North-Western European Journal of Mathematics

Infinite order differential equations in the space of entire functions of minimal type with respect to a proximate order

Xiaoran Jin[1]

Received: July 22, 2020/Accepted: November 13, 2020/Online: April 23, 2021

Abstract

In this paper, we consider the space of entire functions of minimal type growth for a proximate order. We study the surjectivity of corresponding infinite order differential equations in the cases of regular singular type and of Korobeĭnik type.

Keywords: minimal type entire functions in several variable, infinite order partial differential equation, regular singular type, Korobeĭnik type.

MSC: 30D15, 35R50, 32A15, 32W50.

1 Introduction

The topic of characterization of infinite order differential operators in terms of the growth of their coefficients has a long history. The interest in this kind of problems, which lay at the heart of the early study of hyperfunctions, arose from the variations on Koethe's duality theorem. The infinite order differential operator acting on a certain space of holomorphic functions has been studied by several authors, for example: Martineau[2] and Momm[3].

On the other hand, recently, quantum physicists have developed an important theory regarding a phenomenon which is called "superoscillations". The study of superoscillations naturally leads to the analysis of a large class of convolution operators acting on spaces of entire functions. In particular, the key point to address these questions is the continuity of these operators on appropriate spaces. And it leads to the article of Aoki, Colombo, Sabadini, and Struppa[4] in which the authors

[1]Graduate School of Science, Chiba University, Japan

[2]Martineau, 1967, "Équations différentielles d'ordre infini".

[3]Momm, 1990, "Partial differential operators of infinite order with constant coefficients on the space of analytic functions on the polydisc".

[4]Aoki, Colombo, et al., 2018a, "Continuity of some operators arising in the theory of superoscillations".

offer explicit proofs of continuity of such operators by using some recent advances in the study of entire functions.

Furthermore, Aoki, Ishimura, Okada, Struppa, and Uchida[5] characterized continuous endomorphism of the space of entire functions of normal type or minimal type with respect to a given order. Since we established some solvability conditions of the infinite order PDEs for the case of normal type with respect to proximate order in Ishimura and Jin (2019), the duality between the space of normal type entire functions and the space of minimal type entire functions leads us to study the solvability conditions for the case of minimal type.

However, the proofs of main results in Ishimura and Jin (2019) cannot be simply applied because the duality arguments that were central in Aoki, Colombo, et al. (2018b) and Aoki, Ishimura, et al. (2020) cannot be used in this case. Hence, we will establish the solvability conditions in the cases of two types, which are called "regular singular type" and "Korobeĭnik type", by the different approaches.

2 Notations and recall

In this article, we employ the same notations as Ishimura and Miyake 2007: for multi-indexes $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ with $\mathbb{N} := \{0, 1, 2, 3, \dots\}$ and a point $z := (z_1, \dots, z_n) \in \mathbb{C}^n$, we set:

$$|\alpha| := \alpha_1 + \cdots + \alpha_n, \qquad \alpha! := \alpha_1! \cdots \alpha_n!,$$

$$|z| := \sqrt{|z_1|^2 + \cdots + |z_n|^2}, \qquad \overrightarrow{|z|} := \big(|z_1|, \dots, |z_n|\big),$$

$$D_z^\alpha := \frac{\partial^{|\alpha|}}{\partial z_1^{\alpha_1} \cdots \partial z_n^{\alpha_n}}, \qquad \mathrm{H}_k^n := \binom{n+k-1}{k} = \frac{(n+k-1)!}{(n-1)!\,k!}.$$

For any $\sigma > 0$, we define the Banach space

$$B_{w_\sigma} := \left\{ f \in \mathcal{O}(\mathbb{C}^n) \,\middle|\, \|f\|_{w_\sigma} := \sup_{z \in \mathbb{C}^n} \big|f(z)\big| \mathrm{e}^{-w_\sigma(z)} < \infty \right\}$$

with the norm $\|\cdot\|_{w_\sigma}$, where $w_\sigma(z) := \sigma |z|^{\rho(|z|)}$. A differentiable function $\rho(r)\colon \mathbb{R}_+ \to \mathbb{R}_+$ is a *proximate order* of order $\rho > 0$ provided that:

(i) $\lim_{r\to\infty} \rho(r) = \rho$;

(ii) $\lim_{r\to\infty} \rho'(r)\, r \ln r = 0$.

Let $\varphi(q)$ be the inverse function of $q = r^{\rho(r)}$ for all sufficiently large $q \in \mathbb{R}$. Since it is well-known that $r^{\rho(r)}$ is strictly increasing for all sufficiently large $r > 0$, we

[5] Aoki, Ishimura, et al., 2020, "Characterization of Continuous Endomorphisms in the Space of Entire Functions of a Given Order".

may assume the function $\varphi(q)$ is strictly increasing on $q \in [0,\infty)$. For any $q \in \mathbb{N}$, we define

$$A_q := \left(\frac{\varphi(q)^\rho}{e\rho}\right)^{\frac{q}{\rho}}.$$

We define the locally convex space of entire functions of type at most $\sigma \geqslant 0$ with respect to a proximate order $\rho(r)$:

$$E_\sigma^{\rho(r)} := \varprojlim_{\varepsilon\to 0} B_{w_{\sigma+\varepsilon}}.$$

Now we consider the space of entire functions of minimal type with respect to the proximate order $\rho(r)$

$$E_0^{\rho(r)} := \varprojlim_{\varepsilon\to 0} B_{w_{\varepsilon}};$$

and the space of entire functions of normal type with respect to the proximate order $\rho(r)$

$$\mathcal{E}^{\rho(r)} := \varinjlim_{\sigma\to\infty} B_{w_{\sigma}}.$$

By Lemma 1 in Ishimura and Jin 2019, they are (FS)-space and (DFS)-space, respectively.

By the proof of Theorem 1.23 in P. Lelong and L. Gruman Lelong and Gruman 1986, we remark the following lemma:

Lemma 1 – *For every $\delta > 0$ with $\delta < \frac{1}{\rho}$, there exists $T_0 > 0$ such that if $t \geqslant T_0$, we have*

$$\left(\frac{1}{\rho} - \delta\right)\frac{d}{dt}\ln t < \frac{d}{dt}\ln\varphi(t) < \left(\frac{1}{\rho} + \delta\right)\frac{d}{dt}\ln t. \tag{1}$$

We recall the following lemma of Ishimura and Jin 2019 (Corollary 1).

Lemma 2 – *If an entire function $f(z) = \sum_{\alpha\in\mathbb{N}^n} f_\alpha z^\alpha$ belongs to $E_\sigma^{\rho(r)}$, then*

$$\limsup_{|\alpha|=q\to\infty}\left(\left|f_\alpha\right|A_q\right)^{\frac{\rho}{q}} \leqslant (\sqrt{n})^\rho\,\sigma.$$

Conversely, if $f(z)$ satisfies this estimate, then we have $f(z) \in E_{\sqrt{n}^\rho\sigma}^{\rho(r)}$.

In particular, when $\sigma = 0$, we have that:

Proposition 1 – *An entire function* $f(z) = \sum_{\alpha\in\mathbb{N}^n} f_\alpha z^\alpha$ *belongs to* $E_0^{\rho(r)}$ *if and only if we have*

$$\limsup_{|\alpha|\to\infty} \left(\left|f_\alpha\right| A_{|\alpha|}\right)^{\frac{1}{|\alpha|}} = 0.$$

3 Infinite order partial differential equations in $E_0^{\rho(r)}$

We employ the same notation and terminology as Ishimura and Jin 2019: for an infinite order partial differential operator

$$P = P(z, D_z) = \sum_{\alpha\in\mathbb{N}^n} a_\alpha(z) D_z^\alpha, \quad \text{where } a_\alpha(z) := \sum_{\beta\in\mathbb{N}^n} a_\alpha^\beta z^\beta,$$

we define the transpose of $P = P(\zeta, D_\zeta)$:

$${}^tP := {}^tP(z, D_z) := \sum_{\beta\in\mathbb{N}^n} \left(\sum_{\alpha\in\mathbb{N}^n} a_\alpha^\beta z^\alpha \right) D_z^\beta.$$

For a formal power series $f(z) := \sum_\nu f_\nu z^\nu \in \mathbb{C}[[z]]$, the characteristic matrix of operator P is

$$C_P := \left(c_\nu^\mu\right)_{\mu,\nu} := \left(\sum_{\substack{\lambda\leqslant\nu \\ \lambda\leqslant\mu}} \frac{\nu!}{\lambda!} a_{\nu-\lambda}^{\mu-\lambda} \right)_{\mu,\nu} : \mathbb{C}^{\mathbb{N}^n} \to \mathbb{C}^{\mathbb{N}^n}. \tag{2}$$

And the characteristic matrix of the transpose of operator P is

$$C_{{}^tP} = \left(\sum_{\substack{\lambda\leqslant\nu \\ \lambda\leqslant\mu}} \frac{\nu!}{\lambda!} a_{\mu-\lambda}^{\nu-\lambda} \right)_{\mu,\nu} = \left(\frac{\nu!}{\mu!} c_\mu^\nu \right)_{\mu,\nu}. \tag{3}$$

Note that for any formal power series $f(z) = \sum_{\nu\in\mathbb{N}^n} f_\nu z^\nu$ and $g(z) = \sum_{\mu\in\mathbb{N}^n} g_\mu z^\mu$, we have $Pf = g$ if and only if (see Ishimura and Jin 2019 page 80)

$$\sum_{\nu\in\mathbb{N}^n} c_\nu^\mu f_\nu = \sum_{\nu\in\mathbb{N}^n} \left(\sum_{\substack{\lambda\leqslant\nu \\ \lambda\leqslant\mu}} \frac{\nu!}{\lambda!} a_{\nu-\lambda}^{\mu-\lambda} \right) f_\nu = g_\mu \tag{4}$$

for all $\mu \in \mathbb{N}^n$. Similarly, we have ${}^tPf = g$ if and only if

$$\sum_{\nu\in\mathbb{N}^n} \frac{\nu!}{\mu!} c_\mu^\nu f_\nu = \sum_{\nu\in\mathbb{N}^n} \left(\sum_{\substack{\lambda\leqslant\nu\\ \lambda\leqslant\mu}} \frac{\nu!}{\lambda!} a_{\mu-\lambda}^{\nu-\lambda} \right) f_\nu = g_\mu. \tag{5}$$

In this article, as in the paper Ishimura and Jin (2019), we will study the partial differential operator of regular singular type and Korobeĭnik type.

4 Partial differential equations of regular singular type

We recall that a partial differential operator

$$P = \sum_{\alpha\in\mathbb{N}^n} a_\alpha(z) D_z^\alpha$$

is of *regular singular type* provided that

$$a_\alpha(z) = \sum_{\beta\geqslant\alpha} a_\alpha^\beta z^\beta.$$

Theorem 1 – *Let a continuous linear operator $P\colon E_0^{\rho(r)} \to E_0^{\rho(r)}$ be of regular singular type as differential operator. Suppose that the following conditions hold:*

1. *there exist $C, \kappa > 0$ such that for all $\mu \in \mathbb{N}^n$,*

$$C\kappa^{|\mu|} \leqslant \left| \sum_{\lambda\leqslant\mu} \frac{\mu!}{\lambda!} a_{\mu-\lambda}^{\mu-\lambda} \right|;$$

2. *for any $\delta > 0$, there exists $N > 0$ such that*

$$\frac{\left|\sum_{\lambda\leqslant\nu} \frac{\nu!}{\lambda!} a_{\nu-\lambda}^{\mu-\lambda}\right|}{\left|\sum_{\lambda\leqslant\mu} \frac{\mu!}{\lambda!} a_{\mu-\lambda}^{\mu-\lambda}\right|} \leqslant \frac{A_{|\nu|}}{A_{|\mu|}} \cdot \frac{\delta^{|\mu-\nu|}}{|\mu|^{n-1}}$$

whenever $|\mu| > N$ and $\nu < \mu$.

Then $P\colon E_0^{\rho(r)} \to E_0^{\rho(r)}$ is surjective; so is an isomorphism of Fréchet spaces.

Proof. Assume that $Pf = g \in E_0^{\rho(r)}$, where

$$f(z) = \sum_{\mu\in\mathbb{N}^n} f_\mu z^\mu \qquad \text{and} \qquad g(z) = \sum_{\mu\in\mathbb{N}^n} g_\mu z^\mu.$$

In view of Proposition 1, it suffices to show that for any $\varepsilon > 0$, there exists $D > 0$ such that

$$\left|f_\mu\right| \leqslant D\,\frac{\varepsilon^{|\mu|}}{A_{|\mu|}} \tag{6}$$

for all $\left|\mu\right| \in \mathbb{N}$. We prove it by induction on $\left|\mu\right| \in \mathbb{N}$.

By condition (1) and Proposition 1, we have that there exists $D_0 > 0$ such that

$$\frac{\left|g_\mu\right|}{\left|c_\mu^\mu\right|} \leqslant D_0\,\frac{\varepsilon^{|\mu|}}{A_{|\mu|}}$$

for all $\left|\mu\right| \in \mathbb{N}$. Put $\delta := \varepsilon/4$. Condition (2) implies that there exist $N, D_\delta > 0$ such that

(a) if $|\nu| \leqslant N$, then

$$\frac{\left|c_\nu^\mu\right|}{\left|c_\mu^\mu\right|} \leqslant D_\delta\,\frac{A_{|\nu|}}{A_{|\mu|}} \cdot \frac{\delta^{|\mu-\nu|}}{\left|\mu\right|^{n-1}};$$

(b) if $|\nu| > N$, then

$$\frac{\left|c_\nu^\mu\right|}{\left|c_\mu^\mu\right|} \leqslant \frac{A_{|\nu|}}{A_{|\mu|}} \cdot \frac{\delta^{|\mu-\nu|}}{\left|\mu\right|^{n-1}}.$$

Remark that

$$\begin{aligned}\lim_{q\to\infty} \frac{1}{q^{n-1}} \sum_{k=1}^{q} \left(\frac{1}{4}\right)^k \mathsf{H}_{q-k}^n \leqslant \sum_{k=1}^{\infty} \left(\frac{1}{4}\right)^k \lim_{q\to\infty} \frac{\mathsf{H}_{q-1}^n}{q^{n-1}} &= \frac{1}{3\,(n-1)!} \lim_{q\to\infty} \frac{(n+q-2)!}{(q-1)!\,q^{n-1}} \\ &\leqslant \frac{1}{3\,(n-1)!} \lim_{q\to\infty} \left(\frac{n+q-2}{q}\right)^{n-1} = \frac{1}{3\,(n-1)!}.\end{aligned} \tag{7}$$

Hence, there exists $M > N$ such that

$$\sum_{|\nu|=0}^{N} D_\delta \left(\frac{1}{4}\right)^{|\mu-\nu|} < \frac{1}{12} \qquad \text{and} \qquad \frac{1}{\left|\mu\right|^{n-1}} \sum_{j=1}^{|\mu|} \left(\frac{1}{4}\right)^j \mathsf{H}_{|\mu|-j}^n < \frac{5}{12}$$

whenever $\left|\mu\right| > M$. And there exists $D > 2D_0$ such that (6) holds for all $\left|\mu\right| \leqslant M$.

Assume that (6) holds for $|\mu| = q-1$, where $q-1 \geqslant M$. When $|\mu| = q$, by the assumption of induction, we have that

$$\begin{aligned}\frac{1}{|c_\mu^\mu|}\Big|\sum_{\nu<\mu} c_\nu^\mu f_\nu\Big| &\leqslant \sum_{|\nu|=0}^{N} \frac{|c_\nu^\mu|}{|c_\mu^\mu|}|g_\nu| + \sum_{|\nu|=N+1}^{q-1} \frac{|c_\nu^\mu|}{|c_\mu^\mu|}|f_\nu| \\ &\leqslant \sum_{|\nu|=0}^{N} D_\delta \frac{A_{|\nu|}}{A_q}\cdot\frac{\delta^{|\mu-\nu|}}{q^{n-1}}\cdot D\frac{\varepsilon^{|\nu|}}{A_{|\nu|}} + \sum_{|\nu|=N+1}^{|\mu|-1} \frac{A_{|\nu|}}{A_q}\cdot\frac{\delta^{|\mu-\nu|}}{q^{n-1}}\cdot D\frac{\varepsilon^{|\nu|}}{A_{|\nu|}} \\ &= \frac{D}{q^{n-1}}\cdot\frac{\varepsilon^q}{A_q}\left(\sum_{|\nu|=0}^{N} D_\delta\left(\frac{1}{4}\right)^{|\mu-\nu|} + \sum_{|\nu|=N+1}^{q-1}\left(\frac{1}{4}\right)^{|\mu-\nu|}\right) \\ &\leqslant \frac{D}{q^{n-1}}\cdot\frac{\varepsilon^q}{A_q}\left(\frac{1}{12} + \sum_{j=1}^{q}\left(\frac{1}{4}\right)^j \mathrm{H}_{q-j}^n\right) < \frac{D}{2}\cdot\frac{\varepsilon^q}{A_q}.\end{aligned}$$

Finally, applying (4), we have that

$$|f_\mu| \leqslant \frac{|g_\mu|}{|c_\mu^\mu|} + \frac{1}{|c_\mu^\mu|}\Big|\sum_{\nu<\mu} c_\nu^\mu f_\nu\Big| < D_0\frac{\varepsilon^q}{A_q} + \frac{D}{2}\cdot\frac{\varepsilon^q}{A_q} < D\frac{\varepsilon^q}{A_q},$$

as desired. □

We call P an operator of *Euler type* provided that P has the form:

$$P = \sum_{\alpha\in\mathbb{N}^n} a_\alpha z^\alpha D_z^\alpha,$$

where $a_\alpha \in \mathbb{C}$ for all $\alpha \in \mathbb{N}^n$.

Corollary 1 – *If P is of Euler type, then $P\colon E_0^{\rho(r)} \to E_0^{\rho(r)}$ is an isomorphism if and only if there exist $C, \kappa > 0$ such that for all $\mu \in \mathbb{N}^n$, we have*

$$C\kappa^{|\mu|} \leqslant \Big|\sum_{\lambda\leqslant\mu} \frac{\mu!}{\lambda!} a_{\mu-\lambda}^{\mu-\lambda}\Big|.$$

Proof. We only need to prove the necessity. Suppose that $Pf = g$, where

$$f := \sum_{\mu\in\mathbb{N}^n} f_\mu z^\mu \quad\text{and}\quad g := \sum_{\mu\in\mathbb{N}^n} g_\mu z^\mu.$$

First we show that $c_\mu^\mu \neq 0$ for all $\mu \in \mathbb{N}^n$. Assume that there exists $\gamma \in \mathbb{N}^n$ such that $c_\gamma^\gamma = 0$. Set $g := g_\gamma z^\gamma$, where $g_\gamma \neq 0$. Then we have $g \in E_0^{\rho(r)}$ because of Proposition 1.

On the other hand, in view of (4), there doesn't exist $f \in E_0^{\rho(r)}$ such that $Pf = g$, which contradicts the surjectivity of $P\colon E_0^{\rho(r)} \to E_0^{\rho(r)}$.

To show the conclusion, we prove the inequality

$$\limsup_{|\mu|\to\infty} \left(\frac{1}{\left|c_\mu^\mu\right|} \right)^{\frac{1}{|\mu|}} < \infty$$

by contradiction. Assume that there exists an infinite subset $S \subset \mathbb{N}^n$ such that 0 is the only limit point of the sequence $\left(|c_\alpha^\alpha|^{\frac{1}{|\alpha|}}\right)_{\alpha\in S}$. Set

$$g_\mu := \begin{cases} \dfrac{\left|c_\mu^\mu\right|}{A_{|\mu|}} & \mu \in S \\ 0 & \mu \notin S \end{cases}.$$

It follows that $g(z) \in E_0^{\rho(r)}$ from Proposition 1. Since $P\colon E_0^{\rho(r)} \to E_0^{\rho(r)}$ is an isomorphism, we have that $f(z) = P^{-1}g(z) \in E_0^{\rho(r)}$. Therefore,

$$\limsup_{|\mu|\to\infty} \left(\left|f_\mu\right| A_{|\mu|} \right)^{\frac{1}{|\mu|}} = \limsup_{|\mu|\to\infty} \left(\frac{\left|g_\mu\right|}{\left|c_\mu^\mu\right|} \cdot A_{|\mu|} \right)^{\frac{1}{|\mu|}} = 1,$$

which implies a contradiction that $f(z) \notin E_0^{\rho(r)}$. □

Example 1 – Let $n = 1$, $\rho(r) \equiv \rho > 0$. In the following cases, $P = \sum a_\alpha(z) D_z^\alpha$ satisfies the conditions of Theorem 1.

1. with $k > 0$, define

$$a_\alpha^\beta := \begin{cases} \dfrac{k^\alpha}{\alpha!} & \beta = \alpha \\ \dfrac{(-k)^\alpha}{\alpha!} & \beta = \alpha + 1 \\ 0 & \text{otherwise} \end{cases}.$$

2. with $\rho \geqslant 1$ and $k \neq -1$, define

$$a_\alpha^\beta := \begin{cases} \dfrac{k^\alpha}{\alpha!} & \beta = \alpha \\ \dfrac{(-1)^\alpha}{\alpha!\,(\beta-\alpha)!} & \beta > \alpha \\ 0 & \text{otherwise} \end{cases}.$$

5 Partial differential equations of Korobeĭnik type

A partial differential operator

$$P = \sum_{\alpha \in \mathbb{N}^n} a_\alpha(z) D_z^\alpha$$

is of *Korobeĭnik type* provided that

$$a_\alpha(z) = \sum_{\beta \leqslant \alpha} a_\alpha^\beta z^\beta.$$

In the rest of this section, we suppose $\rho > 1$: in this case, the equation $s = r^{\rho(r)-1}$ has the unique solution $r = \lambda(s)$ for all s large enough. A proximate order $\rho^*(s)$ is said to be a *c*onjugate proximate order of $\rho(r)$ if it satisfied for large s (so for large r)

$$\rho^*(s) := \frac{\rho(r)}{\rho(r)-1} \quad \text{i.}e. \quad \frac{1}{\rho(r)} + \frac{1}{\rho^*(s)} = 1. \tag{8}$$

By Lelong and Gruman (1986, Proposition 9.4), we have that $\rho^*(s)$ is indeed a proximate order. Set $\rho^* := \lim_{s\to\infty} \rho^*(s)$. Then we have $\frac{1}{\rho} + \frac{1}{\rho^*} = 1$.

As the case of $r = \varphi(t)$, let $s = \varphi^*(u)$ be a differentiable function being the inverse function of $u = s^{\rho^*(s)}$ when u is sufficiently large. We set also

$$A_q^* := \left(\frac{\varphi^*(q)\rho^*}{\mathrm{e}\rho^*} \right)^{\frac{q}{\rho^*}}.$$

We recall a proposition of Ishimura and Jin (2019, Proposition 3):

Proposition 2 – *Suppose $\rho > 1$. The map*

$$\left(\mathcal{E}^{\rho(r)}\right)' \xrightarrow{\sim} E_0^{\rho^*(s)} \colon T \mapsto \widehat{T}(\zeta) := T_z(\mathrm{e}^{z\cdot\zeta})$$

is a continuous bijection of Fréchet spaces; so an isomorphism.

Since both of $E_0^{\rho(r)}$ and $\mathcal{E}^{\rho(r)}$ are reflexive, we have that

Corollary 2 – *Suppose $\rho > 1$. The map*

$$T \mapsto \widehat{T} \colon \left(E_0^{\rho(r)}\right)' \xrightarrow{\sim} \mathcal{E}^{\rho^*(s)}$$

is a continuous bijection of Fréchet spaces; so an isomorphism.

To prove our main result, we need to characterize the Cauchy sequence in $\mathcal{E}^{\rho(r)}$. So we need the following lemma.

Lemma 3 – *If for each $q \in \mathbb{Z}_+$, $r := r(q)$ is the solution of equation*

$$\frac{\mathrm{d}}{\mathrm{d}r}\Big(r^q\, \mathrm{e}^{-\sigma r^{\rho(r)}}\Big) = 0. \tag{9}$$

Then

$$\lim_{q\to\infty}\left(\frac{A_q\, \mathrm{e}^{\sigma r^{\rho(r)}}}{r^q}\right)^{\frac{1}{q}} = \left(\frac{1}{\sigma}\right)^{\frac{1}{\rho}}.$$

Proof. By computing the equation, we have that

$$\frac{t}{q} = \frac{1}{\sigma\Big(\rho'(r) r \ln r + \rho(r)\Big)}$$

where $t := r^{\rho(r)}$. Since $r \to \infty$ as $q \to \infty$, we have that for any $\varepsilon_1 > 0$, there exists $N_1 > 0$ such that

$$\frac{1}{\sigma\rho} - \varepsilon_1 < \frac{t}{q} < \frac{1}{\sigma\rho} + \varepsilon_1$$

whenever $q, t > N_1$. In view of Lemma 1, for any $\varepsilon_2 > 0$, there exists $N_2 > 0$ such that (1) holds for all $q, t > N_2$.

If $\sigma\rho \leqslant 1$, integrating (1) from q to t, obtain that

$$\left(\frac{1}{\rho} - \varepsilon_2\right)\ln\frac{t}{q} < \ln\frac{\varphi(t)}{\varphi(q)} < \left(\frac{1}{\rho} + \varepsilon_2\right)\ln\frac{t}{q}$$

for all $q, t > N_2$. Hence, for all $q, t > \max\{N_1, N_2\}$, we have

$$\left(\frac{1}{\rho} - \varepsilon_2\right)\ln\left(\frac{1}{\sigma\rho} - \varepsilon_1\right) < \ln\frac{\varphi(t)}{\varphi(q)} < \left(\frac{1}{\rho} + \varepsilon_2\right)\ln\left(\frac{1}{\sigma\rho} + \varepsilon_1\right).$$

When $\sigma\rho > 1$, by the same process, obtain that for all $q, t > \max\{N_1, N_2\}$,

$$\left(\frac{1}{\rho} + \varepsilon_2\right)\ln\left(\frac{1}{\sigma\rho} - \varepsilon_1\right) < \ln\frac{\varphi(t)}{\varphi(q)} < \left(\frac{1}{\rho} - \varepsilon_2\right)\ln\left(\frac{1}{\sigma\rho} + \varepsilon_1\right).$$

Finally, since $r \to \infty$ (so does t) as $q \to \infty$, we have that

$$\begin{aligned}
\lim_{q\to\infty}\frac{1}{q}\ln\frac{r^q \mathrm{e}^{-\sigma r^{\rho(r)}}}{A_q} &= \lim_{q\to\infty}\frac{1}{q}\Big(\ln\varphi(t)^q - \sigma t - \ln A_q\Big) \\
&= \lim_{q\to\infty}\left(-\frac{\sigma t}{q} + \ln\frac{\varphi(t)}{\varphi(q)}\right) + \frac{1}{\rho}\ln(\mathrm{e}\rho) \\
&= -\frac{1}{\rho} + \frac{1}{\rho}\ln\left(\frac{1}{\sigma\rho}\right) + \frac{1}{\rho}\ln(\mathrm{e}\rho) = \frac{1}{\rho}\ln\frac{1}{\sigma}.
\end{aligned}$$

□

Now, by the preceding lemma, we can characterize the Cauchy sequence in $\mathcal{E}^{\rho(r)}$.

Lemma 4 – *Suppose* $f^j(z) := \sum_{\alpha\in\mathbb{N}^n} f^j_\alpha z^\alpha \in \mathcal{E}^{\rho(r)}$ *for each* $j \in \mathbb{Z}_+$. *Then the following three statements are equivalent.*

(a) $f^j(z) \to 0$ *as* $j \to \infty$ *in* $\mathcal{E}^{\rho(r)}$*;*

(b) there exists $\delta > 0$ *such that for any* $\varepsilon > 0$*, there exists* $N > 0$ *such that*

$$\left|f^j_\alpha\right| \leqslant \varepsilon\,\frac{\delta^{|\alpha|}}{A_{|\alpha|}}$$

whenever $j > N$ *and* $\alpha \in \mathbb{N}^n$*;*

(c) there exists $\delta > 0$ *such that for any* $\varepsilon > 0$*, there exists* $N > 0$ *such that*

$$\left|f^j_\alpha\right| \leqslant \varepsilon\,\frac{A^*_{|\alpha|}}{\alpha!}\,\delta^{|\alpha|}$$

whenever $j > N$ *and* $\alpha \in \mathbb{N}^n$*.*

Proof. Note that (a) is equivalent to that:

(a)′ there exists $\sigma > 0$ such that for any $\varepsilon > 0$, there exists $N > 0$ such that $\left\|f^j(z)\right\|_{w_\sigma} < \varepsilon$ whenever $j > N$.

(a)⇒(b). It follows from (a)′ that we have that for any $\varepsilon > 0$, there exists $N_1 > 0$ such that

$$\sup_{|z|\leqslant s}\left|f^j(z)\right| \leqslant \varepsilon\,\mathrm{e}^{\sigma s^{\rho(s)}}$$

whenever $s > 0$ and $j > N_1$. Let $\vec{S} := (s,\ldots,s)$ and $r := s\sqrt{n}$. By Cauchy's inequality, for any $\varepsilon > 0$, there exists $N_2 > N_1$ such that for any $\vec{S} \in \mathbb{R}^n_+$ (so for any $r > 0$), we have that

$$\left|f^j_\alpha\right| = \frac{1}{\alpha!}\left|\partial^\alpha_z f^j(0)\right| \leqslant \frac{1}{\vec{S}^\alpha}\sup_{|\vec{z}|\leqslant\vec{S}}\left|f^j(z)\right| \leqslant \left(\frac{1}{s}\right)^{|\alpha|}\sup_{|z|\leqslant s\sqrt{n}}\left|f^j(z)\right|$$
$$\leqslant \left(\frac{1}{s}\right)^{|\alpha|}\varepsilon\,\mathrm{e}^{\sigma(s\sqrt{n})^{\rho(s\sqrt{n})}} = \left(\frac{\sqrt{n}}{r}\right)^{|\alpha|}\varepsilon\,\mathrm{e}^{\sigma r^{\rho(r)}}$$

whenever $j > N_2$ and $\alpha \in \mathbb{N}^n$. When $\alpha = 0$, it's trivial. For each $\alpha \in \mathbb{N}^n \setminus \{0\}$, we choose some r satisfying (9). Applying Lemma 3, obtain that for any $\tau > 0$, there exist $C, N_3 > N_2$ such that

$$\left|f^j_\alpha\right| \leqslant \left(\frac{\sqrt{n}}{r}\right)^{|\alpha|}\varepsilon\,\mathrm{e}^{\sigma r^{\rho(r)}} = (\sqrt{n})^{|\alpha|}\cdot\frac{\varepsilon}{A_{|\alpha|}}\cdot\frac{A_{|\alpha|}\mathrm{e}^{\sigma r^{\rho(r)}}}{r^{|\alpha|}}$$
$$\leqslant (\sqrt{n})^{|\alpha|}\cdot\frac{C\,\varepsilon}{A_{|\alpha|}}\cdot(\sigma+\tau)^{\frac{|\alpha|}{\rho}} = \frac{C\,\varepsilon}{A_{|\alpha|}}\left((\sigma+\tau)^{\frac{1}{\rho}}\sqrt{n}\right)^{|\alpha|}$$

whenever $j > N_3$ and $\alpha \in \mathbb{N}^n \setminus \{0\}$.

(b)⇒(a). First, we claim that if ρ, $\sigma > 0$, then

$$\limsup_{|\alpha|\to\infty}\left(\frac{\|z^\alpha\|_{w_\sigma}}{A_{|\alpha|}}\right)^{\frac{1}{|\alpha|}} \leqslant \left(\frac{1}{\sigma}\right)^{\frac{1}{\rho}}.$$

In fact, for all $r > 0$ and $|\alpha| \in \mathbb{N}$, we have that

$$\frac{\|z^\alpha\|_{w_\sigma}}{A_{|\alpha|}} = \sup_{|z|=r}\frac{|z^\alpha|\mathrm{e}^{-w_\sigma(z)}}{A_{|\alpha|}} \leqslant \frac{r^{|\alpha|}\,\mathrm{e}^{-\sigma r^{\rho(r)}}}{A_{|\alpha|}}.$$

For each $|\alpha| \in \mathbb{N}$, we choose some $r > 0$ such that the right-hand side of the above inequality attains its maximum, which means (9) is satisfied. And the conclusion follows from Lemma 3.

For any $\delta > 0$, we choose some σ, $\tau > 0$ such that $0 < \tau < \sigma - \delta^\rho$. In view of Lemma 2, we have that there exist D, $C > 0$ such that for any $\varepsilon > 0$, there exists $N > 0$ such that

$$\begin{aligned}\left\|f^j(z)\right\|_{w_\sigma} &\leqslant \sum_{\alpha\in\mathbb{N}^n}\left|f_\alpha^j\right|\|z^\alpha\|_{w_\sigma} \leqslant \sum_{\alpha\in\mathbb{N}^n}\varepsilon\frac{\delta^{|\alpha|}}{A_{|\alpha|}}\cdot D\left(\frac{1}{\sigma-\tau}\right)^{\frac{|\alpha|}{\rho}}A_{|\alpha|}\\ &\leqslant \varepsilon D\sum_{\alpha\in\mathbb{N}^n}\left(\frac{\delta^\rho}{\sigma-\tau}\right)^{\frac{|\alpha|}{\rho}} < \varepsilon C\end{aligned}$$

whenever $j > N$. Since C is independent of ε, we have (a)′.

(b)⇔(c). It suffices to show that there exist C, $K > 0$ such that

$$\frac{1}{CK^{|\mu|}} \leqslant \frac{\mu!}{A_{|\mu|}A^*_{|\mu|}} \leqslant CK^{|\mu|}$$

for all $\mu \in \mathbb{N}^n$. Let $q := \left|\mu\right|$. Then we have

$$\begin{aligned}A_q\,A_q^* &= \frac{\varphi(q)^q}{(\mathrm{e}\rho)^{\frac{q}{\rho}}}\cdot\frac{\varphi^*(q)^q}{(\mathrm{e}\rho^*)^{\frac{q}{\rho^*}}}\\ &= q^{\frac{q}{\rho(r)}}q^{\frac{q}{\rho^*(s)}}\left(\frac{1}{\mathrm{e}}\left(\frac{1}{\rho}\right)^{\frac{1}{\rho}}\left(\frac{1}{\rho^*}\right)^{\frac{1}{\rho^*}}\right)^q = q^q\left(\frac{1}{\mathrm{e}\rho}(\rho-1)^{\frac{\rho-1}{\rho}}\right)^q.\end{aligned}$$

Let $\lfloor\frac{q}{n}\rfloor := \max\left\{j \in \mathbb{Z} : j \leqslant \frac{q}{n}\right\}$. Observe that $\frac{q}{n} \geqslant \lfloor\frac{q}{n}\rfloor \geqslant \frac{q}{n} - 1 \geqslant \frac{q}{n+1}$ holds for all sufficiently large $q \in \mathbb{N}$. Hence, by Stirling's approximation, we have that

$$\begin{aligned}q^q \geqslant \mu! &\geqslant \left(\left\lfloor\frac{q}{n}\right\rfloor!\right)^n \geqslant \left(\left\lfloor\frac{q}{n}\right\rfloor\frac{1}{\mathrm{e}}\right)^{\lfloor\frac{q}{n}\rfloor\cdot n}\\ &\geqslant \left(\frac{q}{\mathrm{e}(n+1)}\right)^{q-n} \geqslant q^q\frac{1}{q^n}\left(\frac{1}{\mathrm{e}(n+1)}\right)^q \geqslant q^q\left(\frac{1}{\mathrm{e}(n+1)^2}\right)^q\end{aligned}$$

for all sufficiently large $q \in \mathbb{N}$. Combining the above equation and these two inequalities, we see the desired inequalities. □

Theorem 2 – *Let $\rho > 1$ and a continuous linear operator $P\colon E_0^{\rho(r)} \to E_0^{\rho(r)}$ be of Korobeĭnik type as partial differential operator. Suppose that the following conditions hold:*

1. *there exist $C, \kappa > 0$ such that for all $\mu \in \mathbb{N}^n$, we have that*

$$C\,\kappa^{|\mu|} \leqslant \left|\sum_{\lambda \leqslant \mu} \frac{\mu!}{\lambda!} a_{\mu-\lambda}^{\mu-\lambda}\right|;$$

2. *there exists $R > 0$ such that for all $\nu < \mu$, we have that*

$$\frac{\left|\sum_{\lambda\leqslant\mu} \frac{1}{\lambda!} a_{\mu-\lambda}^{\nu-\lambda}\right|}{\left|\sum_{\lambda\leqslant\mu} \frac{1}{\lambda!} a_{\mu-\lambda}^{\mu-\lambda}\right|} \leqslant \frac{A_{|\nu|}^{*}}{A_{|\mu|}^{*}} R^{|\mu-\nu|}.$$

Then $P\colon E_0^{\rho(r)} \to E_0^{\rho(r)}$ is surjective; so is an epimorphism of Fréchet spaces.

Proof. By Corollary 2, we have that $P\colon E_0^{\rho(r)} \to E_0^{\rho(r)}$ is surjective if and only if the following two statements hold:

(a) ${}^tP\colon \mathcal{E}^{\rho^*(s)} \to \mathcal{E}^{\rho^*(s)}$ is injective;

(b) $P\left(\mathcal{E}^{\rho(r)}\right)$ is closed in $\mathcal{E}^{\rho(r)}$.

Since the characteristic matrix $C_{{}^tP}$ is "lower triangular" (see (3)), we have that (a) is satisfied. To prove (b), by closed range theorem, we only need to show that ${}^tP\left(\mathcal{E}^{\rho^*(s)}\right)$ is closed in $\mathcal{E}^{\rho^*(s)}$. Assume that

(i) $\widehat{S^j}, \widehat{T^j} \in \mathcal{E}^{\rho^*(s)}$ for all $j \in \mathbb{Z}_+$, where

$$\widehat{S^j}(\zeta) := \sum_{\alpha\in\mathbb{N}^n} \widehat{S}_\alpha^j \zeta^\alpha, \qquad \widehat{T^j}(\zeta) := \sum_{\alpha\in\mathbb{N}^n} \widehat{T}_\alpha^j \zeta^\alpha;$$

(ii) ${}^tP(\widehat{T^j}) = \widehat{S^j}$ for each $j \in \mathbb{Z}_+$;

(iii) The sequence $(\widehat{S^j})_{j\in\mathbb{Z}_+}$ is a Cauchy sequence in $\mathcal{E}^{\rho^*(s)}$.

We shall show that $(\widehat{T^j})_{j\in\mathbb{Z}_+}$ is a Cauchy sequence in $\mathcal{E}^{\rho^*(s)}$. By Lemma 4 (b), it suffices to show that there exists $\delta > 0$ such that for any $\varepsilon > 0$, there exists $N > 0$ such that

$$\left|\widehat{T}_\mu^i - \widehat{T}_\mu^j\right| \leqslant \varepsilon \frac{\delta^{|\mu|}}{A_{|\mu|}^{*}} \tag{10}$$

for all $i, j > N$ and $|\mu| \in \mathbb{N}$. Prove it by induction on $|\mu| \in \mathbb{N}$.

Since $\widehat{S^i} - \widehat{S^j} \to 0$ as $i, j \to \infty$ in $\mathcal{E}^{\rho^*(s)}$, by condition (1) and Lemma 4 (b), there exist $C, \kappa, K > 0$ such that for any $\varepsilon > 0$, there exists $N_1 > 0$ such that

$$\frac{1}{\left|c_\mu^\mu\right|}\left|\widehat{S}_\mu^i - \widehat{S}_\mu^j\right| \leqslant C\,\kappa^{|\mu|} \cdot \frac{\varepsilon}{2C} \cdot \left(\frac{K}{\kappa}\right)^{|\mu|} \cdot \frac{1}{A^*_{|\mu|}} = \frac{\varepsilon}{2} \cdot \frac{K^{|\mu|}}{A^*_{|\mu|}} \tag{11}$$

whenever $i, j > N_1$ and $\left|\mu\right| \in \mathbb{N}$. We choose some $\delta > \max\{2^{n+1}R, K\}$, where R is given by condition (2).

It's obvious that there exists $N > N_1$ such that (10) holds for all $i, j > N$ and $\left|\mu\right| = 0$. Assume that (10) holds for $\left|\mu\right| = q-1$. When $\left|\mu\right| = q$, we have that

$$\begin{aligned}
\frac{1}{\left|c_\mu^\mu\right|}\left|\sum_{\nu<\mu} \frac{\nu!}{\mu!} c_\mu^\nu \left(\widehat{T}_\nu^i - \widehat{T}_\nu^j\right)\right| &\leqslant \sum_{\nu<\mu} \frac{\nu!}{\mu!} \cdot \frac{A^*_{|\nu|}}{A^*_{|\mu|}} R^{|\mu-\nu|} \cdot \varepsilon\, \frac{\delta^{|\nu|}}{A^*_{|\nu|}} \\
&= \varepsilon\, \frac{\delta^{|\mu|}}{A^*_{|\mu|}} \sum_{\nu<\mu} \frac{\nu!}{\mu!} \left(\frac{R}{\delta}\right)^{|\mu-\nu|} \\
&\leqslant \varepsilon\, \frac{\delta^{|\mu|}}{A^*_{|\mu|}} \cdot \frac{R}{\delta} \cdot 2^n \leqslant \frac{\varepsilon}{2} \cdot \frac{\delta^{|\mu|}}{A^*_{|\mu|}}
\end{aligned}$$

for all $i, j > N$, since for any $\mu \in \mathbb{N}^n$, we have (see Ishimura and Jin (2019, Lemma 3))

$$\sum_{\nu<\mu} \frac{\nu!}{\mu!} \leqslant 2^n.$$

Finally, applying (5), we have that there exists $N > 0$ such that

$$\left|\widehat{T}_\mu^i - \widehat{T}_\mu^j\right| \leqslant \left|\frac{1}{c_\mu^\mu}\left(\widehat{S}_\mu^i - \widehat{S}_\mu^j\right)\right| + \left|\frac{1}{c_\mu^\mu} \sum_{\nu<\mu} \frac{\nu!}{\mu!} c_\mu^\nu \left(\widehat{T}_\nu^i - \widehat{T}_\nu^j\right)\right| \leqslant \varepsilon\, \frac{\delta^{|\mu|}}{A^*_{|\mu|}}$$

whenever $i, j > N$, as desired. □

Theorem 3 – *Let $\rho > 1$ and a continuous linear operator $P\colon E_0^{\rho(r)} \to E_0^{\rho(r)}$ be of Korobeĭnik type as partial differential operator. Suppose that the following conditions hold:*

1. *there exist $C, \kappa > 0$ such that for all $\mu \in \mathbb{N}^n$, we have that*

$$C\,\kappa^{|\mu|} \leqslant \left|\sum_{\lambda\leqslant\mu} \frac{\mu!}{\lambda!} a_{\mu-\lambda}^{\mu-\lambda}\right|;$$

2. *there exists $R > 0$ such that for all $\nu < \mu$, we have that*

$$\frac{\left|\sum_{\lambda\leqslant\mu} \frac{1}{\lambda!} a_{\mu-\lambda}^{\nu-\lambda}\right|}{\left|\sum_{\lambda\leqslant\mu} \frac{1}{\lambda!} a_{\mu-\lambda}^{\mu-\lambda}\right|} \leqslant \frac{A_{|\mu|}}{A_{|\nu|}} \cdot \frac{\nu!}{\mu!} R^{|\mu-\nu|}.$$

Then $P\colon E_0^{\rho(r)} \to E_0^{\rho(r)}$ is surjective; so is an epimorphism of Fréchet spaces.

Proof. Considering Lemma 4 (c) and the proof of Theorem 2, we only need to prove that there exists $\delta > 0$ such that for any $\varepsilon > 0$, there exists $N > 0$ such that

$$\left|\widehat{T}^i_\mu - \widehat{T}^j_\mu\right| \leqslant \varepsilon \frac{A_{|\mu|}}{\mu!} \delta^{|\mu|} \tag{12}$$

for all $i, j > N$ and $|\mu| \in \mathbb{N}$. In the rest proof of Theorem 2, replacing A^*_q (resp., $A^*_{|\nu|}$) by $\mu!/A_{|\mu|}$ (resp., $\nu!/A_{|\nu|}$), we have the desired inequality (12). □

Theorem 4 – *Let $\rho > 1$ and a continuous linear operator $P\colon E_0^{\rho(r)} \to E_0^{\rho(r)}$ be of Korobeĭnik type as partial differential operator. Suppose that the following conditions hold:*

1. *there exist $C, \kappa > 0$ such that for all $\mu \in \mathbb{N}^n$, we have that*

$$C\kappa^{|\mu|} \leqslant \left|\sum_{\lambda \leqslant \mu} \frac{\mu!}{\lambda!} a^{\mu-\lambda}_{\mu-\lambda}\right|;$$

2. *there exists $R > 0$ such that for all $\nu < \mu$, we have that*

$$\frac{\left|\sum_{\lambda\leqslant\mu} \frac{1}{\lambda!} a^{\nu-\lambda}_{\mu-\lambda}\right|}{\left|\sum_{\lambda\leqslant\mu} \frac{1}{\lambda!} a^{\mu-\lambda}_{\mu-\lambda}\right|} \leqslant \frac{A_{|\mu|}}{A_{|\nu|}} \cdot \frac{R^{|\mu-\nu|}}{|\mu|^{n-1}}.$$

Then $P\colon E_0^{\rho(r)} \to E_0^{\rho(r)}$ is surjective; so is an epimorphism of Fréchet spaces.

Proof. Considering Lemma 4 (c) and the proof of Theorem 2, we only need to prove that there exists $\delta > 0$ such that for any $\varepsilon > 0$, there exists $N > 0$ such that (12) holds for all $i, j > N$ and $|\mu| \in \mathbb{N}$. We prove it by induction on $|\mu| \in \mathbb{N}$.

For the similar reason of (11), we have that there exists $K > 0$ such that for any $\varepsilon > 0$, there exists $N_1 > 0$ such that

$$\frac{1}{|c^\mu_\mu|}\left|\widehat{S}^i_\mu - \widehat{S}^j_\mu\right| \leqslant \frac{\varepsilon K^{|\mu|}}{2} \cdot \frac{A_{|\mu|}}{\mu!}$$

whenever $i, j > N_1$ and $|\mu| \in \mathbb{N}$. In view of (7), we may choose some $\delta > \max\{R, K\}$ such that

$$\frac{1}{m^{n-1}} \sum_{k=1}^{m} \left(\frac{R}{\delta}\right)^k \mathsf{H}^n_{m-k} \leqslant \sum_{k=1}^{\infty} \left(\frac{R}{\delta}\right)^k \frac{\mathsf{H}^n_{m-1}}{m^{n-1}} = \frac{R}{\delta - R} \cdot \frac{\mathsf{H}^n_{m-1}}{m^{n-1}} < \frac{1}{2}$$

for all $m \in \mathbb{N}$.

It is obvious that there exists $N > N_1$ such that (12) holds for all $i, j > N$ and $|\mu| = 0$. Assume that (10) holds for $|\mu| = q-1$. When $|\mu| = q$, we have that

$$\frac{1}{|c_\mu^\mu|}\left|\sum_{\nu<\mu}\frac{\nu!}{\mu!}c_\mu^\nu\left(\widehat{T}_\nu^i - \widehat{T}_\nu^j\right)\right| \leqslant \sum_{\nu<\mu}\frac{\nu!}{\mu!}\cdot\frac{A_{|\mu|}}{A_{|\nu|}}\cdot\frac{R^{|\mu-\nu|}}{|\mu|^{n-1}}\cdot\varepsilon\,\frac{A_{|\nu|}}{\nu!}\,\delta^{|\nu|}$$
$$= \varepsilon\,\frac{A_{|\mu|}}{\mu!}\cdot\frac{\delta^{|\mu|}}{|\mu|^{n-1}}\sum_{\nu<\mu}\left(\frac{R}{\delta}\right)^{|\mu-\nu|} \leqslant \varepsilon\,\frac{A_{|\mu|}}{\mu!}\cdot\frac{\delta^{|\mu|}}{2}$$

for all $i, j > N$. Finally, applying (5), we have that there exists $N > 0$ such that

$$\left|\widehat{T}_\mu^i - \widehat{T}_\mu^j\right| \leqslant \left|\frac{1}{c_\mu^\mu}\left(\widehat{S}_\mu^i - \widehat{S}_\mu^j\right)\right| + \left|\frac{1}{c_\mu^\mu}\sum_{\nu<\mu}\frac{\nu!}{\mu!}c_\mu^\nu\left(\widehat{T}_\nu^i - \widehat{T}_\nu^j\right)\right| \leqslant \varepsilon\,\frac{A_{|\mu|}}{\mu!}\,\delta^{|\mu|}$$

whenever $i, j > N$, as desired. □

Remark 1 – Theorem 3 (2) and Theorem 4 (2) are not comparable. For example, let $n \geqslant 3$ and

$$\frac{|c_\mu^\nu|}{|c_\mu^\mu|} := \begin{cases} \dfrac{A_{|\mu|}}{A_{|\nu|}}\cdot\dfrac{\nu!}{\mu!} & |\mu-\nu| = 1 \text{ and } \nu < \mu \\ 0 & \text{otherwise} \end{cases}.$$

Then we see that Theorem 3 (2) is satisfied but Theorem 4 (2) is not satisfied. On the other hand, let $n \geqslant 1$, $N \in \mathbb{N}$, and

$$\frac{|c_\mu^\nu|}{|c_\mu^\mu|} := \begin{cases} \dfrac{A_{|\mu|}}{A_{|\nu|}}\cdot\dfrac{1}{|\mu|^{n-1}} & 0 \leqslant |\nu| \leqslant N \text{ and } \nu < \mu \\ 0 & \text{otherwise} \end{cases}.$$

Then we see that Theorem 4 (2) is satisfied but Theorem 3 (2) is not satisfied.

Example 2 – Let $n = 1$, $\rho(r) \equiv \rho > 1$. In the following cases, $P = \sum_{\alpha\in\mathbb{N}^n} a_\alpha(z)D_z^\alpha$ satisfies the conditions of Theorem 2–4.

1. with $k > 0$, define

$$a_\alpha^\beta := \begin{cases} \dfrac{k^\alpha}{\alpha!} & \beta = \alpha \\ \dfrac{(-k)^{\alpha-1}}{(\alpha-1)!} & \beta = \alpha - 1 \\ 0 & \text{otherwise} \end{cases}.$$

2. with $k \neq -1$, define

$$a_\alpha^\beta := \begin{cases} \dfrac{k^\alpha}{\alpha!} & \beta = \alpha \\ \dfrac{(-1)^\beta}{\beta!}\left(\dfrac{1}{(\alpha-\beta)!}\right)^2 & \beta < \alpha \\ 0 & \text{otherwise} \end{cases}.$$

Acknowledgments

Thank Professor R. Ishimura for giving author a lot of advice.

References

Aoki, T., F. Colombo, et al. (2018a). “Continuity of some operators arising in the theory of superoscillations”. *Quantum Stud. Math. Found.* **5** (3), pp. 463–476 (cit. on p. 49).

Aoki, T., F. Colombo, et al. (2018b). “Continuity theorems for a class of convolution operators and applications to superoscillations”. *Ann. Mat. Pura Appl.* **197** (5), pp. 1533–1545 (cit. on p. 50).

Aoki, T., R. Ishimura, et al. (May 27, 2020). “Characterization of Continuous Endomorphisms in the Space of Entire Functions of a Given Order”. *Complex Variables and Elliptic Equations*. URL: https://doi.org/10.1080/17476933.2020.1767086 (cit. on p. 50).

Ishimura, R. and X. Jin (2019). “Infinite order differential equations in the space of entire functions of normal type with respect to a proximate order”. *North-W. Eur. J. of Math.* **5**, pp. 69–87 (cit. on pp. 50–53, 57, 62).

Ishimura, R. and K. Miyake (2007). “Endomorphisms of the space of entire functions with proximate order and infinite order differential operators”. *Far East J. Math. Sci.* **26** (1), pp. 91–103 (cit. on p. 50).

Lelong, P. and L. Gruman (1986). *Entire functions of several complex variables*. Grund. Math. Wiss., vol.282. New York: Springer-Verlag (cit. on pp. 51, 57).

Martineau, A. (1967). “Équations différentielles d’ordre infini”. *Bull. Soc. Math. France* **95**, pp. 109–154 (cit. on p. 49).

Momm, S. (1990). “Partial differential operators of infinite order with constant coefficients on the space of analytic functions on the polydisc”. *Studia Math.* **96** (1), pp. 51–71 (cit. on p. 49).

Contents