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Complete characterization of bounded composition operators on the general weighted Hilbert spaces of entire Dirichlet series

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Abstract

We establish necessary and sufficient conditions for boundedness of composition operators on the most general class of Hilbert spaces of entire Dirichlet series with real frequencies. Depending on whether or not the space being considered contains any nonzero constant function, different criteria for boundedness are developed. Thus, we complete the characterization of bounded composition operators on all known Hilbert spaces of entire Dirichlet series of one variable.

Keywords: Composition operators, entire Dirichlet series, Hilbert spaces, boundedness.

MSC: 30D15, 47B33.

1 Introduction

Suppose $\Lambda = (\lambda_n)_{n=1}^{\infty}$ is a sequence of real numbers that satisfies $\lambda_n \uparrow +\infty$ (i.e., Λ is unbounded and strictly increasing). Consider a *Dirichlet series with real frequencies*

$$\sum_{n=1}^{\infty} a_n e^{-\lambda_n z} = a_1 e^{-\lambda_1 z} + a_2 e^{-\lambda_2 z} + a_3 e^{-\lambda_3 z} + \dots, \tag{1}$$

where $z \in \mathbb{C}$ and $(a_n) \subset \mathbb{C}$. The series (1) is also called a *generalized Dirichlet series*. When $\lambda_n = \log n$, it becomes a *classical* (or *ordinary*) Dirichlet series, which has various important applications in number theory and complex analysis. If $\lambda_n = n$, with the change of variable $\zeta = e^{-z}$, then (1) becomes the usual power series in ζ.

The classical Dirichlet series and their important role in analytic number theory are studied in the book by Apostol (1976), and the theory of generalized Dirichlet

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series is presented in the excellent monograph by Hardy and Riesz (1915). One important result from the monograph states that the region of convergence of a general Dirichlet series (if exists) is a half-plane (and for entire series, the region is the whole complex plane). Furthermore, the representation (1) is unique and holomorphic on that region of convergence.

For entire Dirichlet series, Ritt (1928) investigated their growth and convergence, based on which Reddy (1966) defined and formulated logarithmic orders. In the second half of the last century, Leont'ev (1983) developed theory of representation for entire functions by Dirichlet series with complex frequencies. Such series are of the form (1) but with complex λ_n's. As uniqueness no longer holds for this representation, we will not consider complex frequencies in the present article.

It is clear that only finitely many elements of Λ are negative, but there is no agreement on further restriction on the sequence. Hardy and Riesz allowed some terms $\lambda_n's$ to be negative. Mandelbrojt (1969) supposed that all terms of Λ are strictly positive, so nonzero constants are not representable in the form (1). Ritt (1928) allowed the possibility for free constants by adding a term a_0 to the series. Whether or not constants are representable by (1) affects our results in this paper, so in order to be consistent with the notations of both Mandelbrojt and Ritt, we follow the convention that $\lambda_1 \geq 0$, i.e., all terms of Λ are *nonnegative*.

In functional analysis and operator theory, construction of Hilbert spaces of Dirichlet series and action of composition operators on them have been attractive topics for mathematicians.

In the general context, let $\mathscr{H}$ be some Hilbert space whose members are holomorphic functions on a domain G of the complex plane that are representable by Dirichlet series, and φ be a holomorphic self-map on G. The *composition operator* C_φ *acting on* $\mathscr{H}$ *induced by* φ is defined by the rule $C_\varphi f = f \circ \varphi$, for $f \in \mathscr{H}$. Researchers are interested in the relation between the function-theoretic properties of φ and the operator-theoretic properties of C_φ. Typical problems in this topic include the invariance of C_φ (i.e., $C_\varphi(\mathscr{H}) \subseteq \mathscr{H}$), the boundedness and compactness of C_φ, computation of its norm and essential norms, etc.

Many studies have been done on composition operators on Hilbert spaces of classical Dirichlet series. Gordon and Hedenmalm (1999) considered the boundedness of such operators on space of classical series with square summable coefficients. There were studies on numerical ranges by Finet and Queffélec (2004) and compactness by Finet, Queffélec, and Volberg (2004). Recently, complex symmetric composition operators have been investigated[3].

Although entire Dirichlet series have been studied in many details, not until recently has the theory of composition operators on Banach spaces of entire Dirichlet series been developed. Hou, Hu, and L. Khoi (2013) proposed the construction of the general Hilbert spaces $\mathcal{H}(E,\beta)$ of entire Dirichlet series by the use of weighted sequence spaces. Amongst the many subclasses of $\mathcal{H}(E,\beta)$, several properties of com-

[3]Yao, 2017, "Complex symmetric composition operators on a Hilbert space of Dirichlet series".

position operators on them were explored, including the boundedness, compactness and compact difference, on the most specific case, namely the spaces $\mathcal{H}(E,\beta_S)$. Later, some results on essential norms of such operators[4], their Fredholmness, Hilbert–Schmidtness, cyclicity and norm computation via reproducing kernels[5] on $\mathcal{H}(E,\beta_S)$ were obtained.

Specifically, the aforementioned authors study $\mathcal{H}(E,\beta)$ with the following condition on the weights $\beta=(\beta_n)$:

$$\exists\ \alpha>0\colon\ \liminf_{n\to\infty}\frac{\log\beta_n}{\lambda_n^{1+\alpha}}=+\infty. \tag{S}$$

It should be noted that Condition (S) imposed on the spaces $\mathcal{H}(E,\beta)$ is a very strict assumption because only a small class of spaces satisfy these weights. Yet this condition is crucial in the study of boundedness of C_φ, since it makes elements of the spaces $\mathcal{H}(E,\beta)$ have finite logarithmic orders, so that by applying a lemma by Pólya (1926), the symbol φ has to be a polynomial. On the other hand, the proof given in the original paper is only applicable if $\mathcal{H}(E,\beta)$ contains *no nonzero constant function* (in particular, $\lambda_1>0$ must hold), while no proof was provided when $\lambda_1=0$, even though constant functions φ clearly induce a bounded operator in this case.

Recently, in M. Doan, Mau, and L. Khoi (2019), a boundedness of composition operators C_φ for spaces $\mathcal{H}(E,\beta_S)$ in the case $\lambda_1=0$ has been studied, on which a complex symmetry of C_φ is developed. However, the proof still depends strongly Condition (S).

Therefore, two natural questions can be asked are: (1) Can we study boundedness C_φ when (S) reduces to a weaker condition, and in particular, using different technique other than Pólya's lemma?, and (2) Are there differences between the case $\lambda_1=0$ and $\lambda_1>0$ for the boundedness of C_φ?

The aim of this research article is to answer positively to both of the proposed questions. We in fact work with the spaces $\mathcal{H}(E,\beta)$, the *most general* class of Hilbert spaces of entire Dirichlet series that we know up to now. Thus, we provide a *complete characterization of the boundedness of composition operators C_φ*. As we will see later, Pólya's lemma fails to be applied to the general spaces $\mathcal{H}(E,\beta)$. Hence, we propose different techniques of proofs from those of M. Doan, Mau, and L. Khoi (2019) and Hou, Hu, and L. Khoi (2013), which covers both cases $\lambda_1=0$ and $\lambda_1>0$. We note that the criteria in those cases are not identical, and their proofs are not trivial applications of each other.

The structure of the paper is as follows. We provide in Section 2 a summary of known results about Hilbert spaces of entire Dirichlet series, most importantly the construction of spaces $\mathcal{H}(E,\beta)$. Section 3 presents important notions of reproducing

[4]Hu and L. Khoi, 2012, "Numerical range of composition operators on Hilbert spaces of entire Dirichlet series".

[5]Wang and Yao, 2015, "Some properties of composition operators on Hilbert spaces of Dirichlet series".

kernels on spaces $\mathcal{H}(E,\beta)$, which is helpful for subsequent sections. In Section 4, we deal with boundedness of composition operators. In particular, we first propose a sufficient condition in Proposition 2, and later prove that this condition is also necessary. In Subsections 4.1 and 4.2, boundedness of C_φ for the most general class $\mathcal{H}(E,\beta)$ is studied, in both cases when a space $\mathcal{H}(E,\beta)$ does not contain nonzero constants (Theorem 2) and when it does (Theorem 3). A summary of our results and some concluding remarks are given in Section 5.

2 Hilbert spaces $\mathcal{H}(E,\beta)$ of entire Dirichlet series

For a given sequence $\Lambda = (\lambda_n)_{n=1}^\infty$ with $0 \le \lambda_n \uparrow +\infty$, define the following constant L,

$$L := \limsup_{n\to\infty} \frac{\log n}{\lambda_n}.$$

We associate to each Dirichlet series (1) the following quantity,

$$D = D((a_n)_{n=1}^\infty) := \limsup_{n\to\infty} \frac{\log|a_n|}{\lambda_n}.$$

It is well-known that L is the upper bound of the distance between the abscissa of convergence and the abscissa of absolute convergence of the series (1). We refer the reader to Hardy and Riesz (1915) for the basic properties of these abscissas. If $L < +\infty$, then the Dirichlet series (1) represents (uniquely) an entire function if and only if $D = -\infty$ (see, e.g., Hou and L. Khoi 2012; Mandelbrojt 1969).

Convention 1 – *Throughout this paper, the condition $L < +\infty$ is always supposed to hold.*

Now, let $\beta = (\beta_n)$ be a sequence of (not necessarily distinct or monotonic) positive numbers. We introduce the following *weighted sequence space with weight β*:

$$\ell_\beta^2 = \left\{\mathbf{a} = (a_n)_{n=1}^\infty \subset \mathbb{C} \colon \|\mathbf{a}\|_{\ell_\beta^2} = \Big(\sum_{n=1}^\infty |a_n|^2\beta_n^2\Big)^{1/2} < +\infty\right\},$$

which is a Hilbert space with the inner product of any $\mathbf{a} = (a_n)$ and $\mathbf{b} = (b_n)$ in ℓ_β^2 given by

$$\langle \mathbf{a}, \mathbf{b}\rangle_{\ell_\beta^2} = \sum_{n=1}^\infty a_n \overline{b_n}\beta_n^2.$$

The sequence spaces ℓ_β^2 play an important role in the construction of many important Hilbert spaces by varying β, such as Hardy spaces, Bergman spaces, Dirichlet spaces, Fock spaces, etc. (see, e.g., the book by Cowen and MacCluer 1995).

Consider the following *function space $\mathcal{H}(\beta)$ of entire Dirichlet series induced by weight β*:

$$\mathcal{H}(\beta) = \left\{ f(z) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n z} \text{ entire} \colon \|f\|_{\mathcal{H}(\beta)} := \|(a_n)\|_{\ell^2_\beta} < +\infty \right\}. \tag{2}$$

Here, when we write $f(z) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n z}$, we mean the entire function f is represented by the series on the right-hand side.

The space $\mathcal{H}(\beta)$ is an inner product space, where

$$\langle f, g \rangle_{\mathcal{H}(\beta)} = \sum_{n=1}^{\infty} a_n \overline{b_n} \beta_n^2,$$

for any $f(z) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n z}$ and $g(z) = \sum_{n=1}^{\infty} b_n e^{-\lambda_n z}$ in $\mathcal{H}(\beta)$.

Depending on β, the induced space $\mathcal{H}(\beta)$ may not be complete in its norm, and so it is not necessarily a Hilbert space. The following theorem from Hou, Hu, and L. Khoi (2013) provides a criterion of the weight β for $\mathcal{H}(\beta)$ to be complete.

Theorem 1 – *The space $\mathcal{H}(\beta)$ of entire Dirichlet series induced by a sequence of positive real numbers $\beta = (\beta_n)$, as defined in* (2)*, is a Hilbert space if and only if the following condition* (E) *holds,*

$$\liminf_{n\to\infty} \frac{\log \beta_n}{\lambda_n} = +\infty. \tag{E}$$

A direct consequence of this theorem is that if (E) holds, the space $\mathcal{H}(\beta)$ automatically becomes a Hilbert space of entire functions, so we can drop the condition "entire" in (2).

Note that when (E) holds, if $0 \in \Lambda$, i.e., $\lambda_1 = 0$, then the space contains all nonzero constants, while it contains no nonzero constants if $\lambda_1 > 0$. Obviously, Theorem 1 is unaffected regardless λ_1 is 0 or not. Hence, we adopt the following convention.

Convention 2 – *Unless otherwise stated, we assume condition* (E) *always holds. We denote by $\mathcal{H}(E,\beta)$ the following Hilbert space of entire Dirichlet series*

$$\mathcal{H}(E,\beta) = \left\{ f(z) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n z} \; : \; \|f\|_{\mathcal{H}(E,\beta)} = \left(\sum_{n=1}^{\infty} |a_n|^2 \beta_n^2 \right)^{1/2} < +\infty \right\},$$

and without ambiguity, we denote the norm of any function $f \in \mathcal{H}(E,\beta)$ simply by $\|f\|$.

Remark 1 – One can easily see that Condition (E) is much weaker than Condition (S) in the Introduction.

3 Reproducing kernel Hilbert spaces $\mathcal{H}(E,\beta)$

A (complex) separable Hilbert space $\mathscr{H}$ of functions from a non-empty set $G \subseteq \mathbb{C}$ to $\mathbb{C}$ is called a *reproducing kernel Hilbert space* (RKHS) if for every $y \in G$, the evaluation functional $\delta_y : f \mapsto f(y)$ $(f \in \mathscr{H})$ is bounded.

By Riesz Representation Theorem, there exists a unique element $k_y \in \mathscr{H}$ such that $f(y) = \langle f, k_y \rangle_{\mathscr{H}}$ for every $f \in \mathscr{H}$. We call k_y the *reproducing kernel at the point y*.

The function $K : G \times G \to \mathbb{C}$ defined by

$$K(x,y) = \langle k_y, k_x \rangle_{\mathscr{H}} = k_y(x), \quad x, y \in \mathscr{H},$$

is called the *reproducing kernel for* $\mathscr{H}$. It is well known that if a collection of elements $\{e_j\}_{j=1}^{\infty}$ is an orthonormal basis for $\mathscr{H}$, then

$$K(x,y) = \sum_{j=1}^{\infty} e_j(x)\overline{e_j(y)}, \tag{3}$$

where the convergence is pointwise for $x, y \in \mathscr{H}$ (see the famous article by Aronszajn 1950).

We show in the following proposition that if all elements of $\mathcal{H}(E,\beta)$ are entire Dirichlet series, i.e., if β satisfies (E), then $\mathcal{H}(E,\beta)$ is a reproducing kernel Hilbert space.

Proposition 1 – *Let $\beta = (\beta_n)$ satisfy condition* (E). *Then the space $\mathcal{H}(E,\beta)$ induced by β is a complex reproducing kernel Hilbert space with the reproducing kernel $K : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ given by*

$$K(z,w) = k_w(z) = \sum_{n=0}^{\infty} \frac{e^{-\lambda_n(\overline{w}+z)}}{\beta_n^2}. \tag{4}$$

The convergence is uniform on compact subsets of $\mathbb{C} \times \mathbb{C}$.

There are many ways to prove this classical result. One can, for instance, adapt the proof in Hou, Hu, and L. Khoi (2013), keeping in mind that $\beta = (\beta_n)$ satisfies (E) instead of (S). Here we provide the reader with a shorter proof.

Proof. Apply Cauchy–Schwarz inequality, we have

$$|f(z)|^2 = \left|\sum_{n=1}^{\infty} a_n e^{-\lambda_n z}\right|^2 \le \left(\sum_{n=1}^{\infty} \frac{e^{-2\lambda_n \mathfrak{Re}(z)}}{\beta_n^2}\right)\left(\sum_{n=1}^{\infty} |a_n|^2 \beta_n^2\right) := M_z \|f\|^2.$$

It suffices to show that the series M_z is convergent absolutely for any $z \in \mathbb{C}$. The conditions $L < +\infty$ and (E) imply that there is a constant $C_1 > 0$ such that $\frac{\log n}{\lambda_n} \le C_1$, and for all $C_2 > 0$ there exists $n_0 \in \mathbb{N}$ such that $\frac{\log \beta_n}{\lambda_n} \geq C_2$ for all $n \geq n_0$. This gives $\frac{e^{-\lambda_n \Re(z)}}{\beta_n} \leq n^{-\Re(z)/C_1 - \lambda_n C_2/\log n} \leq n^{-(\Re(z)+C_2)/C_1}$. We can choose C_2 (depending on z and C_1) such that $\Re(z) + C_2 > C_1$, so that $\frac{e^{-2\lambda_n \Re(z)}}{\beta_n^2} < n^{-2}$. The convergence of M_z then follows comparison test. □

Remark 2 – (a) In the proof above, we can easily see that for any $w \in \mathbb{C}$,

$$\|k_w\|^2 = K(w,w) = \sum_{n=1}^{\infty} \frac{e^{-2\lambda_n \Re(w)}}{\beta_n^2}.$$

(b) By a consequence of closed graph theorem, if a composition operator C_φ is *invariant*, that is, if $C_\varphi(\mathscr{H}) \subseteq \mathscr{H}$, then it is automatically bounded. Thus, we don't have to deal with invariance and boundedness separately, since the two properties are equivalent for C_φ acting on RKHSs.

4 Main results

In the sequel, we fix a sequence $\beta = (\beta_n)$ that satisfies (E) and let $\mathcal{H}(E,\beta)$ be the corresponding Hilbert space of Dirichlet series.

We remind an important point, which is seen later, that the criteria for boundedness of C_φ for the case $\lambda_1 > 0$ and for the case $\lambda_1 = 0$ are different. Recall that if $\lambda_1 = 0$, the space $\mathcal{H}(E,\beta)$ also includes all constants, and that the space contains no nonzero constants if $\lambda_1 > 0$. The proof of the necessary condition in the latter case is also more sophisticated than the former, even though the idea used in the two proofs are similar. This fact is reflected in Propositions 3 and 4.

4.1 Sufficient conditions

We can easily obtain the following sufficiency for the boundedness of C_φ on $\mathcal{H}(E,\beta)$.

Proposition 2 – *Let φ be an entire function. Consider the statements below.*

(C_1) *φ is a constant function,*

(C_2) *$\varphi(z) = z + b$ for some $b \in \mathbb{C}$, $\Re(b) \geq 0$.*

The following are true:

1. *Suppose $\lambda_1 = 0$. If either (C_1) or (C_2) holds, then C_φ is bounded.*
2. *Suppose $\lambda_1 > 0$. If (C_2) holds, then C_φ is bounded.*

Proof. Note that the difference between 1. and 2. is that the case "φ is a constant function" is not included when $\lambda_1 > 0$. This can be seen as follows. Take, for instance, $f(z) = e^{-\lambda_1 z} \in \mathcal{H}(E,\beta)$. If $\varphi(z) = z_0$ for some $z_0 \in \mathbb{C}$, then $C_\varphi f(z) = e^{-\lambda_1 z_0}$, which is a nonzero constant, and thus not representable in $\mathcal{H}(E,\beta)$ if $\lambda_1 > 0$.

Suppose $\lambda_1 = 0$. Clearly if (C_1) happens, i.e., $\varphi(z) = z_0$ for some $z_0 \in \mathbb{C}$, then

$$\|C_\varphi f\| = \|f(z_0)\| = |f(z_0)|\beta_1 \leq \beta_1 \|k_{z_0}\|\|f\|,$$

by Cauchy–Schwarz inequality. Hence, C_φ is bounded in this case.

We will use the following argument to prove that (C_2) implies "C_φ is bounded" in both cases $\lambda_1 > 0$ and $\lambda_1 = 0$.

Suppose (C_2) holds, we have

$$C_\varphi f(z) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n (z+b)} = \sum_{n=1}^{\infty} a_n e^{-\lambda_n b} e^{-\lambda_n z},$$

for any $f(z) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n z} \in \mathcal{H}(E,\beta)$. Since $\mathfrak{Re}(b) \geq 0$ and (λ_n) is increasing, we have

$$\|C_\varphi f\|^2 = \sum_{n=1}^{\infty} |a_n|^2 e^{-2\lambda_n \mathfrak{Re}(b)} \beta_n^2 \leq e^{-2\lambda_1 \mathfrak{Re}(b)} \sum_{n=1}^{\infty} |a_n|^2 \beta_n^2 = e^{-2\lambda_1 \mathfrak{Re}(b)} \|f\|. \tag{5}$$

This shows C_φ is bounded. The proof is complete. □

4.2 Necessary conditions

The sufficient conditions in Proposition 2 turn out to be necessary as well. Our aim is to establish the proof for this necessity.

The following lemma is needed for next results. An outline of proof is given in Ritt (1928), but we also provide a proof here for the sake of completeness.

Lemma 1 – *Suppose $f \in \mathcal{H}(E,\beta)$ has the representation*

$$f(z) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n z} \qquad (a_n, z \in \mathbb{C}).$$

Then for any $\sigma \in \mathbb{R}$, for any $n \geq 1$,

$$a_n = \lim_{t \to +\infty} \frac{1}{2ti} \int_{\sigma - ti}^{\sigma + ti} f(z) e^{\lambda_n z} dz, \tag{6}$$

where the integral is taken on the line segment from $\sigma - ti$ to $\sigma + ti$.

Proof. Fix a particular n. Define $\mu_k = \lambda_n - \lambda_k$. Multiply both sides of f by $e^{\lambda_n z}$, we have

$$f(z)e^{\lambda_n z} = a_1 e^{\mu_1 z} + a_2 e^{\mu_2 z} + a_3 e^{\mu_3 z} + \dots \tag{7}$$

For any $\sigma \in \mathbb{R}$ and $t > 0$, we integrate both sides of (7) on the line segment from $\sigma - ti$ to $\sigma + ti$. Since $f(z)e^{\lambda_n z}$ is uniformly convergence for all z, we can integrate term by term on the right-hand side to obtain

$$\int_{\sigma-ti}^{\sigma+ti} f(z)e^{\lambda_n z} dz = \sum_{k=0}^{\infty} a_n \int_{\sigma-ti}^{\sigma+ti} e^{\mu_k z} dz. \tag{8}$$

Note that for any $\mu \in \mathbb{R}$,

$$\frac{1}{2ti}\int_{\sigma-ti}^{\sigma+ti} e^{\mu z} dz = \begin{cases} 1 & \text{if } \mu = 0, \\ \dfrac{e^{\mu\sigma}}{\mu t}\sin(\mu t) & \text{if } \mu \neq 0. \end{cases}$$

Thus, (8) is equivalent to

$$\frac{1}{2ti}\int_{\sigma-ti}^{\sigma+ti} f(z)e^{\lambda_n z} dz = a_n + \sum_{k\neq n} a_k \frac{e^{\mu_k \sigma}}{\mu_k t}\sin(\mu_k t).$$

Letting $t \to \infty$ on both sides, and taking into account the uniform convergence of the series on the right-hand side, we obtain (6). □

We also need the following familiar fact.

Lemma 2 – *Suppose a composition operator C_φ, induced by an entire function φ, maps $\mathcal{H}(E,\beta)$ to itself. Then the adjoint operator C_φ^* of C_φ satisfies*

$$C_\varphi^* k_w = k_{\varphi(w)}, \quad \forall w \in \mathbb{C},$$

where k_w is the reproducing kernel at w as defined in (4).

Case $\lambda_1 > 0$

We have the following necessary condition:

Proposition 3 – *Suppose $\lambda_1 > 0$. Let φ be an entire function and C_φ be the induced composition operator. If C_φ is bounded on $\mathcal{H}(E,\beta)$, then*

$$\varphi(z) = z + b, \text{ with } b \in \mathbb{C}, \mathfrak{Re}(b) \geq 0.$$

Proof. Suppose C_φ is bounded on $\mathcal{H}(E,\beta)$, then its adjoint operator C_φ^* is also bounded. That is, there is a constant $B > 0$ such that

$$\|C_\varphi^* f\|^2 \le B\|f\|^2, \qquad \forall f \in \mathcal{H}(E,\beta). \tag{9}$$

Without the loss of generality, we may assume $B > 1$.

In particular, for $f = k_w$ where w is an arbitrary complex number, we note that $C_\varphi^* k_w = k_{\varphi(w)}$, so together with Remark 2 (a), the inequality (9) becomes

$$\sum_{n=1}^{\infty} \beta_n^{-2} e^{-2\lambda_n \mathfrak{Re}(\varphi(w))} \le B \sum_{n=1}^{\infty} \beta_n^{-2} e^{-2\lambda_n \mathfrak{Re}(w)}, \qquad \forall w \in \mathbb{C}. \tag{10}$$

Claim 1: *We have $\varphi(z) = z + b$ for some $b \in \mathbb{C}$.*

Assume $\psi(z) := z - \varphi(z)$ is a non-constant entire function, we show the contradiction by finding some $w \in \mathbb{C}$ such that inequality (10) does not hold.

Since ψ is not a constant function, the function $F(w) = e^{\lambda_1 \psi(w)}$ is also a non-constant entire function. By Liouville's theorem, F is not bounded, so we can choose a fixed $w = w_0 \in \mathbb{C}$ so that

$$|F(w_0)|^2 = e^{2\lambda_1 \mathfrak{Re}(\psi(w_0))} \ge 2B > 1.$$

This implicitly means $\mathfrak{Re}(\psi(w_0)) > 0$. Noting that (λ_n) is increasing, from Remark 2 (a), we have

$$\begin{aligned}\|k_{\varphi(w_0)}\|^2 &= \sum_{n=1}^{\infty} \beta_n^{-2} e^{-2\lambda_n \mathfrak{Re}(\varphi(w_0))} = \sum_{n=1}^{\infty} e^{2\lambda_n \mathfrak{Re}(\psi(w_0))} \beta_n^{-2} e^{-2\lambda_n \mathfrak{Re}(w_0)} \\ &\ge |F(w_0)|^2 \sum_{n=1}^{\infty} \beta_n^{-2} e^{-2\lambda_n \mathfrak{Re}(w_0)} \ge 2B \sum_{n=1}^{\infty} \beta_n^{-2} e^{-2\lambda_n \mathfrak{Re}(w_0)} > B\|k_{w_0}\|^2, \end{aligned} \tag{11}$$

which clearly contradicts the inequality (10). Thus, $\varphi(z) = z + b$ for some $b \in \mathbb{C}$.

Claim 2: *We have $\mathfrak{Re}(b) \ge 0$.*

Consider the probe functions $q_k(z) = \beta_k^{-1} e^{-\lambda_k z}$ $(k \ge 1)$. Since C_φ is bounded and $\|q_k\| = 1$, the sequence $(\|C_\varphi q_k\|)_k$ must be bounded. We note that $C_\varphi q_k(z) = \beta_k^{-1} e^{-\lambda_k (z+b)}$, so

$$\|C_\varphi q_k(z)\| = e^{-\lambda_k \mathfrak{Re}(b)}. \tag{12}$$

Since $\lambda_k \uparrow +\infty$, it is necessary that $-\mathfrak{Re}(b) \le 0$, i.e., $\mathfrak{Re}(b) \ge 0$. The proof is complete. □

Remark 3 – Proposition 3 is similar to the necessity of Hou, Hu, and L. Khoi (2013, Thm 4.9). To obtain this result, the authors first proved that the function φ

necessarily has the form $az+b$, then derived two other lemmas, before eventually showed that $a=1$. This proof strongly depends on the Lemma of Pólya and long. Our approach is much simpler, which is applicable to the general spaces $\mathcal{H}(E,\beta)$ and only utilizes fundamental results of functional analysis.

Now we obtain the following criterion for the bounded composition operators in the case $\lambda_1 > 0$.

Theorem 2 (Boundedness in case $\lambda_1 > 0$) – *Let φ be an entire function and C_φ be the induced composition operator. Suppose $\lambda_1 > 0$. Then the composition operator C_φ is bounded on $\mathcal{H}(E,\beta)$ if and only if*

$$\varphi(z) = z + b, \text{ for some } b \in \mathbb{C} \text{ with } \mathfrak{Re}(b) \geq 0.$$

Moreover, the operator norm is given by $\|C_\varphi\| = e^{-\lambda_1 \mathfrak{Re}(b)}$.

Proof. The necessary condition is proved in Proposition 3, while the sufficiency is shown in Proposition 2. Thus, C_φ is bounded if and only if $\varphi(z) = z+b$ for some $b \in \mathbb{C}$ with nonnegative real part.

To compute the norm of C_φ, note that (12) implies

$$\|C_\varphi\| \geq \|C_\varphi q_1\| = e^{-\lambda_1 \mathfrak{Re}(b)}. \tag{13}$$

From (5) and (13), we obtain $\|C_\varphi\| = e^{-\lambda_1 \mathfrak{Re}(b)}$. □

Since the spaces $\mathcal{H}(E,\beta_S)$ are special cases of the spaces $\mathcal{H}(E,\beta)$, we easily recover the boundedness by M. Doan, Mau, and L. Khoi (2019) and Hou, Hu, and L. Khoi (2013).

Case $\lambda_1 = 0$

In order to establish the necessity for the boundedness of composition operators on $\mathcal{H}(E,\beta)$, we again use the adjoint operator C_φ^*, but it turns out that the proof is more complicated than that of Theorem 2. The difference comes the fact that if $\lambda_1 = 0$, then $F(w) \equiv 1$ in the proof of Proposition 3, and so we do not have the second inequality of (11). One might attempt to adjust $F(w) = e^{\lambda_2 \psi(w)}$, but then the first inequality of (11) is not true. Hence, a nontrivial modification to the proof is needed.

Proposition 4 – *Suppose $\lambda_1 = 0$. Let φ be an entire function and C_φ be the induced composition operator. If the operator C_φ is bounded on $\mathcal{H}(E,\beta)$, then exactly one of the following possibilities happens:*

1. *φ is a constant function, or*
2. *$\varphi(z) = z + b$, for some $b \in \mathbb{C}$ with $\mathfrak{Re}(b) \geq 0$.*

Proof. Suppose C_φ is bounded. Since $\lambda_1 = 0$, we have $\lambda_2 > 0$.

If $\varphi(z) = z + b$ for some $b \in \mathbb{C}$, we obtain condition $\mathfrak{Re}(b) \geq 0$ in the same way as in Claim 2 of Theorem 2.

If φ is not of the form $z + b$, we prove that φ must be constant. Since C_φ is bounded, so is the adjoint operator C_φ^*. Hence, there is a constant $B > 1$ such that

$$\|C_\varphi^* f\| \leq B\|f\| \quad \forall f \in \mathcal{H}(E, \beta).$$

Consequently, we can assume inequality (10) holds for the chosen B.

Since φ is not of the form $z + b$, the function $\psi(z) = z - \varphi(z)$ is not constant. Thus, the function $Q(z) = e^{\lambda_2 \psi(z)}$ is entire and not constant either. By Liouville's theorem, Q is not bounded, i.e., there exists $(z_k) \subset \mathbb{C}$ such that $|Q(z_k)| \to \infty$ as $k \to \infty$. This allows us to define the following nonempty set of sequences:

$$\mathcal{S} := \left\{ (z_k)_{k=1}^\infty \subset \mathbb{C} : \lim_{k \to \infty} |Q(z_k)| = +\infty \right\}$$

From this point, our proof is divided into several claims as follows.

Claim 1: *If $(z_k) \in \mathcal{S}$, then $(\mathfrak{Re}(z_k))$ is not bounded above.*

Assume there is a sequence $(z_k) \in \mathcal{S}$ such that $\mathfrak{Re}(z_k) < T$ for some $T > 0$. We have

$$\frac{\beta_1^{-2} + \beta_2^{-2} e^{-2\lambda_2 \mathfrak{Re}(\varphi(z))}}{\beta_1^{-2} + \beta_2^{-2} e^{-2\lambda_2 \mathfrak{Re}(z)}} = 1 + \frac{\frac{\beta_1^2}{\beta_2^2}\left(|Q(z)|^2 - 1\right)}{e^{2\lambda_2 \mathfrak{Re}(z)} + \frac{\beta_1^2}{\beta_2^2}} \geq 1 + \frac{\frac{\beta_1^2}{\beta_2^2}(|Q(z)|^2 - 1)}{e^{2T\lambda_2} + \frac{\beta_1^2}{\beta_2^2}} =: C_T(z).$$

and

$$\sum_{n \geq 3} \beta_n^{-2} e^{-2\lambda_n \mathfrak{Re}(\varphi(z))} \geq |Q(z)|^2 \sum_{n \geq 3} \beta_n^{-2} e^{-2\lambda_n \mathfrak{Re}(z)}.$$

Hence,

$$\|k_{\varphi(z)}\|^2 \geq \left(C_T(z)(\beta_1^{-2} + \beta_2^{-2} e^{-2\lambda_2 \mathfrak{Re}(z)}) + |Q(z)|^2 \sum_{n \geq 3} \beta_n^{-2} e^{-2\lambda_n \mathfrak{Re}(z)} \right)$$

Since $(z_k) \in \mathcal{S}$, we can choose k so that $|Q(z_k)|$ is large enough and $\|k_{\varphi(z_k)}\|^2 > B\|k_{z_k}\|^2$. Again, inequality (10) does not hold, and we obtain a contradiction. Thus, every sequence $(\mathfrak{Re}(z_k)) \in \mathcal{S}$ has no upper bound.

Claim 2: *The function Q is bounded on the half-plane $\mathfrak{Re}(z) < 0$.*

Assume Q is unbounded on the half-plane $\mathfrak{Re}(z) < 0$, then there exists a sequence $(z_k) \subset \mathbb{C}$ such that $\mathfrak{Re}(z_k) < 0$ and $|Q(z_k)| \to \infty$ as $k \to \infty$. Hence, $(z_k) \in \mathcal{S}$ and $(\mathfrak{Re}(z_k))$ is bounded above. This clearly contradicts Claim 1.

Claim 3: *We have the representation* $e^{-\lambda_2\varphi(z)} = a_1 + a_2 e^{-\lambda_2 z}$ *for some* $a_1, a_2 \in \mathbb{C}$.

From Claim 2, there exists some $M > 0$ such that $|Q(z)| < M$, if $\mathfrak{Re}(z) < 0$. Substituting $\varphi(z) = z - \psi(z)$, we have

$$|M^{-1}e^{-\lambda_2\varphi(z)}| < e^{-\lambda_2\mathfrak{Re}(z)}, \qquad \text{for all } z \text{ with } \mathfrak{Re}(z) < 0. \tag{14}$$

Consider the function $f(z) = e^{-\lambda_2 z} \in \mathcal{H}(E,\beta)$. Since C_φ maps $\mathcal{H}(E,\beta)$ to itself, we have

$$C_\varphi f(z) = e^{-\lambda_2\varphi(z)} = \sum_{n=1}^{\infty} a_n e^{-\lambda_n z},$$

for some $(a_n) \subset \mathbb{C}$. Dividing each expression of the equality above by M, we obtain

$$g(z) = M^{-1}e^{-\lambda_2\varphi(z)} = c_1 + c_2 e^{-\lambda_2 z} + c_3 e^{-\lambda_3 z} + \dots, \tag{15}$$

where $c_n = a_n/M$.

From (14) and (15), it follows that $|g(z)| < e^{-\lambda_2\mathfrak{Re}(z)}$ for all z with $\mathfrak{Re}(z) < 0$. For any $n > 2$, we write $z = \sigma + ti$ ($\sigma, t \in \mathbb{R}$) and apply Lemma 1 to get

$$|c_n| = \left|\lim_{t\to\infty} \frac{1}{2ti}\int_{\sigma-ti}^{\sigma+ti} g(z)e^{\lambda_n z}dz\right| \le e^{\sigma(\lambda_n-\lambda_2)}.$$

As the inequality above is true for any $\sigma \in \mathbb{R}$, we have

$$|c_n| = \lim_{\sigma\to-\infty} e^{(\lambda_n-\lambda_2)\sigma} = 0, \quad \text{for all } n > 2.$$

Thus $a_n = 0$ for $n > 2$. From the uniqueness of the representation of $e^{-\lambda_2\varphi(z)}$, we have

$$e^{-\lambda_2\varphi(z)} = a_1 + a_2 e^{-\lambda_2 z}, \qquad \forall z \in \mathbb{C}. \tag{16}$$

Claim 4: *The function* φ *is constant.*

With the same notation as in Claim 3, we have the following cases:

1. If $a_2 \neq 0$ and $a_1 \neq 0$: the right hand side of (16) is zero at

$$z = -\lambda_2^{-1}\left(\log\left|\frac{a_1}{a_2}\right| + i\mathrm{Arg}\frac{a_1}{a_2} + i(2k+1)\pi\right) \ (k \in \mathbb{Z}),$$

   while the left hand side function is never zero, so we obtain a contradiction. This shows a_1 and a_2 cannot be both nonzero.

2. If $a_2 \neq 0$ and $a_1 = 0$: equation (16) implies

$$\varphi(z) = z - \lambda_2^{-1}(\log|a_2| + i(\mathrm{Arg}\ a_2 + 2k\pi)),$$

   for some $k \in \mathbb{Z}$, which contradicts the assumption ψ is not constant. This shows $a_2 = 0$.

3. If $a_2 = 0$, then (16) implies $a_1 \neq 0$. Clearly, φ is constant.

The proof is complete □

We conclude this section with the following theorem, which provides a criterion for a composition operator to be bounded on $\mathcal{H}(E,\beta)$ in case $\lambda_1 = 0$.

Theorem 3 (Criterion for $\lambda_1 = 0$) – *Suppose $\lambda_1 = 0$. Let φ is an entire function and C_φ be the induced composition operator. Then C_φ is bounded on $\mathcal{H}(E,\beta)$ if and only if one of the following cases happens:*

(C_1) *φ is constant, or*

(C_2) *$\varphi(z) = z + b$, for some $b \in \mathbb{C}$ with $\mathfrak{Re}(b) \geq 0$.*

Moreover, $\|C_\varphi\| \geq 1$ in Case (C_1), and $\|C_\varphi\| = 1$ in Case (C_2).

Proof. The sufficiency is proved in Proposition 2, and Proposition 4 establishes the necessity, so φ is either constant or of the affine form $z+b$ with $\mathfrak{Re}(b) \geq 0$. For the norm estimation of C_φ, following Claim 3 of Theorem 2, we obtain $\|C_\varphi\| \geq \|C_\varphi q_1\| = 1$. This is true for both cases (C_1) and (C_2). In addition, in Case (C_2), if $f(z) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n z} \in \mathcal{H}(E,\beta)$ is nonzero, as $0 \leq \lambda_n \uparrow +\infty$ and $\mathfrak{Re}(b) \geq 0$, we have

$$\|C_\varphi f\|^2 = \left\| \sum_{n=1}^{\infty} a_n e^{-\lambda_n (z+b)} \right\|^2 = \sum_{n=1}^{\infty} |a_n|^2 \beta_n^2 e^{-2\lambda_n \mathfrak{Re}(b)} \leq \sum_{n=1}^{\infty} |a_n|^2 \beta_n^2 = \|f\|^2,$$

so $\|C_\varphi\| \leq 1$. Hence $\|C_\varphi\| = 1$ for Case (C_2). □

5 Concluding remarks

In the present paper, we study the relation between an entire function φ and the boundedness of the induced composition operator C_φ acting on spaces of entire Dirichlet series $\mathcal{H}(E,\beta)$. We generalize the result of bounded operators C_φ on spaces $\mathcal{H}(E,\beta_S)$ and include the untreated case $\lambda_1 = 0$.

The following theorem establishes the *complete characterization* of the boundedness of C_φ, which shows that the criteria do not depend on whether the weight sequence $\beta = (\beta_n)$ satisfies condition (E) or any condition stronger than (E), such as (S).

Theorem 4 (Criterion for bounded C_φ any space $\mathcal{H}(E,\beta)$) – *Let β be a sequence of positive real number with condition* (E), *and φ be an entire function. Consider the following statements.*

(i) *φ is constant,*

(ii) *$\varphi(z) = z + b$ for some $b \in \mathbb{C}$, $\mathfrak{Re}(b) \geq 0$.*

The following are true about the boundedness of the composition operator C_φ acting on the induced Hilbert space $\mathcal{H}(E,\beta)$:

1. *If $\lambda_1 = 0$, then C_φ is bounded if and only if exactly one of conditions (i) or (ii) holds.*
2. *If $\lambda_1 > 0$, then C_φ is bounded if and only if (ii) holds.*

Furthermore, in Case (ii), the operator norm is given by $\|C_\varphi\| = e^{\lambda_1 \mathfrak{Re}(b)}$.

This theorem comes from Proposition 2, and Theorems 2 and 3.

Since the proofs of criteria for the compactness, compact difference, Hilbert–Schmidtness, cyclicity, etc. of composition operators C_φ acting on $\mathcal{H}(E,\beta_S)$ in Hou, Hu, and L. Khoi (2013), Hu and L. Khoi (2012), and Wang and Yao (2015) do not directly use condition (S) but the necessary condition $\varphi(z) = z + b$ with $\mathfrak{Re}(b) \geq 0$, these result may still be true for the general spaces $\mathcal{H}(E,\beta)$, with the exception that φ being constants is allowed for the case $\lambda_1 = 0$.

Other findings, such as norm estimation through reproducing kernels in Wang and Yao (2015), which directly uses (S) in their computation, need to be reconsidered when working with condition (E). However, we hope that our discovery and method may inspire readers to investigate further these problems in the future.

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