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# Spectra of non-regular elements in irreducible representations of simple algebraic groups

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**Abstract**

We study the spectra of non-regular semisimple elements in irreducible representations of simple algebraic groups. More precisely, we prove that if $G$ is a simply connected simple linear algebraic group and $\phi : G \to \mathrm{GL}(V)$ is a non-trivial irreducible representation for which there exists a non-regular non-central semisimple element $s \in G$ such that $\phi(s)$ has almost simple spectrum, then, with few exceptions, $G$ is of classical type and $\dim V$ is minimal possible. Here the spectrum of a diagonalizable matrix is called *simple* if all eigenvalues are of multiplicity 1, and *almost simple* if at most one eigenvalue is of multiplicity greater than 1. This yields a kind of characterization of the natural representation (up to their Frobenius twists) of classical algebraic groups in terms of the behavior of semisimple elements.

**Keywords:** semisimple elements, irreducible representations, eigenvalue multiplicities, simple linear algebraic groups.

MSC: 20G05, 20G07, 20E28.

*Dedicated to the memory of Ernest Vinberg*

# Introduction

A rather general problem which has received attention in the literature can be stated as that of classifying irreducible group representations whose image contains a matrix with a certain specified property. In this paper we concentrate on a property of the eigenvalue multiplicities of a semisimple element of simple linear algebraic groups in their irreducible representations. (Henceforth we will use "algebraic group" to mean "linear algebraic group".) Although problems on eigenvalues in group representations are important for many applications, little can be said in full

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generality. In fact, the behavior of individual elements in the image of a representation is quite unpredictable. For a discussion of this and related questions, we refer the reader to A. E. Zalesski (2009).

Here, we consider matrices with almost simple spectrum, that is, matrices having at most one eigenvalue of multiplicity greater than 1. More precisely, we will address the following:

**Problem 1 –** Let $G$ be a simple algebraic group defined over an algebraically closed field. Determine the irreducible representations $\phi$ of $G$ such that $\phi(G)$ contains a non-scalar diagonalizable matrix with almost simple spectrum.

Note that the notion of matrices with almost simple spectrum is a natural generalization of the similar notion of pseudo-reflections, the latter being diagonalizable matrices with two eigenvalues, one of which has multiplicity 1. The classification of irreducible matrix groups generated by pseudo-reflections was an important project enjoying numerous applications. (See Wagner (1978), Wagner (1981), and Zalesskii and Serežkin (1977, 1980).) We note as well that the consideration of Problem 1 is an extension of the analogous question for finite quasi-simple groups of Lie type and their representations in defining characteristic (see Suprunenko and Zalesskii (2000) and Suprunenko and Zalesskii (1998)), as well as the classification (in Seitz (1987) and Zalesskii and Suprunenko (1987)) of irreducible representations of simple algebraic groups for which a maximal torus acts with 1-dimensional weight spaces. A similar problem for irreducible representations of finite simple groups occurring as subgroups of $\mathrm{GL}_n(\mathbb{C})$ has been studied in Katz and Tiep (2021).

While Problem 1 is a question about semisimple elements, there is a natural generalization of the notions of simple and almost simple spectra to matrices that are not diagonalizable. Let $V$ be a finite-dimensional vector space over a field $F$ and $M \in \mathrm{GL}(V)$. Then $M$ is called *cyclic* if, for some $v \in V$, the space $V$ is spanned by the vectors $v, Mv, M^2v, \ldots$, and *almost cyclic* if, for some $\lambda \in F$, $M$ is conjugate to a matrix $\mathrm{diag}(\lambda \cdot \mathrm{Id}, M_1)$, where $M_1$ is a cyclic matrix. Almost cyclic matrices in the images of irreducible representations of finite simple groups are studied in Di Martino, Pellegrini, and A. E. Zalesski (2014), Di Martino, Pellegrini, and A. E. Zalesski (2020), and Di Martino and A. E. Zalesski (2018) (in certain special cases). Now let $G$ be as in Problem 1 above, $g \in G$, and let $\phi$ be an irreducible representation such that $\phi(g)$ is almost cyclic. If $g$ is not semisimple, then $g = su = us$ with $u \neq 1$ unipotent and $s$ semisimple, and one sees that $\phi(u)$ has a single non-trivial Jordan block. Such representations have been determined in Suprunenko (2013) and D. M. Testerman and A. E. Zalesski (2018). On the other hand, if $g$ is semisimple, and $\phi(g)$ is almost cyclic, then $\phi(g)$ has almost simple spectrum; indeed $\phi(g)$ has at most two eigenvalues, one of which has multiplicity 1.

Let us now return to our considerations of semisimple elements of $G$ whose spectrum in some irreducible representation of $G$ is almost simple. As every semisimple element $s \in G$ lies in a maximal torus, the condition for $\phi(s)$ to have simple spectrum implies that all weight multiplicities of $\phi$ are equal to 1. The irreducible representations whose set of weights satifies this property are determined in Seitz (1987) for tensor-indecomposable representations and completed in Zalesskii and Suprunenko (1987). By analogy, one could expect $\phi$ in Problem 1 to have all but one weight multiplicity equal to 1. And indeed this is the case, as the following result, which will be etablished in §3, shows.

**Theorem 1 –** *Let $G$ be a simple algebraic group defined over an algebraically closed field and $\phi$ an irreducible representation of $G$. Then the following statements are equivalent:*

*(1) The matrix $\phi(s)$ has almost simple spectrum for some non-central semisimple element $s \in G$.*

*(2) All non-zero weights of $\phi$ are of multiplicity* 1.

Theorem 1 will be relevant to our consideration of Problem 1, especially as the irreducible representations of simple algebraic groups satisfying 1 have been determined in D. M. Testerman and A. E. Zalesski (2015). The above theorem is best possible in the sense that in order to obtain a more precise result one has to specify the nature of the semisimple element $s$ in question. We recall that an element $g \in G$ is said to be *regular* if $\dim(C_G(g))$ is equal to the rank of $G$; for $g$ semisimple this is equivalent to $C_G(g)^\circ$ being abelian, see Springer and Steinberg (1970, Chapter III, Corollary 1.7). Our investigations show that, with very few exceptions, a non-central semisimple element $s$ having an almost simple spectrum in an irreducible representation $\phi$ must be regular.

**Theorem 2 –** *Let $G$ be a simply connected simple algebraic group defined over an algebraically closed field $F$ of characteristic $p \geq 0$ and let $s \in G$ be a non-regular non-central semisimple element. Let $V$ be a non-trivial irreducible $G$-module. If the spectrum of $s$ on $V$ is almost simple, then one of the following holds:*

*(1) $G$ is of Lie type $A_n, B_n$ $(p \neq 2)$, $C_n$ or $D_n$ and* $\dim V = n+1, 2n+1, 2n, 2n$, *respectively;*

*(2) $G$ is of Lie type $B_n$, $p = 2$ and* $\dim V = 2n$;

*(3) $G = A_3$ and* $\dim V = 6$;

*(4) $G = C_2$, $p \neq 2$ and* $\dim V = 5$.

The irreducible representations of $G$ of the dimensions given in Theorem 2 are well known; a description of elements $s$ which have almost simple spectrum on $V$ is provided in Section 3.

**Notation** We fix an algebraically closed field $F$ of characteristic $p \geq 0$.

Throughout the paper $G$ is a simple simply connected linear algebraic group defined over $F$. All $G$-modules considered are rational finite-dimensional $FG$-

modules. For a $G$-module $V$ (or a representation $\rho$ of $G$), we write $V \in \mathrm{Irr}(G)$ (or $\rho \in \mathrm{Irr}(G)$) to mean that $V$ (or $\rho$) is rational and irreducible. If $H$ is a subgroup of $G$ then we write $V|_H$ for the restriction of a $G$-module $V$ to $H$.

We fix a maximal torus $T$ in $G$, which in turn defines the roots of $G$ as well as the weights of $G$-modules and representations. The $T$-weights of a $G$-module $V$ are the irreducible constituents of the restriction of $V$ to $T$. As $T$ is fixed, we will omit the reference to $T$ and write "weights" in place of "$T$-weights". The set of weights of $V$ is denoted by $\Omega(V)$. For $\mu \in \Omega(V)$, the dimension of the $\mu$-weight space $\{v \in V : tv = \mu(t)v$ for all $t \in T\}$ is called the *multiplicity of $\mu$ in $V$*. The Weyl group of $G$ is denoted by $W$; as $W = N_G(T)/T$, the conjugation action of $N_G(T)$ on $T$ yields an action of $W$ on $T$ and consequently on the set of $T$-weights. The $W$-orbit of $\mu \in \Omega$ is denoted by $W\mu$. The set $\Omega = \mathrm{Hom}(T, F^\times)$ (the rational homomorphisms of $T$ to the multiplicative group of $F$) is called the *weight lattice*, which is a free $\mathbf{Z}$-module of finite rank called *the rank of $G$*.

With an algebraic group $H$ is associated the Lie algebra of $H$ denoted here by $\mathrm{Lie}(H)$. For the simple group $G$, we denote the set of roots (that is, the non-zero weights of the $G$-module $\mathrm{Lie}(G)$) by $\Phi$ or $\Phi(G)$. For the notions of closed subsystems of $\Phi$ and subsystem subgroups see Malle and D. Testerman (2011, §13.1). The $\mathbf{Z}$-span of $\Phi$ is called the *root lattice* and is denoted here by $R$ or $R(G)$. In $\Phi(G)$, we fix a base $\Pi = \{\alpha_1, \dots, \alpha_n\}$ and order the simple roots according to the Dynkin diagrams as in Bourbaki (1968). The associated set of positive roots will be denoted $R^+$ or $R^+(G)$. The weights in $R$ are called *radical*. For each root $\alpha \in \Phi(G)$, we choose a non-zero element $X_\alpha$ in the $\alpha$-weight space of $T$ on $\mathrm{Lie}(G)$. Thus, $FX_\alpha$ is the Lie algebra of a $T$-invariant one-dimensional unipotent subgroup $U_\alpha$ of $G$; see Malle and D. Testerman (2011, Theorem 8.16) for details.

One defines a non-degenerate, $W$-invariant, symmetric bilinear form on $\Omega \otimes_{\mathbf{Z}} \mathbb{R}$, which we express as $(\mu, \nu)$. For $\alpha \in \Phi$, let $w_\alpha \in W$ denote the corresponding reflection. The elements $\omega_i$ satisfying $2(\omega_j, \alpha_i) = (\alpha_i, \alpha_i)\delta_{ij}$ for $1 \le i, j \le n$ belong to $\Omega$ and are called *fundamental dominant weights*, see Bourbaki (1968, Ch. VI, §1, no.10). These form a $\mathbf{Z}$-basis of $\Omega$, so every $\nu \in \Omega$ can be expressed in the form $\sum a_i \omega_i$, for $a_i \in \mathbb{Z}$; the set of $\nu$ with $a_1, \dots, a_n \ge 0$ is denoted by $\Omega^+$, the set of dominant weights. We set $\Omega^+(V) = \Omega^+ \cap \Omega(V)$, so $\Omega^+(V)$ is the set of dominant weights of $V$. In what follows, we will regularly use so-called "Bourbaki weights", when $R(G)$ is of type $A_{r-1}, B_r, C_r$ or $D_r$, which are elements of a $\mathbf{Z}$-lattice containing $\Omega$ with basis $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_r$; the explicit expressions of the fundamental weights and the simple roots of $G$ in terms of $\varepsilon_i$'s are given in Bourbaki (1968, Planches I – IV).

There is a standard partial ordering of elements of $\Omega$: for $\mu, \mu' \in \Omega$ we write $\mu \prec \mu'$ and $\mu' \succ \mu$ if and only if $\mu \ne \mu'$ and $\mu' - \mu \in R^+$. (We write $\mu \preceq \mu'$ and $\mu' \succeq \mu$ to allow $\mu = \mu'$.) If $\mu$ and $\mu'$ are dominant weights such that $\mu' \preceq \mu$, we say *$\mu'$ is subdominant to $\mu$*. For the notion of a minuscule weight see Bourbaki (1975, Ch. VIII, §7.3), where they are tabulated. Every irreducible $G$-module has a unique weight $\omega$ such that $\mu \prec \omega$ for every $\mu \in \Omega(V)$ with $\mu \ne \omega$. This is called the *highest weight of $V$*. There is a bijection between $\Omega^+$ and $\mathrm{Irr}(G)$, so for $\omega \in \Omega^+$ we denote by $V_\omega$

the irreducible $G$-module with highest weight $\omega$. Suppose that $p > 0$; a dominant weight $\sum a_i\omega_i$ is called *p-restricted* if $0 \leq a_i < p$ for all $i = 1,\dots,n$. For uniformity, we often do not separate the cases with $p = 0$ and $p > 0$; by convention, when $p = 0$, a $p$-restricted weight is simply a dominant weight. An irreducible $G$-module is called *p-restricted* if its highest weight is $p$-restricted. For classical groups $G$, that is, those with root system one of $A_n$, $B_n$, $C_n$ or $D_n$, the module with highest weight $\omega_1$ is called the *natural module* and the associated representation the *natural representation*. (There is an exceptional case, when $G = B_n$ and $p = 2$, where the natural module is the Weyl module of highest weight $\omega_1$.)

The maximal height root of $\Phi(G)$ is denoted by $\omega_a$; this is the highest weight of Lie($G$) and affords a non-trivial composition factor of the adjoint module Lie($G$). The short root module for $G$ of type $B_n, C_n, F_4$, and $G_2$ is the irreducible $G$-module all of whose non-zero weights are short roots. This is unique, and the highest weight of the short root module is maximal among short roots (with respect to $\prec$). An irreducible $G$-module is called *tensor-decomposable* if it is a tensor product of two or more non-trivial irreducible modules, similarly for representations.

If $h : G \to G$ is a surjective algebraic group homomorphism and $\phi$ is a representation of $G$ then the $h$-twist $\phi^h$ of $\phi$ is defined as the mapping $g \mapsto \phi(h(g))$ for $g \in G$. Of fundamental importance is the Frobenius mapping $Fr : G \to G$ arising from the mapping $x \mapsto x^p$ $(x \in F)$ when $p > 0$. If $V$ is a $G$-module and $k$ a nonnegative integer, then the modules $V^{Fr^k}$ are called *Frobenius twists of $V$*; if $V$ is irreducible with highest weight $\omega$ then the highest weight of $V^{Fr^k}$ (for $k \geq 0$) is $p^k\omega$.

If $p = 2$, then for every $n$ there is a surjective algebraic group homomorphism $B_n \to C_n$ with trivial kernel (so this is an abstract group isomorphism); for our purposes, the choice between these two groups is irrelevant, so we choose to work with $C_n$ when $p = 2$.

For the natural $2n$-dimensional module $M$ of the group $C_n$, $n \geq 2$, a basis $\{e_i, f_i \mid 1 \leq i \leq n\}$ is called *symplectic* if $\{e_i, f_i\}$ is a hyperbolic pair for all $i$ and $M$ is the orthogonal direct sum of the spaces $\langle e_i, f_i\rangle$, $1 \leq i \leq n$.

Finally, we will assume $n \geq 1$ for $A_n$, $n > 1$ for $C_n$, $n > 2$ for $G = B_n$, and $n > 3$ for $D_n$. For brevity we write $G = A_n$ to say that $G$ is a simple simply connected algebraic group of type $A_n$, and similarly for the other types.

# Preliminaries

**Lemma 1 –** *Let $M = M_1 \otimes M_2$ be a Kronecker product of diagonal non-scalar matrices $M_1, M_2$ of sizes $m \leq n$, respectively. Suppose that $M$ has almost simple spectrum. Then*

*(1) $M_1$ and $M_2$ have simple spectrum, and*

*(2) if $M_i$ is similar to $M_i^{-1}$ for $i = 1,2$, then the eigenvalue multiplicities of $M$ do not exceed 2.*

*Proof.* (1) Suppose that $M_1$ has an eigenvalue $e$, say, of multiplicity $r > 1$. Let $b_1, b_2$ be distinct eigenvalues of $M_2$. Then $eb_1$, $eb_2$ are distinct eigenvalues of $M$, each of multiplicity greater than 1. This implies the claim.

(2) Suppose the contrary, and let $e$ be an eigenvalue of $M$ of multiplicity at least 3. By (1), $M_1$ and $M_2$ have simple spectra so $e = a_i b_i$ for $i = 1,2,3$ and some (distinct) eigenvalues $a_i$ of $M_1$ and $b_i$ of $M_2$. Then $e^{-1} = a_i^{-1} b_i^{-1}$ is an eigenvalue of $M$, of the same multiplicity as that of $e$. As $M$ has almost simple spectrum and is similar to $M^{-1}$ by hypothesis, we have $e = e^{-1}$, so $a_1 b_2 = a_2^{-1} b_1^{-1}$. If $(a_2^{-1}, b_2) \neq (a_1, b_1^{-1})$, then $a_1 b_2$ is an eigenvalue of $M$ of multiplicity at least 2 and so is equal to $e$. But this then implies $a_1 b_2 = a_1 b_1$, contradicting that the $b_i$ are distinct. Hence $a_2 = a_1^{-1}$ and $b_2 = b_1^{-1}$. Similarly, $a_1 b_3 = a_3^{-1} b_1^{-1}$ implies that $a_3 = a_1^{-1}$ and $b_3 = b_1^{-1}$. But now $a_2 = a_3$ contradicting that the $a_i$ are distinct. □

**Definition 1 –** Let $V$ be a $G$-module and $\mu, \nu \in \Omega(V)$, $\mu \neq \nu$. We say that $s \in T$ *separates the weights* $\mu$ *and* $\nu$ if $\mu(s) \neq \nu(s)$. If this holds for every pair of distinct weights $\mu, \nu$ of $V$, we say that $s$ *separates the weights of* $V$.

If $s$ separates the weights of $V$ then the eigenvalue multiplicities of $s$ acting on $V$ are simply the weight multiplicities of $V$.

**Lemma 2 –** *Let* $V$ *be a non-trivial* $G$*-module. Let* $S \subset T$ *be the set of all* $t \in T$ *that separate the weights of* $V$*. Then*

*(1)* $S$ *is a nonempty Zariski open subset of* $T$*.*

*(2) Suppose that at most one weight of* $V$ *has multiplicity greater than* 1*. Then, for all* $s \in S$*, the spectrum of* $s$ *is almost simple.*

*Proof.* (1) Let $\mu, \nu$ be weights of $V$, $\mu \neq \nu$. Then $T_{\mu,\nu} := \{x \in T \mid \mu(x) = \nu(x)\}$ is a Zariski closed subset $T_{\mu,\nu}$ of $T$. The set of elements of $T$ that do not separate some pair of weights of $V$, being the finite union of all $T_{\mu,\nu}$, is a proper closed subset of $T$. Moreover, $S = T \setminus (\cup T_{\mu,\nu})$, and so (1) follows.

(2) Let $s \in S$, so that $\mu(s) \neq \nu(s)$ whenever $\mu \neq \nu$ are weights of $V$. Then the eigenvalues of $s$ on $V$ are exactly $\mu(s)$, where $\mu$ runs over the weights of $V$, and the multiplicity of $\mu(s)$ equals that of $\mu$, giving (2). □

We will require the following characterization of regular semisimple elements.

**Proposition 1 –** *Springer and Steinberg (1970, Ch. III, §1, Corollary 1.7) Let* $G$*,* $T$ *be as usual, and let* $s \in T$*. Then the following conditions are equivalent:*

*(1)* $s$ *is regular;*

*(2)* $C_G(s)$ *consists of semisimple elements;*

*(3) for all* $\alpha \in \Phi(G)$*,* $\alpha(s) \neq 1$*;*

*(4)* $C_G(s)^\circ$ *is a torus.*

**Lemma 3 –** *Let $V, V_1, V_2$ be non-trivial $G$-modules. Let $s \in T \setminus Z(G)$ have almost simple spectrum on $V$.*

*(1) Suppose that $V = V_1 \otimes V_2$. Then all weights of $V_1$ and $V_2$ are of multiplicity 1, and $s$ is regular.*

*(2) Suppose that $\Omega(V_1) + \Omega(V_2) = \Omega(V)$. Then $s$ separates the weights of $V_1$ and $V_2$.*

*Proof.* The first claim of (1) follows from Lemma 1. For the second assertion, suppose that $s$ is not regular. Then by Proposition 1, $C_G(s)$ contains a unipotent element $u \neq 1$. As $u$ stabilizes every eigenspace of $s$ on $V_1$, at least one of them is of dimension greater than 1, contradicting Lemma 1(1).

(2) Suppose the contrary, that the weights of $V_1$, say, are not separated by $s$, so there exist distinct weights $\mu_1, \mu_2 \in \Omega(V_1)$ such that $\mu_1(s) = \mu_2(s)$. Then for every $\lambda, \mu \in \Omega(V_2)$, $\mu_i + \lambda, \mu_i + \mu \in \Omega(V)$ for $i = 1, 2$ and $(\mu_1 + \lambda)(s) = (\mu_2 + \lambda)(s)$ and $(\mu_1 + \mu)(s) = (\mu_2 + \mu)(s)$. As $s \notin Z(G)$, the spectrum of $s$ on $V$ is not almost simple, a contradiction. □

With regards to applying Lemma 3(2), we note that $\Omega(V) = \Omega(V_1) + \Omega(V_2)$ if $V = V_1 \otimes V_2$. For certain choices of $V, V_1, V_2$, and under certain conditions on $p$, we may deduce that $\Omega(V) = \Omega(V_1) + \Omega(V_2)$, for $V$ different from $V_1 \otimes V_2$. See Lemma 4(2) below.

We recall here some basic facts about the set of weights of irreducible representations of a simple algebraic group defined over a field of characteristic 0 (which are derived from analogous statements about the weights of irreducible representations of simple Lie algebras defined over $\mathbb{C}$). Fixing a maximal torus $T_H$ of a simple algebraic group $H$ defined over $\mathbb{C}$, and adopting the notation fixed earlier, so in particular, writing $W(H)$ for the Weyl group of $H$ relative to $T_H$, let $\lambda$ be a dominant $T_H$-weight. Then the set of weights of the irreducible $\mathbb{C}H$-module with highest weight $\lambda$ is precisely the set

$$\{w(\mu) \mid \mu \in \Omega^+, \mu \preceq \lambda, w \in W(H)\},$$

that is, the $W(H)$-conjugates of all weights which are subdominant to the highest weight $\lambda$. From this one directly deduces the following facts:

(1) Let $\lambda, \mu \in \Omega^+$ and $\mu \prec \lambda$. Let $V_\lambda$, respectively $V_\mu$, be the associated irreducible $\mathbb{C}H$-modules; then $\Omega(V_\mu) \subset \Omega(V_\lambda)$.

(2) Bourbaki (1975, Ch. VIII, §7, Proposition 10) Let $\lambda, \mu \in \Omega^+$, with associated irreducible $\mathbb{C}H$-modules $V_\lambda$, $V_\mu$; then $\Omega(V_{\lambda+\mu}) = \Omega(V_\lambda \otimes V_\mu)$.

(3) Bourbaki (1975, Ch. VIII, §7, Propositions 4 and 6) Let $\lambda \in \Omega^+$, $\lambda \neq 0$. If $\lambda$ is a radical weight, then some root is a weight of $V_\lambda$; otherwise $\Omega(V_\lambda)$ contains some minuscule weight.

We now return to the situation where the field $F$ is of arbitrary characteristic. We will use a fundamental result of Premet, which relies on the following definition and notation.

**Definition 2 –** We set $e(G) = 1$ for $G$ of type $A_n, D_n$, or $E_n$, $e(G) = 2$ for $G$ of type $B_n, C_n$, or $F_4$, and $e(G) = 3$ for $G$ of type $G_2$.

**Theorem 3 –** *Premet (1987, Theorem 1) Assume $p = 0$ or $p > e(G)$. Let $\lambda$ be a $p$-restricted dominant weight. Then $\Omega(V_\lambda) = \{w(\mu) \mid \mu \in \Omega^+, \mu \preceq \lambda, w \in W\}$.*

An application of Theorem 3 and the preceding remarks now gives:

**Lemma 4 –** *Assume $p = 0$ or $p > e(G)$. Let $\lambda, \mu \in \Omega^+$, where $\lambda$ is $p$-restricted, and let $V_\lambda$, respectively, $V_\mu$ be the associated irreducible $G$-modules. Then the following hold.*

*(1) If $\mu \prec \lambda$ then $\Omega(V_\mu) \subseteq \Omega(V_\lambda)$.*

*(2) If $\lambda + \mu$ is $p$-restricted then $\Omega(V_{\lambda+\mu}) = \Omega(V_\lambda \otimes V_\mu) = \Omega(V_\lambda) + \Omega(V_\mu)$.*

*(3) If $\lambda$ is a radical weight, then some root is a weight of $V_\lambda$; otherwise $\Omega(V_\lambda)$ contains some minuscule weight.*

For the following result we introduce an additional notation. Let $\Psi \subset \Phi$ be a closed subsystem. Then we set $G(\Psi)$ to be the subgroup generated by the $T$-root subgroups corresponding to roots in $\Psi$.

**Theorem 4 –** *Suprunenko and A. E. Zalesski (2005, Theorem 1) Let $G$ be a simple algebraic group with root system $\Phi$. If $\Phi$ is of type $B_n$, assume* $\mathrm{char}(F) \neq 2$. *Let $R_1, R_2 \subset \Phi$ be closed subsystems such that the subgroups $G_1 := G(R_1)$ and $G_2 := G(R_2)$ are simple and $[G_1, G_2] = 1$. Let $\phi$ be an irreducible representation of $G$. Then one of the following holds:*

*(1) $\phi|_{G_1G_2}$ contains a composition factor which is non-trivial for both $G_1$ and $G_2$;*

*(2) $G$ is classical and $\phi$ is a Frobenius twist of either the natural representation or the dual of the natural representation of $G$;*

*(3) $G = C_n$ with $p = 2$, $G = B_n$ with $n > 2$, or $G = D_n$ with $n \geq 4$, and $\phi$ is a Frobenius twist of the irreducible representation of highest weight $\omega_n$, or one of $\omega_n$ and $\omega_{n-1}$ if $G = D_n$.*

The following lemma will allow us in some cases to reduce our analysis of elements with almost simple spectrum to representations all of whose weights occur with multiplicity one.

**Lemma 5 –** *Let $G$ be a simple algebraic group of rank greater than 1 and $s \in T \setminus Z(G)$. Assume that $p = 0$ or $p > e(G)$. Let $\mu \neq 0$ be a $p$-restricted dominant weight.*

(1) *Let $\mu_m$ be the minimal non-zero weight subdominant to $\mu$. Assume that the spectrum of $s$ on $V_{\mu_m}$ is not almost simple. Then the following hold:*

  (i) *if $\mu$ is not radical, then the spectrum of $s$ on $V_\mu$ is not almost simple;*

  (ii) *if $\mu$ is radical and the multiplicity of the weight $0$ in $V_{\mu_m}$ is at most $1$, then the spectrum of $s$ on $V_\mu$ is not almost simple;*

  (iii) *if $\mu$ is radical and the multiplicity of the weight $0$ on both $V_\mu$ and $V_{\mu_m}$ is greater than $1$, then the spectrum of $s$ on $V_\mu$ is not almost simple;*

  (iv) *if $0 \prec \mu_m \preceq \mu$ and $s$ is non-regular, then the spectrum of $s$ on $V_\mu$ is not almost simple.*

(2) *Suppose that $\omega_a \prec \mu$, the multiplicity of the weight $0$ in $V_\mu$ is greater than $1$, and the spectrum of $s$ on $V_{\omega_a}$ is not almost simple. Then the spectrum of $s$ on $V_\mu$ is not almost simple.*

(3) *Suppose that $\omega_a \prec \mu$, $s$ is non-regular, and the spectrum of $s$ on $V_{\omega_a}$ is not almost simple. Then the spectrum of $s$ on $V_\mu$ is not almost simple.*

*Proof.* By assumption, Theorem 3 applies, and we may apply Lemma 4. If $\mu_m$ is non-radical, then all weight multiplicities of $V_{\mu_m}$ are well known to be equal to 1; (i) follows. Together with the hypothesis in (ii) about the multiplicity of the zero weight, we observe that if the spectrum of $s$ on $V_{\mu_m}$ is not almost simple then there are 4 distinct weights $\lambda_1, \lambda_2, \mu_1, \mu_2$ of $V_{\mu_m}$ such that $\lambda_1(s) = \lambda_2(s) \neq \mu_1(s) = \mu_2(s)$. Then Lemma 4(1) implies that these weights are weights of $V_\mu$, and the result follows.

In case (iii), $\mu_m$ is the maximal height short root and the multiplicity of any non-zero weight in $V_{\mu_m}$ is equal to 1. Saying that the spectrum of $s$ on $V_{\mu_m}$ is not almost simple means that there exist weights $\lambda_1, \lambda_2$ of $V_{\mu_m}$ such that $\lambda_1(s) = \lambda_2(s) \neq 1$. As these weights are weights of $V_\mu$ (again by Lemma 4(1)) and, by hypothesis, the weight 0 occurs in $V_\mu$ with multiplicity greater than 1, the result follows.

(iv) As $s$ is non-regular, there exists $\alpha \in \Phi(G)$ such that $\pm\alpha(s) = 1$ (Proposition 1). Since the spectrum of $s$ on $V_{\mu_m}$ is not almost simple, there are distinct short roots $\beta, \gamma$ such that $\beta(s) = \gamma(s) \neq 1$. Then Lemma 4 implies that $\pm\alpha, \beta, \gamma$ are weights of $V_\mu$, and the result follows.

For (2), first note that the multiplicity of the weight 0 in $V_{\omega_a}$ is greater than 1 unless $(G, p) = (A_2, 3)$ (here we again rely on the prime restrictions in the hypotheses). This case is considered in (ii). In all other cases, saying that the spectrum of $s$ on $V_{\omega_a}$ is not almost simple means that there are two roots $\alpha, \beta$ such that $\alpha(s) = \beta(s) \neq 1$. As the weights of $V_{\omega_a}$ occur as weights of $V_\mu$ and the weight 0 occurs in $V_\mu$ with multiplicity greater than 1, the result follows.

Finally, the case (3) follows as (iv) above, where one has to replace $V_{\mu_m}$ by $V_{\omega_a}$ and "short roots" by "roots". □

We complete this section with a straightforward observation about the natural modules for classical groups.

**Lemma 6 –** *Let $G$ be a classical type group and assume $p \neq 2$ when $G$ is of type $B_n$. Let $V = V_{\omega_1}$ and $s \in G$ be a non-central semisimple element.*

*(1) For $G = A_n$ or $C_n$, if $s$ is regular, then $s$ has simple spectrum on $V$.*

*(2) Let $G = B_n$. Then $s$ is regular if and only if the multiplicity of the eigenvalue $-1$ on $V$ is at most 2 and the other eigenvalue multiplicities are equal to 1.*

*(3) Let $G = D_n$. Then $s$ is regular if and only if the multiplicities of the eigenvalues 1 and $-1$ on $V$ are at most 2 and the other eigenvalue multiplicities are equal to 1. In addition, if the spectrum of $s$ on $V$ is not almost simple then that of $s$ on $V_{\omega_2}$ is not almost simple.*

*(4) If $s$ is regular then the spectrum of $s$ on $V$ is almost simple unless $G = D_n$, $p \neq 2$ and $1, -1$ are eigenvalues of $s$ on $V$, each of multiplicity 2.*

*Proof.* (1) This is straightforward and well known.

For the remainder of the proof, we take $T$ to be the maximal torus consisting of the diagonal matrices in the image of the natural representation of $G$. We now turn to (2) and the first statement of (3). Observe that $\Omega(V)$ consists of the weights $\pm\varepsilon_i$, $1 \leq i \leq n$, together with the weight 0 in case $G = B_n$. In addition, $s$ is regular if and only if $\alpha(s) \neq 1$ for every root $\alpha$. Set $a_i = \varepsilon_i(s)$ and recall that $\Phi(D_n) = \{\pm\varepsilon_i \pm \varepsilon_j \mid 1 \leq i < j \leq n\}$ and $\Phi(B_n) = \{\pm\varepsilon_i \pm \varepsilon_j, \pm\varepsilon_r \mid 1 \leq i < j \leq n, 1 \leq r \leq n\}$. So $s$ is regular if and only if $a_i \neq a_j$ and $a_i \neq a_j^{-1}$ for every $i \neq j$, and if in addition, for $G = B_n$, $a_i \neq 1$ for all $1 \leq i \leq n$. So if $G = B_n$, we see that $s$ is regular if and only if either all of the eigenvalues $a_1^{\pm 1}, a_2^{\pm 1}, \ldots, a_n^{\pm 1}$ are distinct and distinct from 1, or there exists a unique $i$ with $a_i = a_i^{-1}$. If $a_i = a_i^{-1} = -1$, then $s$ is regular if and only if all eigenvalues of $s$ on $V$ different from $-1$ occur with multiplicity 1, and $-1$ occurs with multiplicity at most 2. Now if $G = D_n$, then $s$ is regular only if $a_1^{\pm 1}, \ldots, a_n^{\pm 1}$ are distinct or there exists $1 \leq i \leq n$ such that $a_i = a_i^{-1}$, so $a_i \in \{1, -1\}$. In the latter case, $s$ is regular if and only if all eigenvalues distinct from 1 and $-1$ occur with multiplicity 1 and each eigenvalue $a_i \in \{1, -1\}$ occurs with multiplicity at most 2, as claimed.

For the final statement of (3), let $G = D_n$ and suppose that the spectrum of $s$ on $V$ is not almost simple. Then, without loss of generality, we may assume $a_i = a_j$ for some $1 \leq i \neq j \leq n$. Then $(\varepsilon_i - \varepsilon_k)(s) = (\varepsilon_j - \varepsilon_k)(s)$ and $(-\varepsilon_i - \varepsilon_k)(s) = (-\varepsilon_j - \varepsilon_k)(s)$ for every $k \neq i, j$. Recall that the non-zero weights of $V_{\omega_2}$ are the roots in $\Phi(G)$, and the zero weight occurs with multiplicity at least 2. Assume for a contradiction that the spectrum of $s$ on $V_{\omega_2}$ is almost simple. Then $(\varepsilon_i - \varepsilon_k)(s) = (\varepsilon_j - \varepsilon_k)(s) = (-\varepsilon_i - \varepsilon_k)(s) = (-\varepsilon_j - \varepsilon_k)(s) = 1$, whence $-\varepsilon_i(s) = \varepsilon_i(s) = \varepsilon_k(s)$ for all $1 \leq k \leq n$. As $s \notin Z(G)$, we get a contradiction.

(4) This follows from (1), (2) and (3). □

# 1 Reduction theorem, and proof of Theorem 1

For an abelian group $S$, let $\mathrm{Irr}(S)$ denote the set of irreducible $F$-linear representations of $S$ and write $1_S$ for the trivial representation. For $V$ a finite-dimensional $F$-vector space, and $S \subset \mathrm{GL}(V)$ an abelian subgroup, and $\eta \in \mathrm{Irr}(S)$, set $V_S(\eta) = \{v \in V : sv = \eta(s)v \text{ for all } s \in S\}$. If $V_S(\eta) \neq \{0\}$, we say $\eta$ is an $S$-weight of $V$ and we call $V_S(\eta)$ the *$\eta$-weight space for $S$*. As throughout $G$ is a simple algebraic group defined over $F$ and $T \subset G$ is a maximal torus of $G$. If $V$ is a rational $G$-module then $V$ is a direct sum of $T$-weight spaces and for any subgroup $S \subseteq T$, these weight spaces are $S$-invariant. Thus for $\eta \in \mathrm{Irr}(S)$, $V_S(\eta)$ is a sum of $T$-weight spaces of $V$. We establish here a result about such subgroups $S$ of $T$, and later will apply this to the case where $S$ is the subgroup generated by an element $s \in T$.

Recall (see for instance Malle and D. Testerman (2011, §7)) that for any rational representation $\rho : G \to \mathrm{GL}(V)$, we have a corresponding representation of $\mathrm{Lie}(G)$, namely $d\rho : \mathrm{Lie}(G) \to \mathrm{Lie}(\mathrm{GL}(V))$. For $g \in G$, let $t_g : G \to G$ denote the automorphism induced by conjugation by $g$. Then using the basic definitions and properties of the differential, we have that $t_{\rho(g)} \circ \rho = \rho \circ t_g$ and so

$$\mathrm{Ad}(\rho(g)) \circ d\rho = d\rho \circ \mathrm{Ad}(g).$$

**Theorem 5 (Reduction theorem) –** *Let $G$ be a simple algebraic group, $T$ a maximal torus of $G$, and $S \subseteq T$ a subgroup such that $C_G(S) \neq G$. Let $V$ be an irreducible $G$-module with $p$-restricted highest weight. Let $V_S(\eta)$ be an $S$-weight space of $V$, for some $\eta \in \mathrm{Irr}(S)$. Suppose that $\dim V_S(\eta) = k > 1$ and that all other $S$-weight spaces on $V$ are of dimension 1. Then all non-zero $T$-weights of $V$ are of multiplicity 1.*

*Proof.* Set $E = V_S(\eta)$. For $\mu \in \Omega(V)$, write $M_\mu$ for the $T$-weight space of $V$ associated to $\mu$. Suppose that $\dim M_\mu \geq 2$, for some $\mu \in \Omega(V)$. Then $M_\mu \subset E$. As $\dim M_\mu = \dim M_{w(\mu)}$ for any $w \in W$, we necessarily have $M_{w(\mu)} \subset E$. Now let $\rho : G \to \mathrm{GL}(V)$ be the corresponding rational representation of $G$. For a root $\alpha \in \Phi$, $\alpha$ induces a 1-dimensional representation $\lambda_\alpha$ of the group $S$.

Consider first the case where $\lambda_\alpha \neq 1_S$, for all $\alpha \in \Phi$. Recall the notation $X_\alpha \in \mathrm{Lie}(G)$, a root vector associated to the root $\alpha$, a fixed element which spans the Lie algebra of the associated root group. Then $d\rho(X_\alpha)E \subset V_S(\eta\lambda_\alpha)$. Since $\eta\lambda_\alpha \neq \eta$, this latter $S$-weight space is of dimension at most 1. Hence $K_\alpha := \ker((d\rho(X_\alpha))\,|_E)$ is of dimension at least $k-1$. Setting $K_1 = \cap_{\alpha \in \pm\Pi} K_\alpha$, we see that $K_1 \subset V$ is a proper $\mathrm{Lie}(G)$-submodule on which $\mathrm{Lie}(G)$ acts trivially. But by Curtis (1960), $V$ is an irreducible $\mathrm{Lie}(G)$-module, and so $K_1 = \{0\}$. Therefore, $k = \dim E \leq 2n$, where $n$ is the rank of $G$. We can now show that $\mu = 0$; for otherwise the $W$-orbit of $\mu$ is of length at least $n+1$ (the exact values are in A. E. Zalesski (2009, Table 1)). Therefore, $\dim \sum_{w \in W} M_{w(\mu)} \geq 2(n+1)$, which is a contradiction.

Consider now the case where there exists $\alpha \in \Phi$ such that $\lambda_\alpha = 1_S$. Set $M' := \sum_{w \in W} M_{w(\mu)}$, so that $M' \subseteq E$. Let $R_0 = \{\alpha \in \Phi : \lambda_\alpha = 1_S\}$, $R_2 = \Phi \setminus R_0$. Since $S$ is non-central, $R_0 \neq \Phi$ and $R_2 \neq \varnothing$. Let $R_1$ be the set of roots $\alpha$ such that $\dim(d\rho(X_\alpha)M') \leq 1$.

By the considerations of the first case above, $R_2 \subseteq R_1$. Moreover, we claim that $R_1$ is $W$-stable. Indeed for $w \in W$, choose $\dot{w} \in N_G(T)$ such that $w = \dot{w}T$. Then

$$\rho(\dot{w})d\rho(X_\alpha)M' = \rho(\dot{w})d\rho(X_\alpha)\rho(\dot{w})^{-1}\rho(\dot{w})M' = \mathrm{Ad}(\rho(\dot{w}))(d\rho(X_\alpha))M'.$$

By the remarks preceding the statement of the result, this latter is equal to

$$d\rho(\mathrm{Ad}(\dot{w})X_\alpha)M' = d\rho(X_{w(\alpha)})M'$$

and since $\dim(\rho(\dot{w})(d\rho(X_\alpha)M')) = \dim(d\rho(X_\alpha)M')$, we have the claim. Now, if all roots of $\Phi$ are of the same length then $R_1 = \Phi$, and we conclude as in the first case.

Hence we may assume that $\Phi$ has two root lengths and that the roots of $R_1$ are of a single length. Note that $R_0 = -R_0$ and $\beta, \gamma \in R_0$ implies $\beta + \gamma \in R_0$ provided $\beta + \gamma$ is a root. This implies (see for example Malle and D. Testerman (2011, B.14)) that $R_0$ is a root system, that is, $R_0$ is a closed subsystem of $\Phi$. Moreover, $R_0$ is of maximal rank (equal to the rank of $\Phi$) as otherwise, by Malle and D. Testerman (2011, B.18), $R_0$ lies in some subsystem corresponding to a proper subset of $\Pi$, in which case $R_2$, and so $R_1$ has roots of both lengths. So $R_0$ is a subsystem of maximal rank, and by the classification of such, Malle and D. Testerman (2011, B.18), one checks that in every case $\Phi \setminus R_0 = R_2$ again contains roots of both lengths and we conclude as above. □

**Remark 1 –** If $\omega = p^k\omega'$, with $\omega'$ $p$-restricted, then the weights of $V_\omega$ are $p^k\mu$ for $\mu$ a weight of $V_{\omega'}$. Then $p^k\mu(s) = \mu(s^{p^k})$. As the mapping $x \mapsto x^p$ for $x \in F$ is bijective on $F$, the spectrum of $s$ on $V_\omega$ is almost simple if and only if the spectrum of $s$ on $V_{\omega'}$ is almost simple.

We now take $S$ to be generated by a single element $s \in T$ and consider the case of tensor-decomposable irreducible representations.

**Lemma 7 –** *Let $s \in T$ be a non-central element. Let $\omega$ be a dominant weight which is not p-restricted and not of the form $p^k\mu$ for $\mu$ a p-restricted weight. Suppose that the spectrum of s on $V_\omega$ is almost simple. Then all weights of $V_\omega$ are of multiplicity* 1.

*Proof.* By Steinberg's tensor product theorem, $V_\omega = V_{p^{k_1}\mu_1} \otimes V_{p^{k_2}\mu_2} \otimes \cdots \otimes V_{p^{k_t}\mu_t}$, where $t > 1$ and $\mu_1, \ldots, \mu_k$ are non-zero $p$-restricted weights and $(k_1, \ldots, k_t)$ are distinct non-negative integers. Then Lemma 3 implies that the spectrum of $s$ on each tensor factor is simple so the weights of each tensor factor have multiplicity 1. Furthermore, Zalesskii and Suprunenko (1987, Proposition 2) implies that the weights of $V_\omega$ are of multiplicity 1 unless there exists $1 \le j < t$ such that $k_{j+1} = k_j + 1$ and one of the following holds:

(i) $G = C_n$, $p = 2$, $\mu_j = \omega_n$, $\mu_{j+1} = \omega_1$;

(ii) $G = G_2$, $p = 2$, $\mu_j = \omega_1$, $\mu_{j+1} = \omega_1$;

(iii) $G = G_2$, $p = 3$, $\mu_j = \omega_2$, $\mu_{j+1} = \omega_1$.

Moreover, in each of the cases (i), (ii) and (iii), the module $V_{\mu_j} \otimes V_{p\mu_{j+1}}$ has a weight of multiplicity greater than 1. Hence if one of the three cases occurs, we deduce that $t = 2$ and so we can also assume that $j = 1$ and $k_1 = 0$, that is, $V_\omega = V_{\mu_1} \otimes V_{p\mu_2}$. We consider the above cases in detail.

Case (i): Take $T$ to be the set of diagonal matrices in the image of the natural representation of $G$. Here $\Omega(V_{\omega_n}) = \{\pm\varepsilon_1 \pm \cdots \pm \varepsilon_n\}$ and $\Omega(V_{2\omega_1}) = \{\pm 2\varepsilon_1, \ldots, \pm 2\varepsilon_n\}$. (As usual, we have adopted the notation of Bourbaki (1968, Planche III).) Let $\nu$ be a weight of $V_{\omega_n}$ with positive signs of both $\varepsilon_i$ and $\varepsilon_j$, for some $1 \leq i, j \leq n$, $i \neq j$. As $\nu - 2\varepsilon_i$ and $\nu - 2\varepsilon_j$ are weights of $V_{\omega_n}$, it follows that $\nu$ is also a weight of $V_\omega$ with multiplicity at least 2. This remains true for weights where both $\varepsilon_i$ and $\varepsilon_j$ have coefficient $-1$ or have opposite coefficients. It follows that the restriction of $V_{\omega_n + 2\omega_1}$ to $T$ contains a direct sum of at least two copies of $V_{\omega_n}|_T$. Therefore, every eigenvalue of $s$ on $V_{\omega_n}$ is also an eigenvalue of $s$ on $V_{\omega_n+2\omega_1}$, and occurs with multiplicity at least 2. So this case is ruled out as the spectrum of $s$ on $V_{\omega_n}$ is simple.

Case (ii): Here the weights of $V_{\omega_1}$ are the short roots of $\Phi$, and the following weights occur with multiplicity 2 in $V_\omega$: $3\alpha_1 + \alpha_2$, $3\alpha_1 + 2\alpha_2$. Since the spectrum of $s$ on $V_\omega$ is almost simple, these roots must all take equal value on $s$. In particular, $\alpha_2(s) = 1$. But now the eigenvalue $5\alpha_1(s)$ occurs with multiplicity 2 as well as $3\alpha_1(s)$, implying that $\alpha_1(s) = 1$ as well, contradicting the fact that $s$ is non-central.

Case (iii): This case is similar. Here the weights of $V_{\omega_2}$ are the long roots of $\Phi$ and the zero weight, and the weights of $V_{\omega_1}$ are the short roots and the zero weight. We find that each of the weights $3\alpha_1 + \alpha_2$ and $\alpha_2$ occur with multiplicity 2, and deduce that $\alpha_1(s) = 1$. But now the eigenvalue $\alpha_2(s)$ occurs with multiplicity greater than 1, as well as the eigenvalue 1, and so $\alpha_2(s) = 1$ as well, again contradicting $s$ non-central. □

*Proof (of Theorem 1).* Using Lemma 2, we see that assertion (1) follows from assertion (2). We apply Theorem 5, Remark 1 and Lemma 7 to obtain the reverse implication. □

# 2 Commuting subgroups and a partial proof of Theorem 2

An essential element of our proof of Theorem 2 is an application of Theorem 4, which allows us to treat many of the groups and representations in a uniform way. (See Proposition 2 below.) Let $s \in G$ be a non-regular semisimple element. In order to apply Theorem 4, we need to find a pair of subsystem subgroups $K, Y$ such that $[K, Y] = 1$, $[K, s] = 1$ and $[s, Y] \neq 1$. For technical reasons, it will suffice to do this for groups other than $B_n, D_n$, and $G_2$.

**Lemma 8 –** *Let $G = \mathrm{SL}_n(F)$, $n > 3$, and let $s \in T \setminus Z(G)$ be a non-regular element. Then there are simple subsystem subgroups $K, Y$, normalized by $T$, such that $[K, Y] = 1$,*

$[K,s]=1$ *and* $[s,Y]\neq 1$, *unless* $n=4$ *and, up to conjugacy in* $G$, $s=\mathrm{diag}(a,a,a^{-1},a^{-1})$ *or* $s=\mathrm{diag}(a,a,-a^{-1},-a^{-1})$, *for some* $a\in F^{\times}$.

*Proof.* We take $T$ to be the torus of diagonal matrices in $G$. As $s$ is non-regular and non-central, we may assume that $s=\mathrm{diag}(b,b,a_3,\dots,a_n)$, where $a_3\neq b$. Suppose first that $a_3\neq a_i$ for some $i>3$. Set $K=\mathrm{diag}(\mathrm{SL}_2(F),\mathrm{Id}_{n-2})$, $Y=\mathrm{diag}(\mathrm{Id}_2,\mathrm{SL}_{n-2}(F))$. Next, suppose $a_3=\cdots=a_n$. If $n>4$ then we can take $Y=\mathrm{diag}(1,\mathrm{SL}_2(F),\mathrm{Id}_{n-3})$ and $K=\mathrm{diag}(\mathrm{Id}_{n-2},\mathrm{SL}_2(F))$. If $n=4$, then $s=\mathrm{diag}(b,b,a,a)$ and $b^2a^2=1$, whence $b=\pm a^{-1}$. □

**Remark 2 –** If $G=\mathrm{SL}_4(F)$, and $s=\mathrm{diag}(\lambda,\lambda,\lambda^{-1},\lambda^{-1})$ or $s=\mathrm{diag}(\lambda,\lambda,-\lambda^{-1},-\lambda^{-1})$, for $\lambda\in F$, $\lambda^4\neq 1$, then $s$ is non-regular, non-central, and it is impossible to find a pair of commuting subsystem subgroups $K$, $Y$ such that $[s,K]=1$ and $[s,Y]\neq 1$. Moreover, the Jordan form of $s$ on the exterior square of the natural 4-dimensional module is $\mathrm{diag}(\lambda^2,\lambda^{-2},1,1,1,1)$, which is non-central with almost simple spectrum.

**Lemma 9 –** *Let* $G=C_n$, $n>1$, *and let* $s\in T\setminus Z(G)$ *be a non-regular element. Then there are simple subsystem subgroups* $K$, $Y$ *of* $G$, *normalized by* $T$, *such that* $[K,Y]=1$, $[K,s]=1$ *and* $[s,Y]\neq 1$, *unless* $n=2$ *and with respect to an ordered symplectic basis* $(e_1,f_1,e_2,f_2)$ *of* $V_{\omega_1}$, *the Jordan form of* $s$ *on the natural* $G$*-module is either* $\mathrm{diag}(a,a^{-1},a,a^{-1})$, *for* $\pm 1\neq a\in F$, *or* $s=\pm\,\mathrm{diag}(1,1,-1,-1)$, *for* $p\neq 2$.

*Proof.* The group $G=C_n=\mathrm{Sp}_{2n}(F)$ contains a maximal rank subsystem subgroup $H$ isomorphic to $\mathrm{Sp}_2(F)\times\cdots\times\mathrm{Sp}_2(F)$, so every semisimple element is conjugate to an element of $H$. Therefore, we can write the matrix of $s$ with respect to a suitable basis of the natural $G$-module $V_{\omega_1}$ as $\mathrm{diag}(a_1,a_1^{-1},\dots,a_n,a_n^{-1})$ for some $a_1,\dots,a_n\in F$. By Lemma 6, the diagonal entries of $s$ are not distinct. Hence either $a_i=\pm 1$ for some $i\in\{1,\dots,n\}$, or, replacing some $a_i$ by $a_i^{-1}$, we can assume that $a_i=a_j$ for some $1\leq i<j\leq n$.

Suppose first that $a_i=\pm 1$ for some $i\in\{1,\dots,n\}$ and assume without loss of generality that $i=1$. If there exists $j$ such that $a_j\neq\pm 1$, we can assume $j=n$ and then take $K=\mathrm{diag}(\mathrm{Sp}_2(F),\mathrm{Id}_{2n-2})$, $Y=\mathrm{diag}(\mathrm{Id}_{2n-2},\mathrm{Sp}_2(F))$. Otherwise, $s^2=1$ and $p\neq 2$. We can reorder $a_1,\dots,a_n$ so that $a_1\neq a_2$, and if $n>2$ we take $Y=\mathrm{diag}(\mathrm{Sp}_4(F),\mathrm{Id}_{2n-4})$, $K=\mathrm{diag}(\mathrm{Id}_{2n-2},\mathrm{Sp}_2(F))$. If $n=2$, $s^2=1$ and $p\neq 2$, such a choice is not possible and we have $s$ as in the final statement.

Now suppose that $a_i\neq\pm 1$ for all $i\in\{1,\dots,n\}$, so there exists $1\leq i<j\leq n$ such that $a_i=a_j$. In this case, there exists a 2-dimensional totally isotropic subspace of the underlying $2n$-dimensional symplectic space on which $s$ acts as scalar multiplication. If $n>2$, then $s$ is contained in a Levi subgroup $L=L_1\times L_2$ of $G$, where $L_1\cong\mathrm{GL}_2(F)$ and $L_2\cong\mathrm{Sp}_{2n-4}(F)$. Moreover $[s,L_1]=1$, so we can take $K=L_1$, $Y=L_2$. If $n=2$ then $s=\mathrm{diag}(a,a^{-1},a,a^{-1})$ as in the statement of the result. □

**Lemma 10 –** *Let* $G\in\{E_6,E_7,E_8,F_4\}$. *Let* $s\in T\setminus Z(G)$ *be a non-regular element. Then there exist simple subsystem subgroups* $K$, $Y$, *normalized by* $T$, *such that* $K$ *is of type* $A_1$, $[K,Y]=1$, $[K,s]=1$, $[s,Y]\neq 1$.

*Proof.* As $s$ is not regular, $C_G(s)$ contains root subgroups $U_{\pm\alpha}$ for some root $\alpha \in \Phi$. Clearly, we can assume $\alpha$ to be a simple root. Moreover, we can assume that $\alpha = \alpha_1$ if $G \neq F_4$, otherwise, that $\alpha = \alpha_1$ or $\alpha_4$.

Denote by $R_\alpha$ the set of roots orthogonal to $\alpha$, and observe that $R_\alpha$ is not empty. Set $Y = \langle U_{\pm\beta} : \beta \in R_\alpha \rangle$ and $K = \langle U_{\pm\alpha} \rangle$. Then $[Y,K] = 1$ and $[K,s] = 1$. If $[Y,s] \neq 1$, replacing $Y$ by a suitable simple subgroup of $Y$, we are done.

We now assume $[s,U_\beta] = 1$ for all $\beta \in R_\alpha$. In this situation, as $s$ is non-central, $[s,U_\gamma] \neq 1$ for some simple root adjacent to $\alpha$ in the Dynkin diagram. Moreover, the Dynkin diagram of the above groups contains a node $\beta$, not adjacent to each of $\alpha, \gamma$. In particular, $\beta \in R_\alpha$ and so $[s,U_\beta] = 1$, while $[s,U_\gamma] \neq 1$. So now we can take $K = \langle U_{\pm\beta} \rangle$ and $Y = \langle U_{\pm\gamma} \rangle$.

This completes the proof. □

We now apply the previous three lemmas and Theorem 4 to establish Theorem 2 for certain groups.

**Proposition 2 –** *Let $G$ be of type $A_n$ for $n > 3$, $C_n$ for $n > 2$, or of type $F_4, E_6, E_7$, or $E_8$. Let $V$ be a non-trivial irreducible $G$-module and $s \in T \setminus Z(G)$. Suppose that the spectrum of $s$ on $V$ is almost simple. Then one of the following holds:*

*(1) $s$ is regular,*

*(2) $G = C_n$ with $p = 2$ and the highest weight of $V$ is $2^m\omega_n$, or*

*(3) $G$ is classical and $V$ is a Frobenius twist of the natural or the dual of the natural module for $G$.*

*Proof.* Suppose that $s$ is not regular. By Lemma 8 for $A_n$, Lemma 9 for $C_n$, and Lemma 10 for the other groups in the statement, there are simple subsystem subgroups $K$, $Y$, normalized by $T$, such that $[K,Y] = 1$, $[K,s] = 1$ and $[Y,s] \neq 1$. Then we apply Theorem 4 to $K$, $Y$ in place of $G(R_1), G(R_2)$ to conclude that either (2) or (3) holds or there is a $KY$-composition factor $M$ of $V$ afforded by an irreducible representation $\tau$ of $KY$, such that $\tau$ is non-trivial on both $K$ and $Y$. So we assume neither (2) nor (3) holds, so we are in the latter situation, and aim for a contradiction.

We first note that $TY = Y \cdot Z(TY)$, as $Y$ is simple. Therefore, as $s \in T$, $s = s_1 s_Y$ for some $s_1 \in Z(TY) \subset T$ and $s_Y \in (T \cap Y)$. As $[s,K] = 1$ and $[Y,K] = 1$, we have $[s_1,K] = 1$ and $[s_1,YK] = 1$. Also, as $[s,Y] \neq 1$, we have $[s_Y,Y] \neq 1$.

Now $M$ is a direct sum of eigenspaces for $s_1$. It follows that $\tau$ is realized in one of the $s_1$-eigenspaces $M_1$, say, and hence the spectrum of $s$ on $M_1$ is almost simple if and only if that of $s_Y$ on $M_1$ is almost simple. Therefore, it suffices to show that the spectrum of $\tau(s_Y)$ is not almost simple.

Now $\tau = \tau_K \otimes \tau_Y$, where $\tau_K$, $\tau_Y$ are non-trivial irreducible representations of $K$, $Y$, respectively. As $[s_Y,Y] \neq 1$, there are at least two distinct $s_Y$-eigenspaces on the representation space corresponding to $\tau_Y$, each of them is of dimension at least 2 as $\tau_K(K)$ acts on each eigenspace and all $\tau_K(K)$ composition factors of $M$ are of dimension strictly greater than 1. Hence, the spectrum of $s_Y$ on $M$ is not almost simple, giving the desired contradiction. □

**Remark 3 –** (1) Let $G = C_2$, $p$ odd. If $s$ is not as described in the exceptional cases of Lemma 9, then the conclusion of Proposition 2 remains valid.

(2) Note that the irreducible representation of $G = C_2$ with highest weight $\omega_2$ induces an isomorphism between $\mathrm{PSp}_4(F)$ and $\mathrm{SO}_5(F)$, and the element $s = \pm\mathrm{diag}(1,1,-1,-1)$ in Lemma 9 acts as $\mathrm{diag}(1,-1,-1,-1,-1)$, hence has almost simple spectrum. Similarly, the element $s = \mathrm{diag}(a,a^{-1},a,a^{-1})$ acts as $\mathrm{diag}(a^2,1,1,1,a^{-2})$, which has almost simple spectrum provided $a^2 \neq \pm 1$.

(3) In view of Lemma 3 and Proposition 2, to complete the proof of Theorem 2, it remains to consider $p$-restricted representations (of highest weight $\lambda$) of the groups $B_n$ for $n > 2$, $D_n$ for $n > 3$, $C_n$ for $p = 2$ and $\lambda = \omega_n$, and the small rank groups $A_2$, $A_3$, $C_2$, and $G_2$. We will handle the small rank groups in Section 4.1 and complete the proof in Section 4.2 by dealing with the remaining groups.

# 3 Weight levels

Recall we have $\Omega = \sum_{i=1}^n \mathbb{Z}\omega_i$, the weight lattice associated with $\Phi$, and $\Omega^+$ the set of dominant weights in $\Omega$. In this section we establish some results on $\Omega$ in view of applying the results in Section 2. Recall that a weight is radical if it is an integral linear combination of roots. The irreducible $G$-module whose highest weight is the maximal height short root is called the short root module. If all weights are of the same length then any root is regarded as short, and the short root module is $V_{\omega_a}$.

**Definition 3 –** Let

$$\Lambda_1 = \{\mu \in \Omega^+ \mid \text{ if } \nu \preceq \mu \text{ for some } \nu \in \Omega^+ \text{ then } \mu = \nu\ \}.$$

For $i > 1$, let

$$\Lambda_i = \{\mu \in \Omega^+ , \mu \notin \Lambda_1 \cup \cdots \cup \Lambda_{i-1} \mid \text{if } \nu \prec \mu \text{ for some } \nu \in \Omega^+ \text{ then } \nu \in \Lambda_1 \cup \cdots \cup \Lambda_{i-1}\}.$$

The elements of $\Lambda_i$ are called *weights of level $i$*.

**Lemma 11 –** *Assume $p = 0$ or $p > e(G)$. Let $\omega \neq 0$ be a $p$-restricted dominant weight for $G$. If $\omega \notin \Lambda_1 \cup \cdots \cup \Lambda_i$ for some $i > 0$, then there are weights $\nu_1, \dots, \nu_i$ of $V_\omega$ such that $\nu_j \in \Lambda_j$ for $j = 1, \dots, i$. In addition, the weights of $V_{\nu_j}$ occur as weights of $V_\omega$, for $1 \leq j \leq i$.*

*Proof.* This follows from the definition of $\Lambda_j$ and Lemma 4. □

We conclude this section with some precise information about weights of level 1 or 2, and radical weights of level 3, for certain root systems.

**Lemma 12 –** *The sets $\Lambda_1$ and $\Lambda_2$ for the root systems of types $A_n$, $B_n$, $C_n$ and $D_n$ are as in Table 1. In addition, we have*

*(1) for $\Phi = B_n$, $n > 2$, $\omega_2$ is the only radical weight in $\Lambda_3$;*

*(2) for $\Phi = C_n$, $n > 3$, $2\omega_1, \omega_4$ are the only radical weights in $\Lambda_3$;*

*(3) for $\Phi = C_2$ or $C_3$, $2\omega_1$ is the only radical weight in $\Lambda_3$.*

| $\Phi$ | $\Lambda_1$ | $\Lambda_2$ |
|---|---|---|
| $A_n, n \geq 1$ | $0, \omega_1, \dots, \omega_n$ | $2\omega_1, 2\omega_n, \omega_1 + \omega_n, \omega_1 + \omega_i, \omega_i + \omega_n, i = 2, \dots, n-1$ |
| $B_n, n \geq 3$ | $0, \omega_n$ | $\omega_1, \omega_1 + \omega_n$ |
| $C_n, n > 2$ | $0, \omega_1$ | $\omega_2, \omega_3$ |
| $C_2$ | $0, \omega_1$ | $\omega_2, \omega_1 + \omega_2$ |
| $D_n, n > 4$ | $0, \omega_1, \omega_{n-1}, \omega_n$ | $\omega_2, \omega_3, \omega_1 + \omega_{n-1}, \omega_1 + \omega_n$ |
| $D_4$ | $0, \omega_1, \omega_3, \omega_4$ | $\omega_2, \omega_1 + \omega_3, \omega_1 + \omega_4, \omega_3 + \omega_4$ |

Table 1 – Weights in $\Lambda_1$, $\Lambda_2$

*Proof.* By Lemma 4(3), $\Lambda_1$ consists of minuscule weights and the weight 0, justifying the entries in the column headed $\Lambda_1$ of the table. Furthermore, $\Lambda_2$ contains a unique radical weight, which is the maximal short root (see for instance Suprunenko and A. E. Zalesski (2007, Proposition 10)).

Let now $\omega = \sum a_i \omega_i \in \Lambda_2$ be a non-radical weight. Suppose that $a_i \geq 2$ for some $i$. Then $\omega' = \omega - \alpha_i \in \Omega^+$, so $\omega' \in \Lambda_1$. Inspecting $\Lambda_1$ and the expressions of simple roots in terms of fundamental dominant weights, we observe that $\omega' + \alpha_i$ (for $\omega' \in \Lambda_1$) is dominant only if $\Phi$ is of type $A_n$ and $\omega \in \{2\omega_1, 2\omega_n\}$; furthermore, it is straightforward to see that in this latter case, we have $2\omega_1, 2\omega_n \in \Lambda_2$. So we can assume that $a_i \leq 1$ for all $i$. Next we proceed case-by-case, still assuming $\omega \in \Lambda_2$ a non-radical weight.

Consider first the case where $\Phi = A_n$. If $n = 1, 2$ then the result is clear, so assume now $n > 2$. Note that $\omega_i + \omega_j \succ \omega_{i-1} + \omega_{j+1}$ for $1 \leq i < j \leq n$ as $\omega_i + \omega_j - \omega_{i-1} - \omega_{j+1} = \alpha_i + \cdots + \alpha_j$. (Here $\omega_0$ and $\omega_{n+1}$ are understood to be zero.) So if $a_i, a_j \neq 0$ for some $i \neq j$, then $\omega = \omega' + \omega_{i-1} + \omega_{j+1}$ with $\omega' \in \Lambda_1$. Using the same reasoning for different pairs of non-zero coefficients, we see that either $i = 1$ and $\omega' = \omega_1$ or $j = n$ and $\omega' = \omega_n$. Finally, one observes that no weight obtained is subdominant to another one. So $\Lambda_2$ is as in the table. This completes the consideration of $\Phi = A_n$.

For $\Phi \neq A_n$, the argument differs, as some fundamental dominant weights are radical. Recall that $\omega = \sum a_i\omega_i \in \Lambda_2$ is a non-radical weight and we have seen that $a_i \leq 1$ for all $i$. If $\omega_i$ is a radical weight and $a_i > 0$, then $\omega - \omega_i$ is subdominant to the weight $\omega$, and hence $0 \neq \omega - \omega_i \in \Lambda_1$. So $\omega = \nu + \omega_i$, for some $\nu \in \Lambda_1$, $\nu \neq 0$. Moreover, $\omega_i = \mu$, where $\mu$ is the maximal height short root, as otherwise $\nu + \mu$ is subdominant to $\omega$ and $\omega \notin \Lambda_2$. So either $\omega = \nu + \mu$, for some $\nu \in \Lambda_1$, or $a_i = 0$ for all $i$ such that $\omega_i$ is radical. For each root system, we determine when $\nu + \mu$ lies in $\Lambda_2$.

Consider the case $\Phi = B_n$, $n \geq 3$. Following the notation of the previous paragraph, we have $\nu = \omega_n$, $\mu = \omega_1$. Moreover, $\omega_i$ is radical for every $i < n$. So $\omega \in \Lambda_2$ non-radical implies that $\omega = \omega_1 + \omega_n$. It is straightforward to verify that $\omega_1 + \omega_n \in \Lambda_2$. We deduce that $\Lambda_2 = \{\omega_1, \omega_1 + \omega_n\}$. For the claim of (1), let $\omega \in \Lambda_3$ be a radical weight. If $a_i \geq 2$ for some $i$, then $\omega - \alpha_i$ is a radical dominant weight which must lie in $\Lambda_2$. We deduce that $\omega - \alpha_i = \omega_1$ and we find that $n = 2$, contradicting our hypothesis; so we may now assume $a_i \leq 1$ for all $i$. In particular, as $\omega$ is radical, $a_n = 0$. In addition, $\omega_i = \omega_{i-1} + \alpha_i + \cdots + \alpha_n$, see Bourbaki (1968, Planche II), i.e. $\omega_{i-1} \prec \omega_i$. So $\omega \in \Lambda_3$ then implies that $\omega = \omega_2$.

Consider now the case $\Phi = C_n$, for $n \geq 2$. If $a_i \neq 0$ or some $i$ such that $\omega_i$ is radical (as above), we find that $\nu = \omega_1$, $\mu = \omega_2$. In this case $\mu + \nu = \omega_1 + \omega_2$. But $\omega_1 + \omega_2 - \alpha_1 - \alpha_2$ is subdominant to $\omega$ and lies in $\Lambda_1$ only if $n = 2$. We may now assume $a_i = 0$ if $\omega_i$ is radical, so $a_i = 0$ for $i$ even. Also by the preliminary remarks, $a_i \leq 1$ for all $i$. It is easy to observe that $\omega_i \succ \omega_{i-2}$ for $i > 1$, which implies the result on $\Lambda_2$. We now turn to the claims of (2) and (3), so let $\omega \in \Lambda_3$ be a radical weight. If $a_i \geq 2$ for some $i$, then $\omega - \alpha_i \in \Lambda_2$ if only if $\omega = 2\omega_1$. So we now assume $a_i \leq 1$ for all $i$. Let $1 \leq i \leq n$ be maximal such that $a_i = 1$. Since the dominant weight $\omega - \omega_i + \omega_{i-2} \prec \omega$ must lie in $\Lambda_1 \cup \Lambda_2$ and is a radical weight, we find that $n \geq 4$ and $\omega = \omega_4$. Finally, one checks that $\omega_4$ lies in $\Lambda_3$.

Finally consider the case $\Phi = D_n$, $n \geq 4$. Here, in the case where $a_i \neq 0$ for some $i$ with $\omega_i$ radical, we have (in the previously defined notation) $\nu \in \{\omega_1, \omega_{n-1}, \omega_n\}$ and $\mu = \omega_2$, so $\mu + \nu \in \{\omega_1 + \omega_2, \omega_2 + \omega_{n-1}, \omega_2 + \omega_n\}$. Now $\omega_2 + \omega_n \succ \omega_1 + \omega_{n-1} \notin \Lambda_1$ and $\omega_2 + \omega_{n-1} \succ \omega_1 + \omega_n \notin \Lambda_1$ so $\omega_2 + \omega_n, \omega_2 + \omega_{n-1} \notin \Lambda_2$. Furthermore, as $\omega_1 + \omega_2 - \alpha_1 - \alpha_2 = \omega_3 + \delta_{n,4}\omega_4$, it follows that $\omega_1 + \omega_2 \notin \Lambda_2$. So we now assume that $a_i = 0$ for all $i$ such that $\omega_i$ is radical, that is, $a_i = 0$ if $i < n-1$ is even and as established earlier $a_j \leq 1$ for all $j$. Moreover, there are at most two $a_j$ which are non-zero, as otherwise there exists $\beta \in \Phi$ with $\omega - \beta$ dominant and not lying in $\Lambda_1$. Suppose $a_i = 1$ for some (odd) $i < n-1$. Then $\omega - (\omega_i - \omega_{i-2}) \prec \omega$ must lie in $\Lambda_1$ and so $i = 3$. So finally, recalling that $\omega$ is non-radical we have $\omega \in \{\omega_3 (n > 4), \omega_3 + \omega_{n-1} (n > 4), \omega_3 + \omega_n (n > 4), \omega_1 + \omega_n, \omega_1 + \omega_{n-1}, \omega_{n-1} + \omega_n\}$. It is straightforward to see that $\omega_3 (n > 4), \omega_1 + \omega_{n-1}$ and $\omega_1 + \omega_n$ all lie in $\Lambda_2$. In addition, $\omega_{n-1} + \omega_n \succ \omega_{n-3}$, and the latter lies in $\Lambda_1$ if and only if $n = 4$. So it remains to show that $\omega_3 + \omega_n, \omega_3 + \omega_{n-1} \notin \Lambda_2$ for $n > 4$. This is clear since $\omega_2 + \omega_{n-1}$, respectively $\omega_2 + \omega_n$, is subdominant to the given weight and does not lie in $\Lambda_1$. □

# 4 Proof of Theorem 2

In this section, we prove Theorem 2, so in particular we are concerned with the action of non-central non-regular semisimple elements on certain specific representations (as shown by Theorem 1). As noted earlier, in remark 3(3), we must handle some small rank groups as well as the groups $B_n$, $D_n$, and $C_n$ when $p = 2$ and for certain highest weights; we do this in two separate subsections.

## 4.1 Groups of small rank

**Lemma 13 –** *Let $G = A_2$ and let $s \in T \setminus Z(G)$ be a non-regular element. Let $V = V_\omega$ be the irreducible G-module of p-restricted highest weight $\omega \neq 0$. Then the spectrum of s on $V_\omega$ is almost simple if and only if $\omega = \omega_1$ or $\omega_2$.*

*Proof.* We take $T$ to be the torus of diagonal matrices in $\mathrm{SL}_3(F)$. Since $s$ is non-regular non-central, with respect to an appropriate choice of basis of $V_{\omega_1}$, we may assume $s = \mathrm{diag}(a, a, a^{-2})$, for some $a \in F^\times$ with $a^3 \neq 1$. Clearly the spectrum of $s$ on $V_{\omega_1}$ and $V_{\omega_2}$ is indeed almost simple. So we now assume $\omega \notin \{0, \omega_1, \omega_2\}$. In particular, Lemma 12 implies $\omega \notin \Lambda_1$ and by Lemmas 11 and 12, $\Omega(V_\omega)$ has some weight from $\Lambda_2 = \{\omega_1 + \omega_2, 2\omega_1, 2\omega_2\}$.

Suppose first that $\omega = 2\omega_1$, and so $p \neq 2$. The weights of $V_{\omega_1}$ are $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$, so the weights of $V_{\omega_1} \otimes V_{\omega_1}$ are $2\varepsilon_1, 2\varepsilon_2, 2\varepsilon_3, \varepsilon_1+\varepsilon_2, \varepsilon_1+\varepsilon_3, \varepsilon_2+\varepsilon_3$, which by Lemma 4(2) coincide with the weights of $V_\omega$. Now, $2\varepsilon_1(s) = 2\varepsilon_2(s) = a^2$, and $(\varepsilon_1 + \varepsilon_3)(s) = (\varepsilon_2 + \varepsilon_3)(s) = a^{-1}$. As $a^3 \neq 1$, the eigenvalues $a^2, a^{-1}$ are distinct, so the spectrum of $s$ on $V_{2\omega_1}$ is not almost simple, as claimed. Since $V_{2\omega_2}$ is dual to $V_{2\omega_1}$, the spectrum of $s$ on $V_{2\omega_2}$ is not almost simple as well.

Suppose now that $\omega = \omega_1 + \omega_2$. Then the weights of $V_\omega$ are the roots and the zero weight. Then $(\alpha_1+\alpha_2)(s) = \alpha_2(s) = (\varepsilon_2-\varepsilon_3)(s) = a^3 \neq 1$ and $-(\alpha_1+\alpha_2)(s) = -\alpha_2(s) = a^{-3}$. If $p \neq 3$, the eigenvalue 1 is also of multiplicity 2, and we are done. If $p = 3$ and $a^3 \neq a^{-3}$ then we are done as well. So suppose $p = 3$ and $a^6 = 1$ and hence $a^3 = -1$, that is $a = -1$. Note that $\pm\alpha_2(s) = -1$ and $\pm\alpha_1(s) = 1$, so the result also follows in this case.

We now appeal to Lemma 11 to conclude. □

**Lemma 14 –** *Let $G = A_3$ and let $s \in T \setminus Z(G)$ be a non-regular element. Let $V = V_\omega$ be the irreducible G-module of p-restricted highest weight $\omega \neq 0$. If the spectrum of s on V is almost simple, then either $\omega = \omega_1$ or $\omega_3$, or $\omega = \omega_2$ and there exists $a \in F^\times$, $a^4 \neq 1$ such that with respect to a suitably chosen basis, $s = \mathrm{diag}(a, a, \pm a^{-1}, \pm a^{-1})$.*

*Proof.* Without loss of generality, we take $T$ to be the set of diagonal matrices in $\mathrm{SL}_4(F)$. We may assume $s = \mathrm{diag}(a, a, b, c)$ for some $a, b, c \in F$ such that $a^2bc = 1$. Fix the base of $\Phi$ such that $\alpha_i(\mathrm{diag}(a_1, a_2, a_3, a_4)) = a_i a_{i+1}^{-1}$ for $1 \leq i \leq 3$; in particular $\alpha_1(s) = 1$. It is clear that if $a^2b^2 \neq 1$ then $s$ has almost simple spectrum on $V_{\omega_1}$ and on $V_{\omega_3}$. If $\omega = \omega_2$, then the matrix of $s$ on $V$ is conjugate to $s_1 = \mathrm{diag}(a^2, a^{-2}, ab, ab, (ab)^{-1}, (ab)^{-1})$, so the spectrum of $s$ is almost simple only if $b = \pm a^{-1}$ and $a^4 \neq 1$, and the result easily follows.

Now consider the general case, where $\omega \notin \{\omega_1, \omega_2, \omega_3\}$. Assume $s$ has almost simple spectrum on $V_\omega$. Factor $s$ as

$$s = \mathrm{diag}(a\gamma, a\gamma, a^{-2}\gamma^{-2}, 1) \cdot \mathrm{diag}(\gamma^{-1}, \gamma^{-1}, \gamma^{-1}, c),$$

where $\gamma, c \in F^\times$ with $\gamma^3 = c$. Then viewing $s$ as lying in the maximal parabolic $P = LQ$, $Q = R_u(P)$, corresponding to the root $\alpha_3$, we see that the second factor acts as a scalar on the fixed point space $V_\omega^Q$. Hence the eigenvalue multiplicities of $s$ on this fixed point space are determined by those of the first factor. We now apply Lemma 13 to the element $h = \mathrm{diag}(\gamma a, \gamma a, (\gamma a)^{-2})$ and the weight $\omega \downarrow L'$, which is the highest weight of the irreducible $L'$-module $V_\omega^Q$. In addition, we apply Lemma 13 to $(V_\omega^*)^Q$. By Lemma 13, the only $p$-restricted irreducible representations of $\mathrm{SL}_3(F)$ on which $h$ has an almost simple spectrum are the natural representation and its dual. Writing $\omega = m_1\omega_1 + m_2\omega_2 + m_3\omega_3$, we deduce that $(m_1, m_2), (m_2, m_3) \in \{(0,0), (1,0), (0,1)\}$. We are therefore reduced to considering the case $\omega = \omega_1 + \omega_3$, (a quotient of) the adjoint representation. The multiplicity of the weight 0 is at least 2 and $\alpha_1(s) = 1$. Therefore, $(\alpha_1 + \alpha_2)(s) = \alpha_2(s)$, so $\alpha_2(s) = 1$ as well. But then $(\alpha_2 + \alpha_3)(s) = \alpha_3(s) \neq 1$, as $s$ is non-central; hence $s$ is not almost cyclic on $V_{\omega_1+\omega_3}$. □

**Lemma 15 –** *Let $G = C_2$, $p = 2$, and let $\omega$ be a non-zero 2-restricted dominant weight. Let $s \in T$ be a non-regular element. Suppose that the spectrum of $s$ on $V_\omega$ is almost simple. Then $\omega \in \{\omega_1, \omega_2\}$ and the spectrum of $s$ is almost simple on precisely one of the modules $V_{\omega_1}$ and $V_{\omega_2}$. Assume moreover that $T$ is the torus of diagonal matrices in the group $\mathrm{Sp}_4(F)$, written with respect to a fixed symplectic basis $(e_1, e_2, f_2, f_1)$ of the natural module $V_{\omega_1}$. Let $g \in T$ be non-regular. If the spectrum of $g$ on $V_{\omega_1}$ is almost simple then, up to conjugacy, $\varepsilon_1(g) = a$, $\varepsilon_2(g) = 1$ for $1 \neq a \in F^\times$; if the spectrum of $g$ on $V_{\omega_2}$ is almost simple then, up to conjugacy $\varepsilon_1(g) = \varepsilon_2(g) = a$ for $1 \neq a \in F^\times$.*

*Proof.* As $\omega$ is 2-restricted, if $\omega \notin \{\omega_1, \omega_2\}$ then $\omega = \omega_1 + \omega_2$, and Steinberg (2016, §12, Corollary of Theorem 41) implies that $V_\omega = V_{\omega_1} \otimes V_{\omega_2}$. By Lemma 3, the spectrum of $s$ is simple on $V_{\omega_1}$, and hence $s$ is regular, contradicting our hypothesis. One easily verifies the validity of the additional assertions. □

**Lemma 16 –** *Let $G = C_2$, $p \neq 2$, and fix an ordered symplectic basis $(e_1, e_2, f_2, f_1)$ of the natural module of $G$ and let $T$ be the torus of diagonal matrices of $G$ in the natural representation. Let $s \in T \setminus Z(G)$ be a non-regular element and let $V_\omega \in \mathrm{Irr}(G)$ be a non-trivial $p$-restricted $G$-module. Then $s$ has almost simple spectrum on $V_\omega$ if and only if one of the following holds:*

- *(i) $\omega = \omega_1$, and up to conjugacy, $\varepsilon_1(s) = 1$, $\varepsilon_2(s) = a$ or $\varepsilon_1(s) = -1$, $\varepsilon_2(s) = a$, where $a \in F^\times$, $a^2 \neq 1$;*
- *(ii) $\omega = \omega_2$, and up to conjugacy, $\varepsilon_1(s) = 1$, $\varepsilon_2(s) = -1$ or $\varepsilon_1(s) = \varepsilon_2(s) = a$, where $a \in F^\times$, $a^2 \neq \pm 1$.*

*Proof.* Let $\varepsilon_1(s) = b$, $\varepsilon_2(s) = a$, that is $s = \mathrm{diag}(b,a,a^{-1},b^{-1})$.

We first consider $\omega = \omega_1$, so $\Omega(V_\omega) = \{\pm\varepsilon_1, \pm\varepsilon_2\}$. Since $s$ is non-regular, we may assume that either $a = b$ or $b^2 = 1$. In the first case, $s$ does not have almost simple spectrum on $V_\omega$, while in the second case $s$ has almost simple spectrum on $V_\omega$ if and only if $a^2 \neq 1$.

We now turn to the cases $\omega \neq \omega_1$. By Remark 3(1), we are left with the exceptional cases described in Lemma 9, $s_1 = \mathrm{diag}(a,a,a^{-1},a^{-1})$ with $a^2 \neq 1$, or $s_2 = \pm\mathrm{diag}(1,-1,-1,1)$. Note that $\alpha_1(s_1) = 1$ and $\alpha_2(s_2) = 1$. By Lemma 12, $\Lambda_1 = \{0, \omega_1\}$, and $\Lambda_2 = \{\omega_1 + \omega_2, \omega_2\}$ and $2\omega_1$ is the only radical weight in $\Lambda_3$. We consider these weights in turn, before turning to the general case.

The weights of $V_{\omega_2}$ are $0, \pm\varepsilon_1 \pm \varepsilon_2$. The remarks of the preceding paragraph imply that the cases in the statement are the only possible ones, and they yield the matrices of $s_1, s_2$ on $V_{\omega_2}$ (with respect to a suitable basis) $\mathrm{diag}(a^2,1,1,1,a^{-2})$ and $\mathrm{diag}(-1,-1,1,-1,-1)$, respectively.

Suppose $\omega = \omega_1 + \omega_2$. Then $\Omega(V_\omega) = \Omega(V_{\omega_1} \otimes V_{\omega_2})$, by Lemma 4. In terms of Bourbaki weights, the weights in $\Omega(V_\omega)$ are $\pm\varepsilon_1 + (\pm\varepsilon_1 \pm \varepsilon_2)$, $\pm\varepsilon_2 + (\pm\varepsilon_1 \pm \varepsilon_2)$, $\pm\varepsilon_1$, and $\pm\varepsilon_2$. Then $(\pm\varepsilon_1 + (\pm\varepsilon_1 \pm \varepsilon_2))(s_2) = -1$, $(\pm\varepsilon_2 + (\pm\varepsilon_1 \pm \varepsilon_2)(s_2) = 1$, so the spectrum of $s_2$ on $V_\omega$ is not almost simple. Furthermore, $(\varepsilon_1 + (-\varepsilon_1 + \varepsilon_2))(s_1) = a = (\varepsilon_2 + (\varepsilon_1 - \varepsilon_2))(s_1)$ and $(-\varepsilon_1 + (\varepsilon_1 - \varepsilon_2))(s_1) = a^{-1} = (-\varepsilon_2 + (-\varepsilon_1 + \varepsilon_2))(s_1)$. So the spectrum of $s_1$ on $V_\omega$ is not almost simple.

Finally, suppose $\omega = 2\omega_1$. Then by Lemma 4, the weights of $V_\omega$ are the same as those of $V_{\omega_1} \otimes V_{\omega_1}$. These are $\pm\varepsilon_i \pm \varepsilon_j$, for $i, j \in \{1,2\}$. But now it is easy to see that neither $s_1$ nor $s_2$ has almost simple spectrum on $V_\omega$.

We now turn to the general case and suppose that $\omega$ differs from the weights examined above. Then $\omega \notin \Lambda_1 \cup \Lambda_2$ and $\omega \neq 2\omega_1$. Recall that if $\mu \in \Lambda_i$ for some $i$ then $V_\mu$ has a weight from $\Lambda_j$ for every $j = 1, \ldots, i-1$ (Lemma 11). Then Lemma 12 implies that either $2\omega_1$ or $\omega_1 + \omega_2$ is a weight of $V_\omega$ and by Lemma 4, the weights of $V_{2\omega_1}$ or $V_{\omega_1+\omega_2}$ are weights of $V_\omega$. The above considerations of $V_{\omega_1+\omega_2}$ and $V_{2\omega_1}$ show then that, given $s = s_1$ or $s_2$, there are 4 distinct weights $\lambda_1, \lambda_2, \nu_1, \nu_2$ in $\Omega(V_\omega)$ such that $\lambda_1(s) = \lambda_2(s) \neq \nu_1(s) = \nu_2(s)$. So $s$ is not almost cyclic on $V_\omega$, which completes the proof of the result. □

**Lemma 17 –** *Theorem 2 is true for G of type $G_2$.*

*Proof.* Let $G$ be of type $G_2$, and let $V$ be a non-trivial $G$-module and $1 \neq s \in T$ a non-regular element. We have to show that the spectrum of $s$ on $V$ is not almost simple. Let $\omega$ be the highest weight of $V$. Suppose first that $\omega = \omega_1$ or $p = 3, \omega = \omega_2$, so $\dim V = 7$, or 6 for $p = 2$. The group $G$ contains a maximal rank closed subgroup $H$ isomorphic to $A_2$ such that the restriction of $V_{\omega_1}$ to $H$ has composition factors the natural module for $\mathrm{SL}_3(F)$, and its dual and, if $p \neq 2$, an additional trivial summand. So the matrix of $s$ on $V_{\omega_1}$ can be written as $\mathrm{diag}(a,b,c,1,a^{-1},b^{-1},c^{-1})$ if $p \neq 2$, otherwise $\mathrm{diag}(a,b,c,a^{-1},b^{-1},c^{-1})$, where $abc = 1$ in both cases. This is also true if $p = 3$ and $V = V_{\omega_2}$. If all the entries are distinct, this matrix is a regular element in $\mathrm{SL}(V)$, and hence in $G$, contrary to the assumption.

Suppose that the entries are not distinct. As any permutation of $a,b,c$ can be realized by an inner automorphism of $G$, we may assume that $a$ equals some other diagonal entry and by the same reasoning, we may ignore the possibilities $a = c$ and $a = c^{-1}$. So we examine the cases $a = b$, $a = a^{-1}$, and $a = b^{-1}$.

Let $a = b$. Then $s$ has almost simple spectrum on $V_\omega$ only if $a = a^{-1}$. But then $c = 1$ and $s$ is not almost cyclic on $V_\omega$.

Let $a = a^{-1} \neq b$, so $a = \pm 1$, $c = \pm b^{-1}$. If $a = 1$, then $b \neq 1$, $s$ acts on $V_\omega$ as $\hat{s} = \mathrm{diag}(1, b, b^{-1}, 1, 1, b^{-1}, b)$ (where we drop the 1 in the middle if $p = 2$) which does not have almost simple spectrum. If $a = -1$ then $p \neq 2$ and $s$ acts on $V_\omega$ as $\hat{s} = \mathrm{diag}(-1, b, -b^{-1}, 1, -1, b^{-1}, -b)$. If $b = \pm 1$ then the spectrum of $\hat{s}$ is not almost simple. Let $b \neq \pm 1$. As $V$ is an orthogonal space, $s$ is a regular element of $\mathrm{SO}(V)$ (Lemma 6), and hence in $G$, contrary to the assumption.

Let $a = b^{-1}$. Then $c = 1$. By reordering $a, c$, we arrive at the case $a = 1$, considered above. This completes the analysis of the cases $\omega = \omega_1$, and $(\omega, p) = (\omega_2, 3)$.

Suppose now that $\omega$ is an arbitrary $p$-restricted weight. If $p \neq 2, 3$ then the weights of $V_{\omega_1}$ occur as weights of $V$ (Lemma 4), so the result follows from that for $V_{\omega_1}$. Let $p = 2$; now $0, \omega_1, \omega_2, \omega_1 + \omega_2$ are the only 2-restricted dominant weights of $G$. By A. E. Zalesski (2009, Theorem 15), the weights of $V_\omega$ are the same as in characteristic 0, in particular all weights of $V_{\omega_1}$ are weights of $V_\omega$, and we conclude as above.

Now turn to the case $p = 3$ and $\omega$ still $p$-restricted. By A. E. Zalesski (2009, Theorem 15), if $\omega \neq 2\omega_2$ then the weights of $V_\omega$ are the same as in characteristic 0, and in particular all weights of $V_{\omega_1}$ are weights of $V_\omega$. So the result follows as above. For $p = 3$ and $\omega = 2\omega_2$, we use the tables of Lübeck (2018) to see that the weights of $V_{\omega_2}$ are weights of $V_{2\omega_2}$, and then conclude as before.

Finally, suppose that $\omega$ is not $p$-restricted. By Remark 1, we may assume that $V$ is tensor-decomposable, say, $V = V_1 \otimes V_2$, where the highest weight of $V_1$ is of the form $p^k\omega'$ for some $k$. Then the result follows by Lemma 3. □

## 4.2 Groups $B_n$ with $n > 2$, $D_n$ with $n > 3$, and $C_n$ with $p = 2$ and $n > 2$

In this section, we consider the groups as indicated in the heading of the section. Recall that when $G = B_n$, we may assume $p \neq 2$. Note that for groups $G$ of type $B_n$ and of type $D_n$, the multiplicity of the 0 weight in the adjoint representation $V_{\omega_2}$ is greater than 1. Therefore, if $\omega$ is a dominant weight such that $\omega_2 \prec \omega$ then, by Lemma 5(2), it suffices to observe that a non-central non-regular semisimple element $s \in G$ is not of almost simple spectrum on $V_{\omega_2}$. This is done in Lemma 18 below. The condition $\omega_2 \prec \omega$ holds provided $\omega$ is a radical weight and $\omega \neq 0, \omega_1, \omega_2$ for $G$ of type $B_n$, and $\omega \neq 0, \omega_2$ for $G$ of type $D_n$.

**Lemma 18 –** *Let $G=B_n$, $n>2$, $p\neq 2$, $\omega\in\{\omega_2,\omega_n\}$ or $G=D_n$, $n>3$, $\omega\in\{\omega_2,\omega_{n-1},\omega_n\}$. Let $s\in T\setminus Z(G)$ be a non-regular element. Then the spectrum of $s$ on $V_\omega$ is not almost simple, unless $G=D_4$, $\omega\in\{\omega_3,\omega_4\}$.*

*Proof.* Here we take $T$ to be the preimage in $G$ of the set of diagonal matrices in the image of $G$ under the natural representation. We take $s\in T$ and assume the spectrum of $s$ on $V_\omega$ is almost simple. Since $s$ is not regular, there exists a root $\alpha$ with respect to $T$ such that $\alpha(s)=1$. We will assume without loss of generality that either $\alpha=\alpha_1$, or $G=B_n$ and $\alpha=\alpha_n$.

Suppose first that $\omega=\omega_2$. Set $R_0=\{\alpha\in\Phi\mid\alpha(s)=1\}$. Since $s$ is non-central, there exists $\beta\in\Phi\setminus R_0$. Moreover, since $\Phi$ is an irreducible root system, there exists $\beta\in\Phi\setminus R_0$ which is not orthogonal to $R_0$. So for some $\alpha\in R_0$, $w_\alpha(\beta)\neq\beta$. Then $\beta(s)=w_\alpha(\beta)(s)\neq 1$, while $\alpha(s)=-\alpha(s)=1$. So $s$ is not almost cyclic on $\omega_2$.

Let $\omega\in\{\omega_{n-1},\omega_n\}$, for $G=D_n$ and $n>4$, or $\omega=\omega_n$ for $G=B_n$. Then $\mu=\frac{1}{2}(\alpha_1+\nu)$ is a weight of $V_\omega$, for $\nu\in\{\pm\varepsilon_3\pm\cdots\pm\varepsilon_n\}$, with certain conditions on the parity of the number of minus signs in the $D_n$-case. Suppose that $\alpha=\alpha_1$. Then $\mu-\alpha_1$ is a weight of $V_\omega$ for any admissible choice of the signs. As the spectrum of $s$ on $V_\omega$ is almost simple, we deduce that $\mu(s)$ does not depend on the choice of $\nu$ and so $\varepsilon_3(s)=\cdots=\varepsilon_n(s)=1$. Similarly, this then implies that $(\frac{1}{2}(\varepsilon_1+\varepsilon_2+\nu))(s)$ does not depend on the choice of $\nu$, so again this value must be equal to $(\frac{1}{2}(\pm(\varepsilon_1-\varepsilon_2)+\nu))(s)$, whence $\varepsilon_1(s)=1=\varepsilon_2(s)$ as well. This implies $s\in Z(G)$, a contradiction.

Finally, suppose that $G=B_n$, $\omega=\omega_n$ and $\alpha=\alpha_n$. Then for all $1\leq i\leq n-1$, we have the two distinct weights of $V_\omega$, $\omega-\alpha_i-\alpha_{i+1}-\cdots-\alpha_{n-1}-\alpha_n$ and $\omega-\alpha_i-\alpha_{i+1}-\cdots-\alpha_{n-1}-2\alpha_n$, taking the same value on $s$, and therefore deduce that $\alpha_i(s)=1$ for all $i$, again contradicting the fact that $s$ is non-central. □

**Remark 4 –** If $G=D_4$ then there exist non-central non-regular semisimple elements $s$ with almost simple spectrum on $V_{\omega_3}$ or $V_{\omega_4}$. Indeed, one easily observes that there are non-regular elements $s\in T\setminus Z(G)$ whose spectrum is almost simple on $V_{\omega_1}$. Let $\sigma$ be the triality automorphism of $G$. Then $\sigma(s)$ has almost simple spectrum on $V_{\omega_1}^\sigma$, whence the claim.

**Lemma 19 –** *Let $G=B_n$, $n>2$, $p\neq 2$, or $G=D_n$, $n\geq 4$ , and let $V_\omega\in\mathrm{Irr}(G)$, where $\omega\neq 0$ is $p$-restricted. Let $s\in T\setminus Z(G)$ be a non-regular element with almost simple spectrum on $V_\omega$. Then either $\omega=\omega_1$ or $G=D_4$ and $\omega\in\{\omega_1,\omega_3,\omega_4\}$.*

*Proof.* If $\omega$ is radical, this follows from Lemmas 18, 5(2) and 4, both for $B_n$ and $D_n$.

Suppose that $\omega$ is not radical. If $G=B_n$ then $\omega_n\preceq\omega$ by Lemma 4(2), so again the result follows from Lemmas 18 and 5(1)(i). Let $G=D_n$. By Theorem 5, all non-zero weights of $V_\omega$ are of multiplicity 1. Then, by D. M. Testerman and A. E. Zalesski (2015, Tables 1, 2), $\omega\in\{\omega_1,2\omega_1,\omega_2,\omega_{n-1},\omega_n\}$, where the radical weights $2\omega_1,\omega_2$ are to be dropped. Whence the result for $n=4$. If $n>4$ then the spectrum of $s$ on $V_\omega$ is not almost simple by Lemma 18. □

We now handle the case $G = C_n$, for $n > 2$ and $p = 2$, which is excluded in Proposition 2. Moreover, we only need to consider $V_{\omega_n}$ (see Proposition 2).

**Lemma 20 –** *Let $G = C_n$, $n > 2$, $p = 2$. Let $1 \neq s \in T$ be a non-regular element. Then the spectrum of $s$ on $V_{\omega_n}$ is not almost simple.*

*Proof.* We argue as in the proof of Lemma 18. We can assume that $\alpha(s) = 1$ for $\alpha = \alpha_1$ or $\alpha = 2\varepsilon_1$. The weights of $V_{\omega_n}$ are $\pm\varepsilon_1 \pm \cdots \pm \varepsilon_n$. Then $\mu := \varepsilon_1 \pm \varepsilon_2 + \nu$ are weights of $V_{\omega_n}$ for any $\nu = \pm\varepsilon_3 \pm \cdots \pm \varepsilon_n$. If $\alpha = 2\varepsilon_1$ then $\mu - \alpha$ is a weight of $V_{\omega_n}$, and we conclude (as in the proof of Lemma 18) that $(2\varepsilon_1)(s) = \cdots = (2\varepsilon_n)(s)$. As $p = 2$, we have $\varepsilon_1(s) = \cdots = \varepsilon_1(s)$, whence $s \in Z(G) = 1$, a contradiction.

If $\alpha = \alpha_1$ then for $\mu = \varepsilon_1 - \varepsilon_2 + \nu$ we have $\mu - \alpha_1 \in \Omega(V_{\omega_n})$, whence $(2\varepsilon_3)(s) = \cdots = (2\varepsilon_n)(s) = 1$. This implies that $(\varepsilon_1 + \varepsilon_2 + \nu)(s)$ does not depend on $\nu$, nor does $(\epsilon_1 - \epsilon_2 + \nu)(s)$, whence $(2\varepsilon_1)(s) = 1$, and again we conclude that $s = 1$. □

## 4.3 Completion of the Proof of Theorem 2

*Proof.* Let $G, s$ be as in the statement of Theorem 2. Note that $\mathrm{rank}(G) \geq 2$.

Suppose first that $\lambda$ is $p$-restricted. The groups of rank 2 have been examined in Lemmas 13, 15, 16 and 17, and the group of type $A_3$ in Lemma 14. In Proposition 2, we handled the groups $A_n, n > 3$, $F_4$, $E_6, E_7, E_8$, and all $p$-restricted weights for the group $C_n$, $n > 2$, except the weight $\omega = \omega_n$ when $p = 2$. The latter is handled in Lemma 20.

Groups of type $B_n, n > 2$, and $p \neq 2$, and groups of type $D_n$ are dealt with in Lemma 19.

By Remark 1, we may now assume that $V$ is tensor-decomposable. Let $V = V_1 \otimes V_2$ be a non-trivial tensor decomposition of $V$. By Lemma 3, the spectra of $s$ on $V_1$ and $V_2$ are simple. This contradicts Lemma 3. □

Finally, we conclude with a straightforward corollary of Theorem 2.

**Corollary 1 –** *Let $s \in T \setminus Z(G)$ be a non-regular element and $V$ an irreducible $G$-module. Suppose that the spectrum of $s$ on $V$ is almost simple. Then the eigenvalue multiplicities of $s$ on $V$ do not exceed $m = m_V(s)$, where either $m \leq \mathrm{rank}(G)$ or one of the following holds:*

*(1)* $G = A_3$, $\dim V = 6$, $m = 4$;

*(2)* $G = B_n$, $n > 2$, $p \neq 2$, $\dim V = 2n + 1$, $m = 2n$;

*(3)* $G = C_n$, *and either* $\dim V = 2n$ *and* $m = 2n - 2$ *or* $n = 2$, $p \neq 2$, $\dim V = 5$ *and* $m = 4$;

*(4)* $G = D_n$, $n > 3$, $\dim V = 2n$, $m = 2n - 2$.

*Proof.* This will follow from Theorem 2; we discuss each of the cases of the theorem. To get (1) above, we additionally use Lemma 14. For $G = C_2$, $p \neq 2$, we use Lemma 16. The modules of dimensions indicated in Theorem 2(1) are obtained by Frobenius twisting of $V_{\omega_1}$ (where the statement is clear); $m_V(s)$ remains unchanged under such a twist. This leaves us with $G = D_4$ and $\dim V = 8$. The modules $V_{\omega_1}$, $V_{\omega_3}$, $V_{\omega_4}$, are obtained from each other by a graph automorphism of $G$, and the other modules of dimension 8 as in Theorem 2(4) are Frobenius twists of these. The result follows.□

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