_{(\gamma\to\beta)\in\overleftarrow{\mathcal{D}}}$; since qp and the identity map agree on those projections, qp must equal the identity map. This completes the induction for part 1.

For part 2, for every map $\tau\colon \gamma \to \delta$ in $\overleftarrow{\mathcal{D}}$ the map $\boldsymbol{X}_\tau\colon \boldsymbol{X}_\gamma \to \boldsymbol{X}_\delta$ equals the composition

$$\boldsymbol{X}_\gamma \longrightarrow \lim_{\partial(\gamma\downarrow\overleftarrow{\mathcal{D}})} \boldsymbol{X} \longrightarrow \boldsymbol{X}_\delta$$

where the first map is the matching map of $\boldsymbol{X}$ at γ and the second is the projection from the limit $\lim_{\partial(\gamma\downarrow\overleftarrow{\mathcal{D}})} \boldsymbol{X} \to \boldsymbol{X}_{(\tau\colon \gamma\to\delta)} = \boldsymbol{X}_\delta$ (this is the case for *every* $\mathcal{D}$-diagram in $\mathcal{M}$, not just for $\boldsymbol{X}$). Since the matching map at every object other than β is the identity map, the map $\boldsymbol{X}_\tau\colon \boldsymbol{X}_\gamma \to \boldsymbol{X}_\delta$ is the projection $\lim_{\partial(\gamma\downarrow\overleftarrow{\mathcal{D}})} \boldsymbol{X} \to \boldsymbol{X}_{(\tau\colon \gamma\to\delta)} = \boldsymbol{X}_\delta$.

The discussion in the previous paragraph shows that the projection of $\boldsymbol{X}_\tau\colon \boldsymbol{X}_\gamma \to \boldsymbol{X}_\delta$ onto the factor of $\boldsymbol{X}_\delta$ indexed by $\mu\colon \delta \to \beta$ is the projection of $\boldsymbol{X}_\gamma$ onto the factor indexed by $\mu\tau\colon \gamma \to \beta$. This completes the induction for part 2. □

We now consider the diagram $G^*\boldsymbol{X}$ that $G\colon \mathcal{C} \to \mathcal{D}$ induces on $\mathcal{C}$ from $\boldsymbol{X}$.

Proposition 26 – *The matching object* $\mathrm{M}^{\mathcal{C}}_\alpha G^*\boldsymbol{X} = \lim_{\partial(\alpha\downarrow\overleftarrow{\mathcal{C}})} G^*\boldsymbol{X}$ *of the induced diagram on* $\mathcal{C}$ *at* α *is homeomorphic to a product of unit intervals indexed by the union over the maps* $\tau\colon G\alpha \to \beta$ *in* $\overleftarrow{\mathcal{D}}$ *of the sets of path components of the nerve of the category of inverse* $\mathcal{C}$*-factorizations of* (α,τ)*. That is,*

$$\mathrm{M}^{\mathcal{C}}_\alpha G^*\boldsymbol{X} \approx \prod_{(\tau\colon G\alpha\to\beta)\in\overleftarrow{\mathcal{D}}} \Bigg(\prod_{\pi_0 \mathrm{N}(\mathrm{Fact}_{\overleftarrow{\mathcal{C}}}(\alpha,\tau))} I \Bigg).$$

Proof. Let $S = \coprod_{(\alpha\to\gamma)\in \mathrm{Ob}(\partial(\alpha\downarrow\overleftarrow{\mathcal{C}}))} \overleftarrow{\mathcal{D}}(G\gamma,\beta)$, the disjoint union over all objects $\alpha \to \gamma$ of $\partial(\alpha\downarrow\overleftarrow{\mathcal{C}})$ of the set of maps $\overleftarrow{\mathcal{D}}(G\gamma,\beta)$. An element of S is then an ordered pair $\big((\nu\colon \alpha\to\gamma),(\mu\colon G\gamma\to\beta)\big)$ where $\nu\colon \alpha\to\gamma$ is an object of $\partial(\alpha\downarrow\overleftarrow{\mathcal{C}})$ and $\mu\colon G\gamma\to\beta$ is a map in $\overleftarrow{\mathcal{D}}$, and is thus an object of the category of inverse $\mathcal{C}$-factorizations of the composition $(\alpha, G\alpha \xrightarrow{G\nu} G\gamma \xrightarrow{\mu} \beta)$, i.e., of $(\alpha, \mu\circ G\nu\colon G\alpha\to\beta)$. Every object of the category of inverse $\mathcal{C}$-factorizations of every map $(\alpha,\tau\colon G\alpha\to\beta)$ in $\overleftarrow{\mathcal{D}}$ appears exactly once, and so the set S is the union over all maps $\tau\colon G\alpha\to\beta$ in $\overleftarrow{\mathcal{D}}$ of the set of objects of the category of inverse $\mathcal{C}$-factorizations of (α,τ).

Proposition 25 on p. 62 implies that for every object $\tau\colon \alpha\to\gamma$ in $\partial(\alpha\downarrow\overleftarrow{\mathcal{C}})$ the space $(G^*\boldsymbol{X})_\tau = (G^*\boldsymbol{X})_\gamma = \boldsymbol{X}_{G\gamma}$ is a product of unit intervals, one for each map $G\gamma\to\beta$ in $\overleftarrow{\mathcal{D}}$, and so the product over all objects $\tau\colon\alpha\to\gamma$ of $\partial(\alpha\downarrow\overleftarrow{\mathcal{C}})$ of $(G^*\boldsymbol{X})_\tau = (G^*\boldsymbol{X})_\gamma = \boldsymbol{X}_{G\gamma}$ is homeomorphic to the product of unit intervals indexed by S, i.e.,

$$\prod_{(\alpha\to\gamma)\in\mathrm{Ob}(\partial(\alpha\downarrow\overleftarrow{\mathcal{C}}))} (G^*\boldsymbol{X})_\gamma \approx \prod_S I.$$

The matching object $\mathrm{M}^{\mathcal{C}}_\alpha G^*\boldsymbol{X}$ is a subspace of that product. More specifically, it is the subspace consisting of the points such that, for every map

$$\begin{array}{ccc} & \alpha & \\ {}^{\nu}\swarrow & & \searrow^{\nu'} \\ \gamma & \xrightarrow[\tau]{} & \gamma' \end{array}$$

in $\partial(\alpha\downarrow\overleftarrow{\mathcal{C}})$ from $\nu\colon\alpha\to\gamma$ to $\nu'\colon\alpha\to\gamma'$ and every map $\mu'\colon G\gamma'\to\beta$ in $\overleftarrow{\mathcal{D}}$, the projection onto the factor indexed by $\big((\nu'\colon\alpha\to\gamma'),(\mu'\colon G\gamma'\to\beta)\big)$ equals the projection onto the factor indexed by $\big((\nu\colon\alpha\to\gamma),(\mu'\circ(G\tau)\colon G\gamma\to\beta)\big)$.

Generate an equivalence relation on S by letting $\big((\nu\colon \alpha \to \gamma), (\mu\colon G\gamma \to \beta)\big)$ be equivalent to $\big((\nu'\colon \alpha \to \gamma'), (\mu'\colon G\gamma' \to \beta)\big)$ if there is a map $\tau\colon \gamma \to \gamma'$ in $\overleftarrow{\mathcal{C}}$ such that $\tau\nu = \nu'$ and $\mu' \circ (G\tau) = \mu$, i.e., if there is a map in the category of inverse $\mathcal{C}$-factorizations of $(\alpha, \mu \circ (G\nu)\colon G\alpha \to \beta)$ from $\big((\nu\colon \alpha \to \gamma), (\mu\colon G\gamma \to \beta)\big)$ to $\big((\mu'\colon \alpha \to \gamma'), (\mu'\colon G\gamma' \to \beta)\big)$; let T be the set of equivalence classes. This makes two objects in the category of inverse $\mathcal{C}$-factorizations of a map equivalent if there is a zig-zag of maps in that category from one to the other, i.e., if those two objects are in the same component of the nerve, and so the set T is the disjoint union over all maps $\tau\colon G\alpha \to \beta$ in $\overleftarrow{\mathcal{D}}$ of the set of components of the nerve of the category of inverse $\mathcal{C}$-factorizations of (α, τ), i.e.,

$$T = \coprod_{(\tau\colon G\alpha \to \beta) \in \overleftarrow{\mathcal{D}}} \pi_0 \mathrm{N}\Big(\mathrm{Fact}_{\overleftarrow{\mathcal{C}}}(\alpha, \tau)\Big).$$

Let T' be a set of representatives of the equivalence classes T (i.e., let T' consist of one element of S from each equivalence class); we will show that the composition

$$\mathrm{M}^{\mathcal{C}}_{\alpha} G^{*}\boldsymbol{X} \xrightarrow{\ \subset\ } \prod_{S} I \xrightarrow{\ p'\ } \prod_{T'} I$$

(where p' is the projection) is a homeomorphism. We will do that by constructing an inverse $q\colon \prod_{T'} I \to \mathrm{M}^{\mathcal{C}}_{\alpha} G^{*}\boldsymbol{X}$ to the map $p\colon \mathrm{M}^{\mathcal{C}}_{\alpha} G^{*}\boldsymbol{X} \to \prod_{T'} I$ (where p is the restriction of p' to $\mathrm{M}^{\mathcal{C}}_{\alpha} G^{*}\boldsymbol{X}$).

We first construct a map $q'\colon \prod_{T'} I \to \prod_{S} I$ by letting the projection of q' onto the factor indexed by $s \in S$ be the projection of $\prod_{T'} I$ onto the factor indexed by the unique $t \in T'$ that is equivalent to s. The description above of the subspace $\mathrm{M}^{\mathcal{C}}_{\alpha} G^{*}\boldsymbol{X}$ of $\prod_{S} I$ makes it clear that q' factors through $\mathrm{M}^{\mathcal{C}}_{\alpha} G^{*}\boldsymbol{X}$ and thus defines a map $q\colon \prod_{T'} I \to \mathrm{M}^{\mathcal{C}}_{\alpha}\boldsymbol{X}$.

The composition pq equals the identity of $\prod_{T'} I$ because the composition $p'q'$ equals the identity of $\prod_{T'} I$. To see that the composition qp equals the identity of $\mathrm{M}^{\mathcal{C}}_{\alpha} G^{*}\boldsymbol{X}$, it is sufficient to see that the projection of qp onto the factor I indexed by every element s of S agrees with that of the identity map of $\mathrm{M}^{\mathcal{C}}_{\alpha} G^{*}\boldsymbol{X}$. Since the projections of points in $\mathrm{M}^{\mathcal{C}}_{\alpha} G^{*}\boldsymbol{X}$ onto factors indexed by equivalent elements of S are equal, and it is immediate that the projection of $\mathrm{M}^{\mathcal{C}}_{\alpha} G^{*}\boldsymbol{X}$ onto a factor indexed by an element of the set of representatives T' agrees with the corresponding projection of qp, the projections for every element of S must agree, and so qp equals the identity of $\prod_{T'} I$. □

Proposition 27 – *The diagram $G^{*}\boldsymbol{X}$ induced on $\mathcal{C}$ is not a fibrant $\mathcal{C}$-diagram.*

Proof. We will show that the matching map $(G^{*}\boldsymbol{X})_{\alpha} \to \mathrm{M}^{\mathcal{C}}_{\alpha} G^{*}\boldsymbol{X}$ of the induced $\mathcal{C}$-diagram at α is not a fibration. Since the matching object $\mathrm{M}^{\mathcal{C}}_{\alpha} G^{*}\boldsymbol{X}$ is a product of unit intervals (see Proposition 26 on p. 64), it is path connected, and so if the matching map were a fibration, it would be surjective. We will show that the matching map is not surjective.

Since $\sigma: G\alpha \to \beta$ is a map in $\overleftarrow{\mathcal{D}}$ such that the nerve of the category of inverse $\mathcal{C}$-factorizations of (α, σ) is not connected, we can choose objects $(\nu: \alpha \to \gamma, \mu: G\gamma \to \beta)$ and $(\nu': \alpha \to \gamma', \mu': G\gamma' \to \beta)$ of that category that represent different path components of that nerve. Since $\mu \circ (G\nu) = \mu' \circ (G\nu')$, Proposition 25 on p. 62 implies that the projection of the matching map onto the copies of I indexed by those objects are equal, and so the projection onto the $I \times I$ indexed by that pair of components factors as the composition $\boldsymbol{X}_\alpha \to I \to I \times I$, where that second map is the diagonal map and is thus not surjective. □

Proof (Proof of Theorem 13*).* This follows from Proposition 24 on p. 62 and Proposition 27 on the previous page. □

4.5 Proof of Theorem 2

Since $\mathcal{M}$ is complete, the right adjoint of G^* exists and can be constructed pointwise (see Borceux 1994, Thm. 3.7.2 or MacLane 1971, p. 235), and Theorem 1 on p. 22 implies that $(G^{\mathrm{op}})^*: (\mathcal{M}^{\mathrm{op}})^{\mathcal{D}^{\mathrm{op}}} \to (\mathcal{M}^{\mathrm{op}})^{\mathcal{C}^{\mathrm{op}}}$ is a right Quillen functor for every model category $\mathcal{M}^{\mathrm{op}}$ if and only if G^{op} is fibering (because every model category $\mathcal{N}$ is of the form $\mathcal{M}^{\mathrm{op}}$ for $\mathcal{M} = \mathcal{N}^{\mathrm{op}}$).

Proposition 8 on p. 32 implies that the functor $G: \mathcal{C} \to \mathcal{D}$ is cofibering if and only if its opposite $G^{\mathrm{op}}: \mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$ is fibering, and Theorem 1 on p. 22 implies that this is the case if and only if $(G^{\mathrm{op}})^*: (\mathcal{M}^{\mathrm{op}})^{\mathcal{D}^{\mathrm{op}}} \to (\mathcal{M}^{\mathrm{op}})^{\mathcal{C}^{\mathrm{op}}}$ is a right Quillen functor for every model category $\mathcal{M}^{\mathrm{op}}$, which is the case if and only if $G^*: \mathcal{M}^{\mathcal{D}} \to \mathcal{M}^{\mathcal{C}}$ is a left Quillen functor for every model category $\mathcal{M}$ (see Proposition 12 on p. 35 and Proposition 9 on p. 34).

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