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Composition operators with surjective symbol and small approximation numbers

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Abstract

We give a new proof of the existence of a surjective symbol whose associated composition operator on $H^2(\mathbb{D})$ is in all Schatten classes, with the improvement that its approximation numbers can be, in some sense, arbitrarily small. We show, as an application, that, contrary to the 1-dimensional case, for $N \geq 2$, the behavior of the approximation numbers $a_n = a_n(C_\phi)$, or rather of $\beta_N^- = \liminf_{n\to\infty}[a_n]^{1/n^{1/N}}$ or $\beta_N^+ = \limsup_{n\to\infty}[a_n]^{1/n^{1/N}}$, of composition operators on $H^2(\mathbb{D}^N)$ cannot be determined by the image of the symbol.

Keywords: Approximation numbers, cusp map, composition operator, Hardy space, lens map, polydisk.

MSC: Primary 47B33; Secondary 32A35, 46B28.

1 Introduction

We start by recalling some notations and facts.

Let $\mathbb{D}$ be the open unit disk, H^2 the Hardy space on $\mathbb{D}$, and $\varphi\colon \mathbb{D} \to \mathbb{D}$ a non-constant analytic self-map. It is well known[4] that φ induces a composition operator $C_\varphi\colon H^2 \to H^2$ by the formula:

$$C_\varphi(f) = f \circ \varphi,$$

and the connection between the "symbol" φ and the properties of the operator $C_\varphi\colon H^2 \to H^2$, in particular its compactness, can be further studied[5].

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[4] J. Shapiro, 1993, *Composition operators and classical function theory*.

[5] Ibid.

We also recall that the nth approximation number $a_n(T)$, $n = 1, 2, \ldots$, of an operator $T\colon H_1 \to H_2$, between Hilbert spaces H_1 and H_2, is defined as the distance of T to operators of rank $< n$, for the operator-norm:

$$a_n(T) = \inf_{\operatorname{rank} R < n} \|T - R\|. \tag{1}$$

The p-Schatten class $S_p(H_1, H_2)$, $p > 0$ consists of all $T\colon H_1 \to H_2$ such that $\big(a_n(T)\big)_n \in \ell^p$. The approximation numbers have the ideal property:

$$a_n(ATB) \le \|A\|\, a_n(T)\, \|B\|.$$

Let now, for $\xi \in \mathbb{T} = \partial\mathbb{D}$ and $h > 0$, the Carleson window $S(\xi, h)$ be defined as:

$$S(\xi, h) = \{z \in \mathbb{D}\,;\ |z - \xi| \le h\}. \tag{2}$$

For a symbol φ, we define $m_\varphi = \varphi^*(m)$ where m is the Haar measure of $\mathbb{T}$ and $\varphi^*\colon \mathbb{T} \to \overline{\mathbb{D}}$ the (almost everywhere defined) radial limit function associated with φ, namely:

$$\varphi^*(\xi) = \lim_{r \to 1^-} \varphi(r\xi).$$

Finally, we set for $h > 0$:

$$\rho_\varphi(h) = \sup_{\xi \in \mathbb{T}} m_\varphi[S(\xi, h)]. \tag{3}$$

It is known[6] that $\rho_\varphi(h) = \mathrm{O}(h)$ and that C_φ[7] is compact if and only if $\rho_\varphi(h) = \mathrm{o}(h)$ as $h \to 0$. Simpler criteria[8] exist when φ is injective, or even p-valent, meaning that for any $w \in \mathbb{D}$, the equation $\varphi(z) = w$ has at most p solutions.

A measure μ on $\mathbb{D}$ is called α-Carleson, $\alpha \ge 1$, if $\sup_{|\xi|=1} \mu[S(\xi, h)] = \mathrm{O}(h^\alpha)$.

B. MacCluer and J. Shapiro showed in MacCluer and H. Shapiro (1986, Example 3.12) the following result, paradoxical at first glance.

Theorem 1 (MacCluer-Shapiro) – *There exists a* surjective and four-valent *symbol $\varphi\colon \mathbb{D} \to \mathbb{D}$ such that the composition operator $C_\varphi\colon H^2 \to H^2$ is compact.*

Observe that such a symbol φ cannot be one-valent (injective), because it would be an automorphism of $\mathbb{D}$, and C_φ would be invertible and therefore not compact. In Lefèvre et al. (2012, Theorem 4.1), we gave the following improved statement.

[6]J. Shapiro, 1993, *Composition operators and classical function theory*.
[7]MacCluer, 1984, "Spectra of compact composition operators on $H^p(B_N)$".
[8]J. Shapiro, 1993, *Composition operators and classical function theory*.

Theorem 2 – *For every non-decreasing function $\delta\colon (0,1) \to (0,1)$, there exists a two-valent symbol and nearly surjective (i.e. $\varphi(\mathbb{D}) = \mathbb{D} \setminus \{0\}$) symbol ϕ, and $0 < h_0 < 1$, such that:*

$$m(\{z \in \mathbb{T}\,;\ |\phi^*(z)| \geq 1 - h\}) \leq \delta(h) \quad \textit{for } 0 < h \leq h_0. \tag{4}$$

As a consequence, there exists a surjective and four-valent *symbol $\psi\colon \mathbb{D} \to \mathbb{D}$ such that the composition operator $C_\psi\colon H^2 \to H^2$ is in every Schatten class $S_p(H^2)$, $p > 0$.*

Our proof was rather technical and complicated, and based on arguments of barriers and harmonic measures.

The goal of this paper is to give a more precise statement of Theorem 2 in terms of approximation numbers $a_n(C_\varphi)$, and not only in terms of Schatten classes, and with a simpler proof. We then apply this result to show that for the polydisk $\mathbb{D}^N$, $N \geq 2$, the nature (boundedness, compactness, asymptotic behavior of approximation numbers) of the composition operator cannot be determined by the geometry of the image $\phi(\mathbb{D}^N)$ of its symbol ϕ. For certain asymptotic behavior of approximation numbers, this is contrary to the 1-dimensional case (see Li, Queffélec, and Rodríguez-Piazza 2015, Theorem 3.1 and Theorem 3.14).

The notation $A \lesssim B$ means that $A \leq C\,B$ for some positive constant C, and $A \approx B$ that $A \lesssim B$ and $B \lesssim A$.

2 Background and preliminary results

We initiated the study of approximation numbers of composition operators on H^2 in Li, Queffélec, and Rodríguez-Piazza (2012), and proved the following basic results:

Theorem 3 – *If φ is any symbol, then, for some $\delta > 0$ and $r > 0$, or $a > 0$:*

$$a_n(C_\varphi) \geq \delta\, r^n = \delta\, e^{-an}.$$

Moreover, as soon as $\|\varphi\|_\infty = 1$, there exists some sequence ε_n tending to 0 such that:

$$a_n(C_\varphi) \geq \delta\, e^{-n\varepsilon_n}.$$

We also proved in Li, Queffélec, and Rodríguez-Piazza (2012, Theorem 5.1) that:

Proposition 1 – *For any symbol φ, we have:*

$$a_n(C_\varphi) \lesssim \inf_{0<h<1} \left[e^{-nh} + \sqrt{\frac{\rho_\varphi(h)}{h}} \right].$$

We also recall (see Li, Queffélec, and Rodríguez-Piazza 2012) that, for $\gamma > -1$, the weighted Bergman space $\mathcal{B}_\gamma$ is the space of functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ such that:

$$\|f\|_\gamma^2 := \sum_{n=0}^{\infty} \frac{|a_n|^2}{(n+1)^{\gamma+1}} < \infty . \tag{5}$$

Equivalently, $\mathcal{B}_\gamma$ is the space of analytic functions $f \colon \mathbb{D} \to \mathbb{C}$ such that:

$$\int_{\mathbb{D}} |f(z)|^2 \, (\gamma+1)(1-|z|^2)^\gamma \, dA(z) < \infty , \tag{6}$$

where dA is the normalized area measure on $\mathbb{D}$, and then:

$$\int_{\mathbb{D}} |f(z)|^2 \, (\gamma+1)(1-|z|^2)^\gamma \, dA(z) \approx \|f\|_\gamma^2 . \tag{7}$$

The case $\gamma = 0$ corresponds to the usual Bergman space $\mathcal{B}^2$, and the limiting case $\gamma = -1$ to the Hardy space H^2. We wish to note in passing (we will make use of that elsewhere) that the proof of Theorem 5.1 in Li, Queffélec, and Rodríguez-Piazza (2012) easily gives the following result.

Proposition 2 – *Let $\gamma > -1$ and φ a symbol inducing a bounded composition operator $C_\varphi \colon \mathcal{B}_\gamma \to H^2$. Then:*

$$a_n(C_\varphi \colon \mathcal{B}_\gamma \to H^2) \lesssim \inf_{0<h<1} \left((n+1)^{(\gamma+1)/2} \, e^{-nh} + \sup_{0<t\leq h} \sqrt{\frac{\rho_\phi(t)}{t^{2+\gamma}}} \right).$$

Proof. Take $E = z^n \mathcal{B}_\gamma$; this is a subspace of $\mathcal{B}_\gamma$ of codimension $\leq n$. Let $f \in E$ with $\|f\|_\gamma = 1$. Writing $f = z^n g$ with $\|g\|_\gamma^2 \leq (n+1)^{\gamma+1}$ and splitting the integral into two parts, we have, for $0 < h < 1$:

$$\|C_\varphi f\|_{H^2}^2 = \int_{\mathbb{D}} |f|^2 \, dm_\phi \leq (1-h)^{2n} \int_{(1-h)\mathbb{D}} |g|^2 \, dm_\phi + \int_{\mathbb{D} \setminus (1-h)\mathbb{D}} |f|^2 \, dm_\phi .$$

For the first integral, we have:

$$\int_{(1-h)\mathbb{D}} |g|^2 \, dm_\phi \leq \int_{\mathbb{D}} |g|^2 \, dm_\phi = \|C_\phi \, g\|_{H^2}^2 \leq \|C_\phi\|_{\mathcal{B}_\gamma \to H^2}^2 \|g\|_\gamma^2 . \tag{8}$$

For the second integral, we have:

$$\int_{\mathbb{D} \setminus (1-h)\mathbb{D}} |f|^2 \, dm_\phi \leq \|J \colon \mathcal{B}_\gamma \to L^2(\mu_h)\|^2 ,$$

where μ_h is the restriction of m_φ to the annulus $\{z \in \mathbb{D};\ 1-h < |z| < 1\}$ and J the canonical injection of $\mathcal{B}_\gamma$ into $L^2(\mu_h)$. Hence Stegenga's version of the Carleson embedding theorem for $\mathcal{B}_\gamma$ (Stegenga 1980, Theorem 1.2; see Hastings 1975 for the unweighted case; see also Duren and Schuster 2004, p. 62 or Zhu 2007, p. 167) gives us:

$$\int_{\mathbb{D}\setminus(1-h)\mathbb{D}} |f|^2\, dm_\phi \lesssim \sup_{0<t\leq h} \frac{\rho_\phi(t)}{t^{2+\gamma}} \cdot \tag{9}$$

Putting (8) and (9) together, that gives:

$$\|C_\varphi f\|_{H^2} \lesssim \mathrm{e}^{-nh}(n+1)^{(\gamma+1)/2} + \sup_{0<t\leq h} \sqrt{\frac{\rho_\phi(t)}{t^{2+\gamma}}} \cdot$$

In other terms, using the Gelfand numbers c_k:

$$c_{n+1}(C_\phi \colon \mathcal{B}_\gamma \to H^2) \lesssim (n+1)^{(\gamma+1)/2}\, \mathrm{e}^{-nh} + \sup_{0<t\leq h} \sqrt{\frac{\rho_\phi(t)}{t^{2+\gamma}}} \cdot$$

As $a_{n+1} = c_{n+1}$ and as we can ignore the difference between a_n and a_{n+1}, that finishes the proof. □

As an application, we mention the following result. We refer to Li, Queffélec, and Rodríguez-Piazza (2013, Section 4.1) for the definition of the cusp map, denoted χ.

Theorem 4 – *Let $\chi\colon \mathbb{D} \to \mathbb{D}$ be the cusp map and $\Phi\colon \mathbb{D}^N \to \mathbb{D}^N$ the diagonal map defined by:*

$$\Phi(z_1, z_2, \dots, z_N) = \big(\chi(z_1), \chi(z_1), \dots, \chi(z_1)\big). \tag{10}$$

Then, the composition operator C_Φ maps $H^2(\mathbb{D}^N)$ to itself and:

$$a_n(C_\Phi) \lesssim e^{-d\sqrt{n}} \tag{11}$$

where d is a positive constant depending only on N.

Remark 1 – We have to compare with Bayart et al. (2018, Theorem 6.2) where, for:

$$\Psi(z_1, \dots, z_N) = \big(\chi(z_1), \dots, \chi(z_N)\big),$$

it is shown that, for constants $b \geq a > 0$ depending only on N:

$$\mathrm{e}^{-b\,(n^{1/N}/\ln n)} \lesssim a_n(C_\Psi) \lesssim \mathrm{e}^{-a\,(n^{1/N}/\ln n)}.$$

Note also that for $N = 1$, the estimate of Theorem 4 is very crude.

Proof (Proof of Theorem 4 on the previous page). Take $\gamma = N - 2$. As in Li, Queffélec, and Rodríguez-Piazza (n.d.[a], Section 4), we have thanks to the Cauchy-Schwarz inequality, and the fact that $\sum_{|\alpha|=n} 1 \approx (n+1)^{N-1}$, a factorization:

$$C_\Phi = J C_\chi M\,,$$

where $M\colon H^2(\mathbb{D}^N) \to \mathcal{B}_\gamma$ is defined by $Mf = g$ with:

$$g(z) = f(z, z, \dots, z) = \sum_{n=0}^{\infty} \Bigg(\sum_{|\alpha|=n} a_\alpha \Bigg) z^n\,, \quad z \in \mathbb{D}\,, \tag{12}$$

for

$$f(z_1, z_2, \dots, z_N) = \sum_\alpha a_\alpha z_1^{\alpha_1} \cdots z_N^{\alpha_N}\,,$$

and where $J\colon H^2(\mathbb{D}) \to H^2(\mathbb{D}^N)$ is the canonical injection given by:

$$(Jh)(z_1, z_2, \dots, z_N) = h(z_1)\,. \tag{13}$$

This corresponds to a diagram:

$$H^2(\mathbb{D}^N) \xrightarrow{M} \mathcal{B}_\gamma \xrightarrow{C_\chi} H^2(\mathbb{D}) \xrightarrow{J} H^2(\mathbb{D}^N)\,, \tag{14}$$

where $C_\chi\colon \mathcal{B}_\gamma = \mathcal{B}_{N-2} \to H^2(\mathbb{D})$ is a bounded operator. Indeed, we have the behavior[9]:

$$|1 - \chi^*(\mathrm{e}^{i\theta})| \approx \frac{1}{\ln(1/|\theta|)}\,,$$

and this implies, with c an absolute constant:

$$\begin{aligned} m_\chi[S(\xi, h)] &\lesssim m_\chi[S(1, h)] = m(\{|\chi^*(\mathrm{e}^{i\theta}) - 1| < h) \\ &\lesssim m[\{c/\ln(1/|\theta|) < h\}] \leq \mathrm{e}^{-c/h}\,; \end{aligned} \tag{15}$$

in particular $\rho_\chi(h) \leq \mathrm{e}^{-c/h} = \mathrm{O}(h^N)$, so m_χ is an N-Carleson measure and the Stegenga-Carleson theorem[10] says that the operator $C_\chi\colon \mathcal{B}_{N-2} \to H^2(\mathbb{D})$ is bounded.

Now Proposition 2 on p. 4 with (15) give:

$$a_n(C_\chi\colon \mathcal{B}_\gamma \to H^2) \lesssim \inf_{0<h<1} \Big[(n+1)^{(N-1)/2}\, \mathrm{e}^{-nh} + \mathrm{e}^{-c/h} h^{-N/2} \Big].$$

Adjusting $h = 1/\sqrt{n}$, we get $a_n(C_\chi\colon \mathcal{B}_\gamma \to H^2) \lesssim \mathrm{e}^{-d\sqrt{n}}$ for some positive constant d. Finally, the factorization $C_\Phi = J C_\chi M$ and the ideal property of approximation numbers give the result. □

In the case of lens maps, Proposition 2 on p. 4 gives very poor estimates. We avoid using this theorem in Li, Queffélec, and Rodríguez-Piazza (n.d.[a], Section 4), when $N = 2$, using the semi-group property of those lens maps. The same proof gives for arbitrary $N \geq 2$ the following result.

Theorem 5 – *Let λ_θ the lens map with parameter θ, $0 < \theta < 1$, and let $\Phi\colon \mathbb{D}^N \to \mathbb{D}^N$ be the diagonal map defined by:*

$$\Phi(z_1, z_2, \dots, z_N) = \Big(\lambda_\theta(z_1), \lambda_\theta(z_1), \dots, \lambda_\theta(z_1)\Big). \tag{16}$$

Then:

1) *if $\theta > 1/N$, C_Φ is unbounded on $H^2(\mathbb{D}^N)$;*
2) *if $\theta = 1/N$, C_Φ is bounded and not compact on $H^2(\mathbb{D}^N)$;*
3) *if $\theta < 1/N$, C_Φ is compact on $H^2(\mathbb{D}^N)$ and moreover:*

$$a_n(C_\Phi) \lesssim e^{-d\sqrt{n}} \tag{17}$$

for a constant $d > 0$ depending only on θ and N.

Remark 2 – In Bayart et al. (2018, Theorem 6.1), it is shown that, for:

$$\Psi(z_1, \dots, z_N) = \Big(\lambda_\theta(z_1), \dots, \lambda_\theta(z_N)\Big),$$

we have, for constants $b \geq a > 0$, depending only on θ and N:

$$\mathrm{e}^{-b\, n^{1/(2N)}} \lesssim a_n(C_\Psi) \lesssim \mathrm{e}^{-a\, n^{1/(2N)}}.$$

Proof (Proof of Theorem 5). That had been proved, for $N = 2$ in Li, Queffélec, and Rodríguez-Piazza (n.d.[a], Theorem 4.2 and Theorem 4.4). For convenience of the reader, we sketch the proof.

Assume first $\theta \leq 1/N$, and write $\lambda_\theta = \lambda_{N\theta} \circ \lambda_{1/N}$, where we set, for convenience, $\lambda_1(z) = z$, so $C_{\lambda_1} = \mathrm{I}d$. As in the proof of Theorem 4 on p. 5 (see Li, Queffélec, and Rodríguez-Piazza n.d.(a), Section 4), we have a factorization:

$$C_\Phi = J C_{\lambda_{N\theta}} C_{\lambda_{1/N}} M,$$

where M and J are defined in (12) and (13).

[9]Li, Queffélec, and Rodríguez-Piazza, 2013, "Estimates for approximation numbers of some classes of composition operators on the Hardy space", Lemma 4.2.

[10]Stegenga, 1980, "Multipliers of the Dirichlet space", Theorem 1.2.

This corresponds to a diagram (recall that $\gamma = N-2$):

$$H^2(\mathbb{D}^N) \xrightarrow{M} \mathcal{B}_\gamma \xrightarrow{C_{\lambda_{1/N}}} H^2(\mathbb{D}) \xrightarrow{C_{\lambda_{N\theta}}} H^2(\mathbb{D}) \xrightarrow{J} H^2(\mathbb{D}^N).$$

The second arrow is bounded, since we know[11] that the pullback measure $m_{\lambda_{1/N}}$ is N-Carleson, so that $C_{\lambda_{1/N}}$ maps $\mathcal{B}_{N-2}$ to $H^2(\mathbb{D})$ by the Stegenga-Carleson embedding theorem[12].

For $\theta < 1/N$, we have $N\theta < 1$ and $C_{\lambda_{N\theta}}$ is compact and, for some constant $b = b(\theta)$, we have $a_n(C_{\lambda_{N\theta}}) \lesssim \mathrm{e}^{-b\sqrt{n}}$[13]. Hence C_Φ is compact and $a_n(C_\Phi) \lesssim \mathrm{e}^{-b\sqrt{n}}$.

Now, for $\theta \geq 1/N$, we consider the reproducing kernels:

$$K_{a_1,\ldots,a_N}(z_1,\ldots,z_N) = \prod_{j=1}^{N} \frac{1}{1-\bar{a}_j z_j} \cdot$$

We have:

$$\|K_{a_1,\ldots,a_N}\|^2 = \prod_{j=1}^{N} \frac{1}{1-|a_j|^2}$$

and:

$$C_\Phi^*(K_{a_1,\ldots,a_N}) = K_{\lambda_\theta(a_1),\ldots,\lambda_\theta(a_1)},$$

so:

$$\|C_\Phi^*(K_{a_1,\ldots,a_N})\|^2 = \left(\frac{1}{1-|\lambda_\theta(a_1)|^2}\right)^N \cdot$$

Since:

$$1-|\lambda_\theta(a_1)|^2 \approx 1-|\lambda_\theta(a_1)| \approx (1-|a_1|)^\theta,$$

we see that $\|C_\Phi^*(K_{a_1,\ldots,a_N})\|/\|K_{a_1,\ldots,a_N}\|$ is not bounded for $\theta > 1/N$, so C_ϕ is then not bounded; and it does not converge to 0 for $\theta = 1/N$, so C_Φ is then not compact. □

[11] Lefèvre et al., 2013b, “Some new properties of composition operators associated with lens maps”, Lemma 3.3.

[12] Stegenga, 1980, “Multipliers of the Dirichlet space”, Theorem 1.2.

[13] Lefèvre et al., 2013b, “Some new properties of composition operators associated with lens maps”, Theorem 2.1.

3 Surjectivity

Let us come back to our surjectivity issues.

Let us first remark that Theorem 2 on p. 3 gives the following result.

Theorem 6 – *For every non-decreasing function* $\delta\colon (0,1) \to (0,1)$, *there exists a* surjective and four-valent *symbol* ψ, *and* $0 < h_0 < 1$, *such that, for* $0 < h \leq h_0$*:*

$$m(\{z \in \mathbb{T}\,;\, |\phi^*(z)| \geq 1 - h\}) \leq \delta(h). \tag{18}$$

Proof. Just observe that the passage from "φ two-valent and nearly surjective" to "ψ four-valent and surjective" is harmless: for this, consider the Blaschke product:

$$B(z) = \left(\frac{z-a}{1-az}\right)^2,$$

where $0 < a < 1$, and take $\psi = B \circ \varphi$; we observe that $B(\mathbb{D} \setminus \{0\}) = \mathbb{D}$ since $a^2 = B\Big(\frac{2a}{1+a^2}\Big)$, and, for $z \in \mathbb{D}$:

$$\frac{1 - |B(z)|}{1 - |z|} \geq \frac{1 - |\frac{z-a}{1-az}|^2}{1 - |z|^2} = \frac{1 - a^2}{|1 - az|^2} \geq \frac{1 - a^2}{4},$$

so that:

$$m(|\psi^*| > 1 - h) = m(1 - |B \circ \varphi^*| < h) \leq m\Big(1 - |\varphi^*| \leq \kappa_a h\Big),$$

with $\kappa_a = 4/(1 - a^2)$. Hence, this map ψ is surjective, four-valent, and satisfies (18), as well, up to a change of $\delta(h)$ to $\delta(h/\kappa_a)$ for φ at the beginning. □

3.1 A more precise statement

Our new statement is as follows.

Theorem 7 – *For every positive sequence* $(\varepsilon_n)_n$ *with limit* 0, *there exists a* surjective and four-valent *symbol* φ *such that:*

$$a_n(C_\varphi) \lesssim e^{-n\varepsilon_n}.$$

Consequently, there exists a surjective and four-valent *symbol* $\varphi\colon \mathbb{D} \to \mathbb{D}$ *such that the composition operator* $C_\varphi\colon H^2 \to H^2$ *is in every Schatten class* $S_p(H^2)$, $p > 0$.

Proof. Observe first that $\|\varphi\|_\infty = 1$ when φ is surjective, so that, in view of Theorem 3 on p. 3, we cannot dispense with the numbers ε_n, even if they can tend to 0 arbitrarily slowly.

Now, we can choose $\delta\colon (0,1) \to (0,1)$ non-decreasing such that $\delta(\varepsilon_n) \le \mathrm{e}^{-n\varepsilon_n}$ for all n, and then, using Theorem 6 on the previous page, we get a surjective and four-valent symbol φ, satisfying for all h small enough:

$$\rho_\varphi(h) \le h\,\delta^2(h)\,.$$

Proposition 1 on p. 3 gives:

$$a_n(C_\varphi) \lesssim \inf_{0<h<1} \Big[\mathrm{e}^{-nh} + \delta(h)\Big].$$

Adjusting $h = \varepsilon_n$, we get $a_n(C_\varphi) \lesssim \mathrm{e}^{-n\varepsilon_n}$.

To get the second part of the theorem, just take $\varepsilon_n = n^{-1/2}$. □

3.2 A simplified proof of Theorem 2 on p. 3

We give here the announced simplified proof of Theorem 2 on p. 3. This proof is based on the following key lemma, in which $\mathcal{H}(\mathbb{D})$ denotes the set of holomorphic functions on $\mathbb{D}$.

Lemma 1 (Lefèvre et al. 2013a, Lemma 2.11) – *There exists a numerical constant C such that, if $f \in \mathcal{H}(\mathbb{D})$ satisfies, for some $\alpha \in \mathbb{R}$:*

$$\begin{cases} \mathfrak{Im}[f(0)] < \alpha \\ f(\mathbb{D}) \subseteq \{z \in \mathbb{C}\,;\ 0 < \mathfrak{Re}\, z < \pi\} \cup \{z \in \mathbb{C}\,;\ \mathfrak{Im}\, z < \alpha\}, \end{cases}$$

then:

$$m(\{\mathfrak{Im}\, f^* > y\}) \le C\, e^{\alpha - y}\,, \quad \textit{for } y \ge \alpha\,.$$

Proof (Proof of Theorem 2 on p. 3). Let $g\colon (0,\infty) \to (0,\infty)$ be a continuous decreasing function such that:

$$\lim_{t\to 0^+} g(t) = +\infty\,, \quad g(\pi) = \pi\,, \quad \lim_{t\to+\infty} g(t) = 0\,.$$

Then let Ω be the simply connected region defined by:

$$\Omega = \{x + iy\,;\ x > 0\,,\ g(x) < y < g(x) + 4\pi\}\,,$$

and $f : \mathbb{D} \to \Omega$ be a Riemann map such that $f(0) = \pi + 3i\pi$. Observe that we can apply Lemma 1 to f with $\alpha = 5\pi$ since $\mathfrak{Im}\, f(0) = 3\pi$ and if $f(z) = x + iy$ with $x \ge \pi$; hence:

$$\mathfrak{Im}\, f(z) = y < g(x) + 4\pi \le g(\pi) + 4\pi = 5\pi\,.$$

Finally, consider the symbol $\varphi = \mathrm{e}^{-f}$. It is nearly surjective: $\phi(\mathbb{D}) = \mathbb{D} \setminus \{0\}$, and two-valent, as easily checked.

For $0 < h \leq 1/2$, we have for $\xi \in \mathbb{T}$ and $|\phi^*(\xi)| > 1 - h$:

$$\mathrm{e}^{-2h} \leq 1 - h < |\phi^*(\xi)| = \exp\Big(- \mathfrak{Re}\, f^*(\xi)\Big);$$

hence $\mathfrak{Re}\, f^*(\xi) < 2h$.

But if $2h > x = \mathfrak{Re}\, f^*(\xi)$, we have $g(x) > g(2h)$. As $f^*(\xi) = x + iy \in \overline{\Omega}$, we get $\mathfrak{Im}\, f^*(\xi) = y \geq g(x) > g(2h)$. Lemma 1 on the preceding page now gives:

$$m(\{\xi\,;\, |\varphi^*(\xi)| > 1 - h\}) \leq m(\{\xi\,;\, \mathfrak{Im}\, f^*(\xi) > g(2h)\}) \leq C\,\mathrm{e}^{5\pi - g(2h)}. \tag{19}$$

It is now enough to adjust g so as to have $\mathrm{e}^{g(t)} \geq C\,\mathrm{e}^{5\pi}/\delta(t/2)$ for t small enough to get (4) from (19). □

For sake of completeness, we give the proof of Lemma 1 on the preceding page.

Proof (Proof of Lemma 1 on the preceding page). We now prove Lemma 1 on the preceding page. If $\mathrm{e}^{y-\alpha} < 2$, there is nothing to prove, since then:

$$m(\mathfrak{Im}\, f^* > y) \leq 1 \leq 2\,\mathrm{e}^{\alpha - y}.$$

We can hence assume that $\mathrm{e}^{y-\alpha} \geq 2$. First, we make a comment. If the Riemann mapping theorem is very general and flexible, it gives very few informations on the parametrization $t \mapsto f^*(\mathrm{e}^{it})$ when $f\colon \mathbb{D} \to \Omega$ is a conformal map, except in some specific cases (lens maps, cusps, etc.: see Li, Queffélec, and Rodríguez-Piazza 2013). Here, the Kolmogorov weak type inequality provides a substitute. Write:

$$f = u + iv$$

and set:

$$f_1 = -if + i\frac{\pi}{2} - \alpha = v - \alpha + i\left(\frac{\pi}{2} - u\right)$$

and:

$$F_1 = 1 + \mathrm{e}^{f_1} = (1 + \mathrm{e}^{v-\alpha}\sin u) + i\mathrm{e}^{v-\alpha}\cos u.$$

If $v < \alpha$, then $\mathfrak{Re} F_1 > 1 - |\sin u| \geq 0$. If $v \geq \alpha$, then $0 < u < \pi$ and $\mathfrak{Re} F_1 \geq 1$. Hence F_1 maps $\mathbb{D}$ to the right half-plane $\mathbb{C}_0 = \{z\,;\, \mathfrak{Re}\, z > 0\}$. Finally, let $F = U + iV\colon \mathbb{D} \to \mathbb{C}_0$ be defined by:

$$F = F_1 - i\,\mathfrak{Im} F_1(0),$$

so that $V(0) = 0$. By the Kolmogorov inequality for the conjugation map $U \mapsto V$, and the harmonicity of U, we have, for all $\lambda > 0$ (a designating an absolute constant):

$$m(|F^*| > \lambda) \leq \frac{a}{\lambda} \|U^*\|_1 = \frac{a}{\lambda} \int_{\mathbb{T}} U^* \, dm = \frac{a}{\lambda} U(0). \tag{20}$$

Next, we claim that:

$$|\mathfrak{Im} F_1(0)| < 1 \quad \text{and} \quad U(0) < 2. \tag{21}$$

Indeed, $v(0) < \alpha$ by hypothesis, so that $|\mathfrak{Im} F_1(0)| = \mathrm{e}^{v(0)-\alpha} |\cos u(0)| < 1$, and $U(0) = 1 + \mathrm{e}^{v(0)-\alpha} \sin u(0) < 2$. Suppose now that, for some $y > \alpha$ and $z \in \mathbb{D}$, we have $v(z) > y$. Then, $0 < u(z) < \pi$ by our second assumption, and this implies $\mathfrak{Re}\, \mathrm{e}^{f_1(z)} = \mathrm{e}^{v(z)-\alpha} \sin u(z) > 0$, so that, using $|1 + w| \geq |w|$ if $\mathfrak{Re}\, w > 0$ and (21), and remembering that $\mathrm{e}^{y-\alpha} \geq 2$:

$$\begin{aligned} |F(z)| &= \left|1 + \mathrm{e}^{f_1(z)} - i\, \mathfrak{Im} F_1(0)\right| \geq \left|1 + \mathrm{e}^{f_1(z)}\right| - 1 \\ &\geq \left|\mathrm{e}^{f_1(z)}\right| - 1 = \mathrm{e}^{v(z)-\alpha} - 1 > \mathrm{e}^{y-\alpha} - 1 \geq \frac{1}{2} \mathrm{e}^{y-\alpha}. \end{aligned}$$

Taking radial limits and using (20) and (21), we get:

$$m(\mathfrak{Im} f^* > y) \leq m(|F^*| > \mathrm{e}^{y-\alpha}/2) \leq 4a\, \mathrm{e}^{\alpha-y}.$$

This ends the proof of Lemma 1 on p. 10 with $C = \max(2, 4a)$. □

4 Application to the multidimensional case

In this section, we apply Theorem 6 on p. 9 and Theorem 7 on p. 9 to show that, for $N \geq 2$, the image of the symbol cannot determine the behavior of the approximation numbers, or rather of $\beta_N(C_\phi)$, of the associated composition operator $C_\phi \colon H^2(\mathbb{D}^N) \to H^2(\mathbb{D}^N)$.

Recall that for an operator $T \colon H_1 \to H_2$, we set:

$$\beta_N^-(T) = \liminf_{n\to\infty} [a_n(T)]^{1/n^{1/N}} \quad \text{and} \quad \beta_N^+(T) = \limsup_{n\to\infty} [a_n(T)]^{1/n^{1/N}}, \tag{22}$$

and write $\beta_N(T)$ when $\beta_N^-(T) = \beta_N^+(T)$.

Theorem 8 – *For $N \geq 2$, there exist pairs of symbols $\Phi_1, \Phi_2 \colon \mathbb{D}^N \to \mathbb{D}^N$, such that $\Phi_1(\mathbb{D}^N) = \Phi_2(\mathbb{D}^N)$ and:*

1) *C_{Φ_1} is not bounded, but C_{Φ_2} is compact, and even $\beta_N(C_{\Phi_2}) = 0$;*
2) *C_{Φ_1} is bounded but not compact, so $\beta_N(C_{\Phi_1}) = 1$, and C_{Φ_2} is compact, with $\beta_N(C_{\Phi_2}) = 0$;*

3) *C_{Φ_1} is compact, with $\beta_N^-(C_{\Phi_1}) > 0$ and $\beta_N^+(C_{\Phi_1}) < 1$, and C_{Φ_2} is compact, with $\beta_N(C_{\Phi_2}) = 0$;*

4) *C_{Φ_1} is compact, with $\beta_N(C_{\Phi_1}) = 1$, and C_{Φ_2} is compact, but with $\beta_N(C_{\Phi_2}) = 0$.*

Proof. Let $\sigma\colon \mathbb{D} \to \mathbb{D}$ be a surjective symbol such that $\rho_\sigma(h) \le h^N \,\mathrm{e}^{-2/h^2}$ given by Theorem 6 on p. 9. By Proposition 2 on p. 4, we have, with $\gamma = N-2$:

$$a_n(C_\sigma\colon \mathcal{B}_\gamma \to H^2) \lesssim \inf_{0<h<1} (n^{(N-1)/2} \mathrm{e}^{-nh} + \mathrm{e}^{-1/h^2}),$$

and, with $h = 1/n^{1/3}$, we get $a_n(C_\sigma\colon \mathcal{B}_\gamma \to H^2) \lesssim \mathrm{e}^{-d\, n^{2/3}}$.

We choose the exponent 2/3 for fixing the ideas, but every exponent $\alpha > 1/2$, with $\alpha < 1$, (i.e. $a_n(C_\sigma\colon \mathcal{B}_\gamma \to H^2) \lesssim \mathrm{e}^{-d\, n^{\alpha}}$) would be suitable.

1) We take $\Phi_1(z_1, z_2, z_3, \ldots, z_N) = (z_1, z_1, \ldots, z_1)$. The composition operator C_{Φ_1} is not bounded because if $f_n(z_1, \ldots, z_N) = \left(\frac{z_1+z_2}{2}\right)^n$, then $\|f_n\|_2^2 = 4^{-n} \sum_{k=0}^n \binom{n}{k}^2 = 4^{-n}\binom{2n}{n} \approx 1/\sqrt{n}$, though $(C_{\Phi_1} f_n)(z_1, \ldots, z_N) = z_1^n$ and $\|C_{\Phi_1} f_n\|_2 = 1$.

We define Φ_2 by:

$$\Phi_2(z_1, z_2, \ldots, z_N) = \big(\sigma(z_1), \sigma(z_1), \ldots, \sigma(z_1)\big).$$

Since σ is surjective, we have $\Phi_2(\mathbb{D}^N) = \Phi_1(\mathbb{D}^N)$. Now, as in the proof of Theorem 4 on p. 5, we have $C_{\Phi_2} = J C_\sigma M$, so:

$$a_n(C_{\Phi_2}) \le a_n(C_\sigma\colon \mathcal{B}_{N-2} \to H^2) \lesssim \mathrm{e}^{-d\, n^{2/3}},$$

by the ideal property. Hence $[a_n(C_{\Phi_2})]^{1/n^{1/N}} \lesssim \mathrm{e}^{-d\, n^{\frac{2}{3}-\frac{1}{N}}}$ and therefore $\beta_N(C_{\Phi_2}) = 0$ since $\frac{2}{3} - \frac{1}{N} > 0$.

2) We consider the lens map $\lambda = \lambda_{1/N}$ of parameter $1/N$. We define:

$$\begin{cases} \Phi_1(z_1, \ldots, z_N) = \big(\lambda(z_1), \lambda(z_1), \ldots, \lambda(z_1)\big) \\ \Phi_2(z_1, \ldots, z_N) = \big(\lambda[\sigma(z_1)], \lambda[\sigma(z_1)], \ldots, \lambda[\sigma(z_1)]\big). \end{cases}$$

Since σ is surjective, we have $\Phi_1(\mathbb{D}^N) = \Phi_2(\mathbb{D}^N)$ and we saw in Theorem 5 on p. 7 that C_{Φ_1} is bounded but not compact.

On the other hand, we have the factorization $C_{\Phi_2} = J C_\sigma C_\lambda M$. Hence C_{Φ_2} is compact, and, as in 1), $\beta_N(C_{\Phi_2}) = 0$.

3) For this item, the map σ does not suffice, and we will use another surjective symbol $s\colon \mathbb{D} \to \mathbb{D}$. By Theorem 6 on p. 9, there exists such a map s with:

$$\rho_s(t) \le t^2 \mathrm{e}^{-2/t^2} \tag{23}$$

and

$$\rho_s(t) \le t\,\delta^2(t) \tag{24}$$

for t small enough, where $\delta\colon (0,1) \to (0,1)$ is a non-decreasing function such that $\delta(\varepsilon_n) \le \mathrm{e}^{-n\varepsilon_n}$ and:

$$\varepsilon_n = n^{-\frac{1}{4N-7}}\,. \tag{25}$$

By the proof of Theorem 7 on p. 9, (24) implies that:

$$a_n(C_s) \le \mathrm{e}^{-n\varepsilon_n}\,. \tag{26}$$

We also consider a lens map $\lambda = \lambda_\theta$, with parameter $\theta < 1/N$, and we set:

$$\begin{cases} \Phi_1(z_1,\dots,z_N) = \Big(\lambda(z_1), \lambda(z_1), \dfrac{z_3}{2}, \dots, \dfrac{z_N}{2}\Big) \\ \Phi_2(z_1,\dots,z_N) = \Big(\lambda[s(z_1)], \lambda[s(z_1)], \dfrac{s(z_3)}{2}, \dots, \dfrac{s(z_N)}{2}\Big). \end{cases}$$

Since s is surjective, we have $\Phi_1(\mathbb{D}^N) = \Phi_2(\mathbb{D}^N)$.

a) Let us prove that $\beta_N^-(C_{\Phi_1}) > 0$ and $\beta_N^+(C_{\Phi_1}) < 1$.

Note that:

$$C_{\Phi_1} = C_u \otimes C_{v_3} \otimes \cdots \otimes C_{v_N}\,,$$

where $u\colon \mathbb{D}^2 \to \mathbb{D}^2$ is defined by $u(z_1,z_2) = \big(\lambda(z_1), \lambda(z_1)\big)$ and $v_j\colon \mathbb{D} \to \mathbb{D}$ is defined by $v_j(z_j) = z_j/2$. In fact, if $f \in H^2(\mathbb{D}^2)$ and $g_j \in H^2(\mathbb{D})$, $3 \le j \le N$, we have:

$$\begin{aligned} &[C_{\Phi_1}(f \otimes g_3 \otimes \cdots \otimes g_N)](z_1, z_2, z_3, \dots, z_N) \\ &\qquad = (f \otimes g_3 \otimes \cdots \otimes g_N)\big(u(z_1,z_2), v_3(z_3), \dots, v_N(z_N)\big) \\ &\qquad = f[\lambda(z_1), \lambda(z_1)]\, g_3[v_3(z_3)] \cdots g_N[v_N(z_N)] \\ &\qquad = (C_u f)(z_1,z_2)\,(C_{v_3} g_3)(z_3) \cdots (C_{v_N} g_N)(z_N) \\ &\qquad = [(C_u \otimes C_{v_3} \otimes \cdots \otimes C_{v_N})(f \otimes g_3 \otimes \cdots \otimes g_N)](z_1, z_2, z_3, \dots, z_N), \end{aligned}$$

hence the result since $H^2(\mathbb{D}^2) \otimes H^2(\mathbb{D}) \otimes \cdots \otimes H^2(\mathbb{D})$ is dense in $H^2(\mathbb{D}^N)$. That proves in particular that C_{Φ_1} is compact since C_u and $C_{v_3}, \dots, C_{v_N}$ are (by Theorem 5 on p. 7 for C_u).

By the supermultiplicativity of singular numbers of tensor products (see Li, Queffélec, and Rodríguez-Piazza n.d.(a), Lemma 3.2), it ensues that:

$$a_{n^N}(C_{\Phi_1}) \ge a_{n^2}(C_u) \prod_{j=3}^{N} a_n(C_{v_j}) = a_{n^2}(C_u) \Big(\frac{1}{2}\Big)^{n(N-2)}.$$

By Li, Queffélec, and Rodríguez-Piazza (n.d.[a], Remark at the end of Section 4), we have $a_{n^2}(C_u) \gtrsim \mathrm{e}^{-bn}$ for some positive constant $b = b(\theta)$. Indeed, if $J = J_2 \colon H^2(\mathbb{D}) \to H^2(\mathbb{D}^2)$ is the canonical injection defined by $(Jh)(z_1, z_2) = h(z_1)$ and $Q \colon H^2(\mathbb{D}^2) \to H^2(\mathbb{D})$ is defined by $(Qf)(z_1) = f(z_1, 0)$, we have $C_\lambda = QC_uJ$. Hence $a_k(C_u) \gtrsim a_k(C_\lambda) \gtrsim \mathrm{e}^{-b\sqrt{k}}$.

Therefore we get:

$$a_{n^N}(C_{\Phi_1}) \gtrsim \mathrm{e}^{-cn}$$

for some positive constant depending only on θ and N. It follows that $\beta_N^-(C_{\Phi_1}) > 0$.

To see that $\beta_N^+(C_{\Phi_1}) < 1$, we need the following lemma, whose proof is postponed.

Lemma 2 – *Let $S \colon H_1 \to H_1$ and $T \colon H_2 \to H_2$ be two operators between Hilbert spaces and A, B a pair of positive numbers. Then, whenever:*

$$a_{[n^A]}(S) \leq e^{-cn} \quad \textit{and} \quad a_{[n^B]}(T) \leq e^{-cn},$$

where $[\,.\,]$ stands for the integer part, we have, for some constant integer $M = M(A,B) > 0$:

$$a_{M\,[n^{A+B}]}(S \otimes T) \leq e^{-cn}.$$

Let $S = C_u$ and $T = C_{v_3} \otimes \cdots \otimes C_{v_N}$. For c small enough, we have $a_{n^{N-2}}(T) \leq C\,(1/2)^n \leq \mathrm{e}^{-cn}$ and, using (17), $a_{n^2}(S) \leq \mathrm{e}^{-dn} \leq \mathrm{e}^{-cn}$. Hence, with $A = 2$, $B = N-2$, Lemma 2 gives:

$$a_{Mn^N}(C_{\Phi_1}) \lesssim \mathrm{e}^{-cn}.$$

Therefore $\beta_N^+(C_{\Phi_1}) \leq \mathrm{e}^{-c/M^{1/N}} < 1$.

b) Define $\Psi \colon \mathbb{D}^N \to \mathbb{D}^N$ by:

$$\Psi(z_1, z_2, z_3, \ldots, z_N) = \big(s(z_1), s(z_1), s(z_3), \ldots, s(z_N)\big).$$

If $\tau_1 \colon \mathbb{D}^2 \to \mathbb{D}^2$ is defined by $\tau_1(z_1, z_2) = \big(s(z_1), s(z_1)\big)$ and the map $\tau_2 \colon \mathbb{D}^{N-2} \to \mathbb{D}^{N-2}$ by $\tau_2(z_3, \ldots, z_N) = \big(s(z_3), \ldots, s(z_N)\big)$, we have:

$$C_\Psi = C_{\tau_1} \otimes C_{\tau_2}.$$

As in the proof of Theorem 4 on p. 5, we have the factorization:

$$\tau_1 \colon H^2(\mathbb{D}^2) \xrightarrow{M} \mathcal{B}_0 = \mathcal{B}^2 \xrightarrow{C_s} H^2(\mathbb{D}) \xrightarrow{J} H^2(\mathbb{D}^2).$$

Hence $a_n(C_{\tau_1}) \leq \|M\|\,\|J\|\, a_n(C_s \colon \mathcal{B}^2 \to H^2)$.

By Proposition 2 on p. 4, we have:

$$a_n(C_s\colon \mathcal{B}^2 \to H^2) \lesssim \inf_{0<h<1} \left(\sqrt{n}\,\mathrm{e}^{-nh} + \sup_{0<t\leq h} \sqrt{\frac{\rho_s(t)}{t^2}}\right);$$

so (23) implies that $a_n(C_s\colon \mathcal{B}^2 \to H^2) \lesssim \inf_{0<h<1}(\sqrt{n}\,\mathrm{e}^{-nh} + \mathrm{e}^{-1/h^2})$ and, taking $h = n^{-1/3}$, we get, with some c small enough:

$$a_n(C_s\colon \mathcal{B}^2 \to H^2) \lesssim \mathrm{e}^{-cn^{2/3}}.$$

It follows that $a_n(C_{\tau_1}) \lesssim \mathrm{e}^{-c\,n^{2/3}}$ and hence:

$$a_{[n^{3/2}]}(C_{\tau_1}) \lesssim \mathrm{e}^{-c\,n}. \tag{27}$$

On the other hand, Bayart et al. (2018, Theorem 5.5) says that:

$$a_n(C_{\tau_2}) \leq 2^{N-3} \|C_s\|^{N-2} \inf_{n_3 \cdots n_N \leq n} \Big(a_{n_3}(C_s) + \cdots + a_{n_N}(C_s)\Big).$$

Taking $n_3 = \cdots = n_N = n^{\frac{1}{N-2}}$, we get, using (26):

$$a_n(C_{\tau_2}) \leq K^N N \exp\Big(-n^{\frac{1}{N-2}}\,\varepsilon_{n^{\frac{1}{N-2}}}\Big).$$

Using (25), that gives:

$$a_n(C_{\tau_2}) \lesssim \exp\Big(-n^{\frac{1}{N-2}(1-\frac{1}{4N-7})}\Big) = \exp\Big(-n^{\frac{4}{4N-7}}\Big),$$

or:

$$a_{\left[n^{N-\frac{7}{4}}\right]}(C_{\tau_2}) \lesssim \mathrm{e}^{-n} \leq \mathrm{e}^{-cn}. \tag{28}$$

Now, (27) and (28) allow to use Lemma 2 on the previous page with $A = 3/2$ and $B = N - 7/4$, and we get:

$$a_{M\left[n^{N-\frac{1}{4}}\right]}(C_\Psi) \lesssim \mathrm{e}^{-cn}.$$

Equivalently:

$$a_k(C_\Psi) \lesssim \exp\Big(-c' k^{\frac{4}{4N-1}}\Big)$$

and:

$$\Big(a_k(C_\Psi)\Big)^{1/k^{1/N}} \lesssim \exp\Big(-c' k^{\frac{4}{4N-1} - \frac{1}{N}}\Big) = \exp\Big(-c' k^{\frac{1}{N(4N-1)}}\Big),$$

which gives $\beta_N(C_\Psi) = 0$.

To end the proof, it suffices to remark that $C_{\Phi_2} = C_\Psi \circ C_{\Phi_1}$, since $\Phi_2 = \Phi_1 \circ \Psi$, and hence $\beta_N^+(C_{\Phi_2}) \leq \beta_N^+(C_\Psi) = 0$, so $\beta_N(C_{\Phi_2}) = 0$.

4) We use a Shapiro-Taylor map. This one-parameter map ς_θ, $\theta > 0$, was introduced by J. Shapiro and P. Taylor in 1973[14] and was further studied, with a slightly different definition, in Lefèvre et al. (2008, Section 5). J. Shapiro and P. Taylor proved that $C_{\varsigma_\theta} \colon H^2 \to H^2$ is always compact, but is Hilbert-Schmidt if and only if $\theta > 2$. Let us recall their definition.

For $0 < \varepsilon < 1$, we set $V_\varepsilon = \{z \in \mathbb{C};\ \mathfrak{Re}\, z > 0 \text{ and } |z| < \varepsilon\}$. For $\varepsilon = \varepsilon_\theta > 0$ small enough, one can define:

$$f_\theta(z) = z(-\ln z)^\theta,$$

for $z \in V_\varepsilon$, where $\ln z$ will be the principal determination of the logarithm. Let now g_θ be the conformal mapping from $\mathbb{D}$ onto V_ε, which maps $\mathbb{T} = \partial\mathbb{D}$ onto ∂V_ε, defined by $g_\theta(z) = \varepsilon\, \phi_0(z)$, where ϕ_0 is given by:

$$\phi_0(z) = \frac{\left(\dfrac{z-i}{iz-1}\right)^{1/2} - i}{-i\left(\dfrac{z-i}{iz-1}\right)^{1/2} + 1} \cdot$$

Then, we define:

$$\varsigma_\theta = \exp(-f_\theta \circ g_\theta).$$

We proved in Li, Queffélec, and Rodríguez-Piazza (2013, Section 4.2) (though it is not sharp) that:

$$a_n(C_{\varsigma_\theta}) \gtrsim \frac{1}{n^{\theta/2}} \cdot \tag{29}$$

We define $\Phi_1 \colon \mathbb{D}^N \to \mathbb{D}^N$ as:

$$\Phi_1(z_1, z_2, \ldots, z_N) = \big(\varsigma_\theta(z_1), 0, \ldots, 0\big). \tag{30}$$

If $J = J_N \colon H^2(\mathbb{D}) \to H^2(\mathbb{D}^N)$ is the canonical injection defined by $(Jh)(z_1, \ldots, z_N) = h(z_1)$ and $Q = Q_N \colon H^2(\mathbb{D}^N) \to H^2(\mathbb{D})$ is defined by $(Qf)(z_1) = f(z_1, 0, \ldots, 0)$, then $C_{\Phi_1} = J C_{\varsigma_\theta} Q$; hence C_{Φ_1} is compact. On the other hand, we also have $Q C_{\Phi_1} J = C_{\varsigma_\theta}$, which implies that $a_n(C_{\Phi_1}) \gtrsim a_n(C_{\varsigma_\theta}) \gtrsim n^{-\theta/2}$. It follows that:

$$\beta_N(C_{\Phi_1}) \geq \lim_{n\to\infty} (n^{-\theta/2})^{1/n^{1/N}} = 1,$$

and hence $\beta_N(C_{\Phi_1}) = 1$.

Now, if:

$$\Phi_2(z_1,\ldots,z_N) = \Big(\varsigma_\theta[\sigma(z_1)], 0, \ldots, 0\Big),$$

since σ is surjective, we have $\Phi_1(\mathbb{D}^N) = \Phi_2(\mathbb{D}^N)$. Moreover, we have $C_{\Phi_2} = J C_{\varsigma_\theta \circ \sigma} Q = J C_\sigma C_{\varsigma_\theta} Q$, so $a_n(C_{\Phi_2}) \lesssim a_n(C_\sigma)$. Since $\rho_\sigma(h) \leq h^{N+1}\,\mathrm{e}^{-2/h^2}$, Proposition 1 on p. 3 gives, with $h = 1/n^{1/3}$:

$$a_n(C_\sigma) \lesssim \mathrm{e}^{-c n^{2/3}},$$

so $[a_n(C_{\Phi_2})]^{1/n^{1/N}} \lesssim \exp(-c\, n^{\frac{2}{3} - \frac{1}{N}})$ and $\beta_N(C_{\Phi_2}) = 0$. □

Proof (Proof of Lemma 2 on p. 15). In Li, Queffélec, and Rodríguez-Piazza (n.d.[a]), we observed that the singular numbers of $S \otimes T$ are the non-increasing rearrangement of the numbers $s_j t_k$, where s_j and t_k denote respectively the j-th and the k-th singular number of S and T. We can assume $s_1 = t_1 = 1$. Using this observation, we will majorize the number of pairs (j,k) such that $s_j t_k > \mathrm{e}^{-cn}$. Let (j,k) be such a pair. Since $s_j \leq s_1 = 1$, we have $t_k \geq \mathrm{e}^{-cn}$ so that $k \leq [n^B] \leq n^B$. Hence, for some $2 \leq l \leq n$, we have $(l-1)^B < k \leq l^B$. Then, due to the assumption on T, $t_k < \mathrm{e}^{-c(l-1)}$ and $s_j \geq \mathrm{e}^{-cn} t_k^{-1} \gtrsim \mathrm{e}^{-c(n-l+1)}$, implying that $j \lesssim (n-l+1)^A$, thanks to the assumption on S. As a consequence, since the number of integers k such that $(l-1)^B < k \leq l^B$ is dominated by l^{B-1}, the number ν_n of pairs (j,k) such that $s_j t_k > \mathrm{e}^{-cn}$ is dominated by:

$$\sum_{l=1}^{n} (n-l+1)^A l^{B-1} \sim n^{A+B} \int_0^1 t^A (1-t)^B \, dt,$$

by a Riemann sum argument. Next, let $M \in \mathbb{N}$ big enough to have:

$$\sum_{l=1}^{n} (n-l+1)^A l^{B-1} \leq M n^{A+B} - 1, \quad \text{for all } n.$$

By definition, $a_{M[n^{A+B}]}(S \otimes T) \leq a_{\nu_n + 1}(S \otimes T) \leq \mathrm{e}^{-cn}$, giving the result. □

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Contents