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On the sum of dilates in $\mathbb{R}^d$

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Abstract

Let A be a nonempty finite subset of $\mathbb{Z}^d$ which is not contained in a hyperplane, $q \in \mathbb{Z}$ with $|q| > 1$ and $m \in \mathbb{Z}$ such that $|q| + 2d - 1 \leq m \leq (|q| + 2d - 1)^2$. In this paper it is shown that

$$|A + q \cdot A| \geq \left(\frac{m}{|q| + 2d - 1}\right)|A| - c$$

where c depends only on q, d and m. In particular, taking $m = (|q| + 2d - 1)^2$, this results confirms a conjecture of A. Balog and G. Shakan.

Keywords: sumsets, dimension, affine hull.

MSC: 11B30, 11B13.

1 Introduction

We denote by $\mathbb{Z}, \mathbb{N}$ and $\mathbb{R}$ the set of integers, natural numbers and real numbers, respectively; we consider $0 \notin \mathbb{N}$. In this paper, d will denote a nonnegative integer. For any nonempty subsets A and A' of $\mathbb{R}^d$ and $q \in \mathbb{R}$, set

$$\begin{aligned} A + A' &:= \{\mathbf{a} + \mathbf{a}' : \ \mathbf{a} \in A, \mathbf{a}' \in A'\} \\ -A &:= \{-\mathbf{a} : \ \mathbf{a} \in A\} \\ q \cdot A &:= \{q\mathbf{a} : \ \mathbf{a} \in A\}. \end{aligned}$$

Let V be a $\mathbb{R}$-vector space. An *affine subspace* W of V is a translation of a linear subspace W' of V; we write $\dim W := \dim W'$. The minimal affine subspace containing A is known as the *affine hull* and we will denote it by $\mathrm{aff}A$; set $\dim A := \dim \mathrm{aff}A$. The canonical basis of $\mathbb{R}^d$ will be denoted by $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_d\}$. Set $\mathbb{Z}^d := \left\{\sum_{j=1}^d n_j \mathbf{e}_j : \ n_1, n_2, \ldots, n_d \in \mathbb{Z}\right\}$ and $\mathbf{0} := \sum_{i=1}^d 0\mathbf{e}_i$.

The study of the sum of dilates has had new and interesting results in the last few years, see Balog and Shakan 2014, Balog and Shakan 2015, Cilleruelo, Hamidoune, and O. 2009, Cilleruelo, M., and Vinuesa 2010, Du, Cao, and Sun 2015,

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Fiz Pontiveros 2013, Hamidoune and Rué 2011, Ljujic 2013, Plagne 2011, Shakan 2016. Most of these results deal with the sum of dilations in one-dimensional spaces. However, A. Balog and G. Shakan proved the following high-dimensional result.

Theorem 1 – *Let A be a nonempty finite subset of $\mathbb{Z}^d$ with $\dim A = d > 1$ and $q \in \mathbb{Z}$ such that $|q| > 1$. Then*

$$|A + q \cdot A| \geq (|q| + d + 1)|A| - c \tag{1}$$

where c is a constant which depends only on q and d.

Proof. See Balog and Shakan 2015, Thm. 3. □

Balog and Shakan conjectured that the coefficient $|q|+d+1$ in (1) could be improved to $|q| + 2d - 1$, see Balog and Shakan 2015, Conj. 1. Furthermore, they gave an example which shows that $|q| + 2d - 1$ is the best possible. This conjecture is the main motivation of this paper. To state the main result, write for all $i, j \in \mathbb{Z}$,

$$c_{i,j} := 4^{j^{4^i}}.$$

Theorem 2 – *Let A be a nonempty finite subset of $\mathbb{Z}^d$, $q \in \mathbb{Z}$ with $|q| > 1$ and $m \in \mathbb{Z}$ such that $|q| + 2\dim A - 1 \leq m \leq (|q| + 2\dim A - 1)^2$. Then*

$$|A + q \cdot A| \geq \frac{m}{|q| + 2\dim A - 1}|A| - c_{\dim A, m}.$$

The particular case $\dim A = d$ and $m = (|q| + 2\dim A - 1)^2$ confirms the conjecture of Balog and Shakan.

We sketch the content of this paper. In Section 2 we state auxiliary results that will be needed later. In Section 3 we shall study the partitions $A = A_1 \uplus A_2$ such that

$$(A_1 + q \cdot A) \cap (A_2 + q \cdot A) = \varnothing. \tag{2}$$

Also, in Section 3, we study the partitions $A = A_1 \uplus A_2$ which satisfy the stronger condition: for all $i, j, i', j' \in \{1, 2\}$ with $(i, j) \neq (i', j')$, we have that

$$(A_i + q \cdot A_j) \cap (A_{i'} + q \cdot A_{j'}) = \varnothing.$$

The most important results of Section 3 will be Lemma 7 and Lemma 8, and they will be fundamental tools in the proof of Theorem 2. The proof of Theorem 2 will be done by induction on $\dim A$ and m. We will take a partition $A = A_1 \uplus A_2$ with A_1 and A_2 nonempty satisfying (2), and we will proceed using the hypothesis of induction depending on whether

i) $\dim A_1 < \dim A$ or $\dim A_2 < \dim A$.

ii) $\dim A_1 = \dim A_2 = \dim A$.

Case ii) follows from the ideas that Balog and Shakan used in Balog and Shakan 2014, Balog and Shakan 2015 and Shakan 2016. Case i) is the one where more work needs to be done. The conclusion of the proof of Theorem 2 is done in Section 4.

2 Preliminaries

In this section we will state some auxiliary results that will be needed in the proof of Theorem 2. We start with a fundamental result.

Theorem 3 – *Let A_1 and A_2 be nonempty finite subsets of $\mathbb{R}^d$. Then*

$$|A_1 + A_2| \geq |A_1| + |A_2| - 1.$$

Proof. See Grynkiewicz 2013, Thm. 3.1. □

For any affine subspace V of $\mathbb{R}^d$, we denote by $\mathbb{R}^d/V$ the set of equivalence classes with respect to the relation $\mathbf{a} \sim \mathbf{b}$ if $\{\mathbf{a}\} + V = \{\mathbf{b}\} + V$; we consider $\mathbb{R}^d/V$ with its usual $\mathbb{R}$–vector space structure, and we denote by $\pi_V : \mathbb{R}^d \to \mathbb{R}^d/V$ the canonical projection. For any subset A of $\mathbb{R}^d/V$, we have that the dimension of the affine hull of A is at most $|A| - 1$. This implies easily the following fact.

Remark 1 – Let A_1 and A_2 be nonempty finite subsets of $\mathbb{R}^d$. Then

$$\begin{aligned}|\pi_{\operatorname{aff} A_1}(A_1 + A_2)| &\geq 1 + \dim \pi_{\operatorname{aff} A_1}(\operatorname{aff}(A_1 + A_2)) \\ &= 1 + \dim \operatorname{aff}(A_1 + A_2) - \dim \operatorname{aff} A_1 \\ &= 1 + \dim(A_1 + A_2) - \dim A_1.\end{aligned}$$

We shall need two consequences of Theorem 3.

Corollary 1 – *Let A_1 and A_2 be nonempty finite subsets of $\mathbb{R}^d$. Then*

$$|A_1 + A_2| \geq |\pi_{\operatorname{aff} A_1}(A_2)|(|A_1| - 1) + |A_2| \geq |\pi_{\operatorname{aff} A_1}(A_2)||A_1|.$$

Proof. Since $|\pi_{\operatorname{aff} A_1}(A_2)| \leq |A_2|$, it suffices to prove the left-hand side inequality. Set $\pi := \pi_{\operatorname{aff} A_1}$ and $\pi(A_2) = \{z_1, z_2, \dots, z_n\}$. For each $i \in \{1, 2, \dots, n\}$, write $B_i := \pi^{-1}(z_i) \cap A_2$. On the one hand, for each $i, j \in \{1, 2, \dots, n\}$ with $i \neq j$, we have that B_i and B_j are contained in distinct translations of $\operatorname{aff} A_1$; therefore $A_1 + B_i$ and $A_1 + B_j$ are contained in distinct translations of $\operatorname{aff} A_1$ and hence

$$|A_1 + A_2| = \sum_{i=1}^{n} |A_1 + B_i|. \tag{3}$$

On the other hand, for each $i \in \{1, 2, \dots, n\}$, Theorem 3 leads to

$$|A_1 + B_i| \geq |A_1| + |B_i| - 1. \tag{4}$$

Finally

$$\begin{aligned}
|A_1 + A_2| &= \sum_{i=1}^{n} |A_1 + B_i| && \big(\text{by (3)}\big)\\
&\geq \sum_{i=1}^{n} (|A_1| - 1) + |B_i| && \big(\text{by (4)}\big)\\
&= \sum_{i=1}^{n} (|A_1| - 1) + \sum_{i=1}^{n} |B_i|\\
&= |\pi(A_2)|(|A_1| - 1) + |A_2|.
\end{aligned}$$

□

Corollary 2 – *Let* A_1 *and* A_2 *be nonempty finite subsets of* $\mathbb{R}^d$*. Set* $d_1 := \dim A_1$, $d_2 := \dim A_2$ *and* $d_3 := \dim(A_1 + A_2)$*. Then*

$$|A_1 + A_2| \geq (1 + d_3 - d_1)|A_1| + (1 + d_3 - d_2)|A_2| - (1 + d_3 - d_1)(1 + d_3 - d_2).$$

Proof. The proof is by induction on d_3. If $d_3 = 0$, then $|A_1 + A_2| = |A_1| = |A_2| = 1$ and $d_1 = d_2 = 0$ so the statement is trivial. We assume from now on that $d_3 > 0$ and that the statement holds for all $0 \leq d' < d_3$. If $d_3 = d_1$ and $d_3 = d_2$, then the statement follows from Theorem 3. It remains to complete the induction when $d_1 < d_3$ or $d_2 < d_3$; without loss of generality assume that $d_1 < d_3$. Also, translating if necessary, we assume that A_1 and A_2 contain the origin; hence

$$\dim(\operatorname{aff}A_1 \cap \operatorname{aff}A_2) + \dim(\operatorname{aff}A_1 + \operatorname{aff}A_2) = \dim \operatorname{aff}A_1 + \dim \operatorname{aff}A_2. \tag{5}$$

Set $\pi := \pi_{\operatorname{aff}A_1}$. Write $\pi(A_2) = \{z_1, z_2, \ldots, z_n\}$ and $B_i := \pi^{-1}(z_i) \cap A_2$ for each $i \in \{1, 2, \ldots, n\}$. Take $i \in \{1, 2, \ldots, n\}$, and note that B_i is contained in a translation of $\operatorname{aff}A_1$. Moreover, since $B_i \subseteq A_2$, it is also contained in $\operatorname{aff}A_2$, and therefore B_i is contained in a translation of $\operatorname{aff}A_1 \cap \operatorname{aff}A_2$. Insomuch as $\operatorname{aff}(A_1 + A_2) = \operatorname{aff}A_1 + \operatorname{aff}A_2$, we conclude by (5) that

$$\dim B_i \leq \dim(\operatorname{aff}A_1 \cap \operatorname{aff}A_2) = d_1 + d_2 - d_3. \tag{6}$$

Since B_i is contained in a translation of $\operatorname{aff}A_1$, the affine hull of $A_1 + B_i$ is a translation of $\operatorname{aff}A_1$; in particular $\dim A_1 + B_i = d_1 < d_3$. Thus we can apply the induction hypothesis on the pair (A_1, B_i), and we obtain that

$$\begin{aligned}
|A_1 + B_i| &\geq |A_1| + (1 + d_1 - \dim B_i)|B_i| - (1 + d_1 - \dim B_i)\\
&\geq |A_1| + (1 + d_3 - d_2)|B_i| - (1 + d_3 - d_2). && \big(\text{by (6)}\big)
\end{aligned} \tag{7}$$

Set $\pi_i := \pi_{\operatorname{aff}B_i}$. Recall that $\dim(A_1 + B_i) = d_1$. Hence Remark 1 applied to the pair (B_i, A_1) leads to

$$|\pi_i(B_i + A_1)| \geq 1 + d_1 - \dim B_i. \tag{8}$$

Corollary 1 applied to (B_i, A_1) leads to

$$|B_i + A_1| \geq |\pi_i(A_1)||B_i|, \tag{9}$$

and thus

$$\begin{aligned} |A_1 + B_i| &\geq |\pi_i(A_1)||B_i| && \big(\text{by } (9)\big) \\ &= |\pi_i(B_i + A_1)||B_i| \\ &\geq (1 + d_1 - \dim B_i)|B_i| && \big(\text{by } (8)\big) \\ &\geq (1 + d_3 - d_2)|B_i|. && \big(\text{by } (6)\big) \end{aligned} \tag{10}$$

From Remark 1 applied to the pair (A_1, A_2),

$$n = |\pi(A_1 + A_2)| \geq 1 + d_3 - d_1.$$

On the one hand, (7) yields

$$\begin{aligned} \sum_{i=1}^{1+d_3-d_1} |A_1 + B_i| \geq (1 + d_3 - d_1)|A_1| + (1 + d_3 - d_2)\left(\sum_{i=1}^{1+d_3-d_1} |B_i|\right) \\ - (1 + d_3 - d_1)(1 + d_3 - d_2). \end{aligned} \tag{11}$$

On the other hand, (10) leads to

$$\sum_{i=2+d_3-d_1}^{n} |A_1 + B_i| \geq (1 + d_3 - d_2) \sum_{i=2+d_3-d_1}^{n} |B_i|. \tag{12}$$

For each $i, j \in \{1, 2, \ldots, n\}$ with $i \neq j$, $A_1 + B_i$ and $A_1 + B_j$ are contained in distinct translations of $\operatorname{aff} A_1$; in particular they are disjoint and therefore

$$|A_1 + A_2| = \sum_{i=1}^{n} |A_1 + B_i|. \tag{13}$$

Finally

$$\begin{aligned} |A_1 + A_2| &= \sum_{i=1}^{n} |A_1 + B_i| && \big(\text{by } (13)\big) \\ &\geq (1 + d_3 - d_1)|A_1| + (1 + d_3 - d_2)|A_2| \\ &\quad - (1 + d_3 - d_1)(1 + d_3 - d_2), && \big(\text{by } (11),(12)\big) \end{aligned}$$

and this completes the induction. □

For any nonempty subsets A_1 and A_2 of $\mathbb{R}^d$, set

$$\delta(A_1,A_2) := \begin{cases} 1 & \text{if } \dim A_1 = \dim(A_1 \cup A_2) \text{ or } \dim A_2 = \dim(A_1 \cup A_2); \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 1 – *Let* A_1 *and* A_2 *be nonempty subsets of* $\mathbb{R}^d$ *such that* $\dim(A_1 \cup A_2) = d$ *and* $q \in \mathbb{Z}$. *Then*

$$\dim(A_1 + q \cdot A_2) \geq d - 1 + \delta(A_1, A_2)$$

Proof. Since $A_1 + q \cdot A_2$ contains a copy of A_1, if $\dim A_1 = d$, then $d = \dim A_1 \leq \dim(A_1 + q \cdot A_2)$. In the same way, if $\dim A_2 = d$, then $\dim q \cdot A_2 = d$ and thereby $d = \dim q \cdot A_2 \leq \dim(A_1 + q \cdot A_2)$.

Now we prove that $\dim(A_1 + q \cdot A_2) \geq d - 1$ for arbitrary A_1 and A_2. Set $\pi := \pi_{\mathrm{aff} A_1}$. Hence

$$\begin{aligned} \dim \pi(A_1 + q \cdot A_2) &= \dim(A_1 + q \cdot A_2) - \dim A_1 \\ \dim \pi(A_1 \cup A_2) &= \dim(A_1 \cup A_2) - \dim A_1. \end{aligned} \tag{14}$$

Thus, since π is linear, we have that in $\mathbb{R}^d / \mathrm{aff} A_1$

$$\dim \pi(A_1 + q \cdot A_2) = \dim \pi(q \cdot A_2) = \dim \pi(A_2). \tag{15}$$

Also, insomuch as $|\pi(A_1)| = 1$,

$$\dim \pi(A_1 \cup A_2) = \dim(\pi(A_1) \cup \pi(A_2)) \leq (\dim \pi(A_2)) + 1. \tag{16}$$

Then

$$\begin{aligned} \dim(A_1 + q \cdot A_2) &= \dim \pi(A_1 + q \cdot A_2) + \dim A_1 && \big(\text{by } (14)\big) \\ &= \dim \pi(A_2) + \dim A_1 && \big(\text{by } (15)\big) \\ &\geq (\dim \pi(A_1 \cup A_2)) - 1 + \dim A_1 && \big(\text{by } (16)\big) \\ &= (d - \dim A_1) - 1 + \dim A_1 && \big(\text{by } (14)\big) \\ &= d - 1. \end{aligned}$$

□

For any affine subspace V of $\mathbb{R}^d$, we say that V is *defined over* $\mathbb{Z}$ if there are linear equations over $\mathbb{Z}$ such that V is its solution space. Notice that if there is $A \subseteq \mathbb{Z}^d$ such that $\mathrm{aff} A = V$, then V is defined over $\mathbb{Z}$. We will need some results of the geometry of numbers. We start with an easy consequence of Cassels 1997, Ch. 1 Cor. 3.

Lemma 2 – *Let V be a linear subspace of $\mathbb{R}^d$ defined over $\mathbb{Z}$. Then there is $\{\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_d\}$ a basis of $\mathbb{Z}^d$ such that $\{\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_{\dim V}\}$ is a basis of $\mathbb{Z}^d \cap V$.*

For any nonempty subset A of $\mathbb{Z}^d$, we denote by $\langle A \rangle$ the subgroup of $\mathbb{Z}^d$ generated by A. We say that A is $\mathbb{Z}^d$*-reduced* if $\langle A - A \rangle = \mathbb{Z}^d$; if no confusion is possible with d, we simply say that A is *reduced*. Note that if A is $\mathbb{Z}^d$-reduced, then $\dim A = d$. For arbitrary subsets A of $\mathbb{Z}^d$ with $\dim A = d$, $\langle A - A \rangle$ is a sublattice of $\mathbb{Z}^d$. Thus we get the following fact.

Remark 2 – Let A be a nonempty finite subset of $\mathbb{Z}^d$. If $\dim A = d$, there is a bijective affine map $\phi : \mathbb{R}^d \to \mathbb{R}^d$ such that $\phi(A)$ is reduced.

Given an ordered basis $\mathcal{B} = \{\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_d\}$ of $\mathbb{Z}^d$ and an ordered subset $\mathcal{B}' = \{\mathbf{f}_{i_1}, \mathbf{f}_{i_2}, \ldots, \mathbf{f}_{i_k}\}$ of $\mathcal{B}$, define

$$\pi_{\mathcal{B},\mathcal{B}'} : \mathbb{R}^d \longrightarrow \mathbb{R}^k, \qquad \pi_{\mathcal{B},\mathcal{B}'}\left(\sum_{i=1}^{d} z_i \mathbf{f}_i\right) = \sum_{j=1}^{k} z_{i_j} \mathbf{e}_j.$$

Note that $\pi_{\mathcal{B},\mathcal{B}'}(\mathbb{Z}^d) = \mathbb{Z}^k$. Moreover, we have the following trivial fact.

Remark 3 – Let $\mathcal{B}$ be an ordered basis of $\mathbb{Z}^d$ and $\mathcal{B}'$ be a nonempty ordered subset of $\mathcal{B}$ with $k := |\mathcal{B}'|$. If A is $\mathbb{Z}^d$-reduced, then $\pi_{\mathcal{B},\mathcal{B}'}(A)$ is $\mathbb{Z}^k$-reduced.

Before we conclude this section, we recall two results of Balog and Shakan. Let A be a nonempty subset of $\mathbb{Z}^d$, $q \in \mathbb{Z}$ and $\pi : \mathbb{Z}^d \to \mathbb{Z}^d / q \cdot \mathbb{Z}^d$ the canonical projection. We say that A is *q-domain* if $\pi(A) = \mathbb{Z}^d / q \cdot \mathbb{Z}^d$.

Lemma 3 – *Let A be a nonempty subset of $\mathbb{Z}^d$ and $q \in \mathbb{Z}$ with $|q| > 1$. If A is q-domain, then*

$$|A + q \cdot A| \geq \left(d + |q|^d\right)|A| - \frac{d(d+1)}{2}|q|^d.$$

Proof. See Balog and Shakan 2015, Lemma 1. □

Lemma 4 – *Let A be a nonempty subset of $\mathbb{Z}^d$, $q \in \mathbb{Z}$ with $|q| > 1$ and $\pi : \mathbb{Z}^d \to \mathbb{Z}^d / q \cdot \mathbb{Z}^d$ be the canonical projection. Set $\pi(A) = \{z_1, z_2, \ldots, z_n\}$ and take $\mathbf{b}_1, \mathbf{b}_2, \ldots \mathbf{b}_n \in A$ such that $\pi(\mathbf{b}_i) = z_i$ for each $i \in \{1, 2, \ldots, n\}$. Write $B_i := \pi^{-1}(z_i) \cap A$ and $B_i' := q^{-1} \cdot (B_i - \{\mathbf{b}_i\})$ for each $i \in \{1, 2, \ldots, n\}$. For all $i \in \{1, 2, \ldots, n\}$, if B_i' is not q-domain, then*

$$|B_i + q \cdot A| \geq |B_i + q \cdot B_i| + \min_{1 \leq j \leq n} |B_j|.$$

Proof. See Balog and Shakan 2015, Lemma 4. □

3 q-weak partitions and q-partitions

Let A_1 and A_2 be nonempty subsets of $\mathbb{Z}^d$ and $q \in \mathbb{Z}$. We say that (A_1, A_2) is a *q-weak pair* if

$$(A_1 + q\cdot(A_1 \cup A_2)) \cap (A_2 + q\cdot(A_1 \cup A_2)) = \varnothing.$$

We say that (A_1, A_2) is a *q-pair* if, for all $i, j, i', j' \in \{1,2\}$ with $(i,j) \neq (i',j')$, we have that

$$(A_i + q\cdot A_j) \cap (A_{i'} + q\cdot A_{j'}) = \varnothing.$$

Hence we have that if (A_1, A_2) is a q-pair, then it is a q-weak pair.

Remark 4 – Let $B_1, B_2, \ldots, B_n$ be nonempty subsets of $\mathbb{Z}^d$, $\{1, 2\ldots, n\} = I_1 \uplus I_2$ a partition with I_1 and I_2 nonempty, and $q \in \mathbb{Z}$.

i) Assume that for all $i, j \in \{1, 2, \ldots, n\}$ with $i \neq j$,

$$\left(B_i + q\cdot\bigcup_{i=1}^{n} B_i\right) \cap \left(B_j + q\cdot\bigcup_{i=1}^{n} B_i\right) = \varnothing.$$

Then $\left(\bigcup_{i\in I_1} B_i, \bigcup_{i\in I_2} B_i\right)$ is a q-weak pair.

ii) Assume that for all $i, j, i', j' \in \{1, 2, \ldots, n\}$ with $(i,j) \neq (i',j')$,

$$\left(B_i + q\cdot B_j\right) \cap \left(B_{i'} + q\cdot B_{j'}\right) = \varnothing.$$

Then $\left(\bigcup_{i\in I_1} B_i, \bigcup_{i\in I_2} B_i\right)$ is a q-pair.

Let A be a subset of $\mathbb{Z}^d$, $A = A_1 \uplus A_2$ a partition and $q \in \mathbb{Z}$. We say that $A = A_1 \uplus A_2$ is a *q-weak partition* if (A_1, A_2) is a q-weak pair. We say that $A = A_1 \uplus A_2$ is a *q-partition* if (A_1, A_2) is a q-pair.

Remark 5 – Let $\mathcal{B}$ be an ordered basis of $\mathbb{Z}^d$, $\mathcal{B}'$ be a nonempty ordered subset of $\mathcal{B}$ and A a subset of $\mathbb{Z}^d$.

i) If $\pi_{\mathcal{B},\mathcal{B}'}(A) = A_1 \uplus A_2$ is a q-weak partition, then

$$A = \left(\pi^{-1}_{\mathcal{B},\mathcal{B}'}(A_1) \cap A\right) \uplus \left(\pi^{-1}_{\mathcal{B},\mathcal{B}'}(A_2) \cap A\right)$$

is a q-weak partition.

ii) If $\pi_{\mathcal{B},\mathcal{B}'}(A) = A_1 \uplus A_2$ is a q-partition, then

$$A = \left(\pi^{-1}_{\mathcal{B},\mathcal{B}'}(A_1) \cap A\right) \uplus \left(\pi^{-1}_{\mathcal{B},\mathcal{B}'}(A_2) \cap A\right)$$

is a q-partition.

The next two results study the existence of q-weak partitions and q-partitions of A when A is $\mathbb{Z}^d$-reduced and $|A|$ is small.

Lemma 5 – *Let A be a reduced subset of $\mathbb{Z}^d$ and $q \in \mathbb{Z}$ with $|q| > 1$. If $|A| \leq 2d$, then there is a q-partition $A = A_1 \uplus A_2$.*

Proof. The proof will be done by induction on d. First assume that $d = 1$. Since A is $\mathbb{Z}$-reduced, $\langle A - A \rangle = \mathbb{Z}$, and therefore $|A| > 1$. Insomuch as $|A| \leq 2d = 2$, we conclude that $|A| = 2$; write $A = \{\mathbf{a}_1, \mathbf{a}_2\}$. We have that $A = \{\mathbf{a}_1\} \uplus \{\mathbf{a}_2\}$ is a q-partition and the basis of induction is proven. From now on we assume that $d > 1$ and that the claim holds for all $1 \leq d' < d$. Let $\pi : \mathbb{Z}^d \to \mathbb{Z}^d / q \cdot \mathbb{Z}^d$ be the canonical projection and write $\pi(A) = \{z_1, z_2, \ldots, z_n\}$. Set $B_i := \pi^{-1}(z_i) \cap A$ for each $i \in \{1, 2, \ldots, n\}$. Inasmuch as $\langle A - A \rangle = \mathbb{Z}^d$, we have that $\pi(A) - \pi(A)$ generates $\mathbb{Z}^d / q \cdot \mathbb{Z}^d$ and therefore $n \geq 2$ (since $|q| > 1$). We deal with two cases.

- Assume that for all $i, j, i', j' \in \{1, 2, \ldots, n\}$ with $(i, j) \neq (i', j')$,

  $$\left(B_i + q \cdot B_j\right) \cap \left(B_{i'} + q \cdot B_{j'}\right) = \varnothing.$$

  Then Remark 4 ii) implies that $A = B_1 \uplus (A \setminus B_1)$ is a q-partition.

- Assume that there are $i, j, i', j' \in \{1, 2, \ldots, n\}$ with $(i, j) \neq (i', j')$ such that

  $$\left(B_i + q \cdot B_j\right) \cap \left(B_{i'} + q \cdot B_{j'}\right) \neq \varnothing. \tag{17}$$

  Since $\pi(B_i + q \cdot B_j) = \{z_i\}$ and $\pi(B_{i'} + q \cdot B_{j'}) = \{z_{i'}\}$, we get from (17) that $i = i'$. Let $\mathbf{a}, \mathbf{a}' \in B_i$, $\mathbf{b} \in B_j$ and $\mathbf{b}' \in B_{j'}$ be such that

  $$\mathbf{a} + q\mathbf{b} = \mathbf{a}' + q\mathbf{b}'. \tag{18}$$

  Insomuch as $(i, j) \neq (i', j')$ and $i = i'$, we notice that $\mathbf{a} \neq \mathbf{a}'$ (otherwise $\mathbf{b} = \mathbf{b}'$ and thereby $j = j'$). Let V be the linear subspace of $\mathbb{R}^d$ generated by $\mathbf{a} - \mathbf{a}'$. Since $\mathbf{a} - \mathbf{a}' \in \mathbb{Z}^d \setminus \{\mathbf{0}\}$, note that V is a 1-dimensional subspace of $\mathbb{R}^d$ defined over $\mathbb{Z}$. From Lemma 2, there is an ordered basis $\mathcal{B} := \{\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_d\}$ of $\mathbb{Z}^d$ such that $\{\mathbf{f}_d\}$ is a basis of $\mathbb{Z}^d \cap V$. Set the ordered set $\mathcal{B}' := \{\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_{d-1}\}$. On the one hand, $\pi_{\mathcal{B},\mathcal{B}'}(A)$ is $\mathbb{Z}^{d-1}$-reduced by Remark 3. On the other hand, since $\mathbf{a} \neq \mathbf{a}'$, we also have that $\mathbf{b} \neq \mathbf{b}'$. However, (18) implies that $\mathbf{a} - \mathbf{a}', \mathbf{b} - \mathbf{b}' \in V$, and hence

  $$\pi_{\mathcal{B},\mathcal{B}'}(\mathbf{a}) = \pi_{\mathcal{B},\mathcal{B}'}(\mathbf{a}') \qquad \text{and} \qquad \pi_{\mathcal{B},\mathcal{B}'}(\mathbf{b}) = \pi_{\mathcal{B},\mathcal{B}'}(\mathbf{b}'). \tag{19}$$

  Since $\mathbf{a} \neq \mathbf{a}'$, $\mathbf{b} \neq \mathbf{b}'$, and $|A| \leq 2d$, we get that $|\pi_{\mathcal{B},\mathcal{B}'}(A)| \leq 2d - 2$ from (19). Then $\pi_{\mathcal{B},\mathcal{B}'}(A) \subseteq \mathbb{Z}^{d-1}$ satisfies the hypothesis of induction, and therefore there is a q-partition $\pi_{\mathcal{B},\mathcal{B}'}(A) = A_1' \uplus A_2'$. Set $A_1 := \pi_{\mathcal{B},\mathcal{B}'}^{-1}(A_1') \cap A$ and $A_2 := \pi_{\mathcal{B},\mathcal{B}'}^{-1}(A_2') \cap A$. Remark 5 ii) implies that $A = A_1 \uplus A_2$ is a q-partition, and this concludes the induction. $\square$

Lemma 6 – *Let A be a reduced subset of $\mathbb{Z}^d$ and $q \in \mathbb{Z}$ with $|q| > 1$. If $|A| \leq 2d+1$, then there is a q-weak partition $A = A_1 \uplus A_2$ such that $|A_1| = 1$.*

Proof. Translating if necessary, we assume that $\mathbf{0} \in A$. Let $\pi : \mathbb{Z}^d \to \mathbb{Z}^d / q \cdot \mathbb{Z}^d$ be the canonical projection. Since

$$\langle A \rangle = \langle A - \{\mathbf{0}\} \rangle = \langle A - A \rangle = \mathbb{Z}^d,$$

we have that $\pi(A)$ generates the group $\mathbb{Z}^d / q \cdot \mathbb{Z}^d$. Insomuch as $|q| > 1$, the group $\mathbb{Z}^d / q \cdot \mathbb{Z}^d$ has rank d. Therefore any subset which generates $\mathbb{Z}^d / q \cdot \mathbb{Z}^d$ has at least d nonneutral elements; hence, since $\mathbf{0} \in A$, we obtain that

$$|\pi(A)| \geq d+1. \tag{20}$$

Write $\pi(A) = \{z_1, z_2, \ldots, z_n\}$ and $B_i := \pi^{-1}(B_i) \cap A$ for each $i \in \{1, 2, \ldots, n\}$. Without loss of generality assume that

$$|B_1| \leq |B_2| \leq \ldots \leq |B_n|. \tag{21}$$

On the one hand, for all $i, j \in \{1, 2, \ldots, n\}$ with $i \neq j$,

$$(B_i + q \cdot A) \cap (B_j + q \cdot A) = \varnothing.$$

Thus Remark 4 i) implies that $A = B_1 \uplus (A \setminus B_1)$ is a q-weak partition. On the other hand, insomuch as $|A| \leq 2d+1$, we get from (20) and (21) that $|B_1| = 1$. □

Let A be a nonempty subset of $\mathbb{R}^d$, and $A = A_1 \uplus A_2$ a partition with A_1 and A_2 nonempty. We say that $A = A_1 \uplus A_2$ is *low-dimensional* if $\dim A_1 < \dim A$ or $\dim A_2 < \dim A$. We say that $A = A_1 \uplus A_2$ is *flat* if there is $i \in \{1, 2\}$ such that $\dim A_i < \dim A$ and $|\pi_{\operatorname{aff} A_i}(A)| \leq 2(\dim A - \dim A_i)$. The previous two lemmas will have useful applications as we shall see in the next two results.

Lemma 7 – *Let A be a reduced subset of $\mathbb{Z}^d$ and $q \in \mathbb{Z}$ with $|q| > 1$. Assume that there is a flat partition of A. Hence there is a q-partition $A = A_1 \uplus A_2$ such that, if $d_1 := \dim A_1$ and $d_2 := \dim A_2$, then*

$$\begin{aligned} |A + q \cdot A| \geq {} & |A_1 + q \cdot A_1| + |A_2 + q \cdot A_2| + 2\delta(A_1, A_2)|A| \\ & + 2(d - d_1)|A_1| + 2(d - d_2)|A_2| - 2(1 + d - d_1)(1 + d - d_2). \end{aligned}$$

Proof. Let $A = A_1' \uplus A_2'$ be a partition with $\dim A_1' < \dim A$ and $|\pi_{\operatorname{aff} A_1'}(A)| \leq 2(d - \dim A_1')$. Set $V := \operatorname{aff} A_1'$ and $k := d - \dim V$; translating if necessary, assume that $\mathbf{0} \in A_1'$ so V is a linear subspace of $\mathbb{R}^d$ defined over $\mathbb{Z}$. From Lemma 2, there is an ordered basis $\mathcal{B} := \{\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_d\}$ of $\mathbb{Z}^d$ such that $\{\mathbf{f}_{k+1}, \mathbf{f}_{k+2}, \ldots, \mathbf{f}_d\}$ is an ordered basis of $V \cap \mathbb{Z}^d$. Set $\mathcal{B}' := \{\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_k\}$ and $\pi := \pi_{\mathcal{B}, \mathcal{B}'}$. The function

$$\phi : \mathbb{R}^k \to \mathbb{R}^d / V, \qquad \phi\left(\sum_{i=1}^{k} a_i \mathbf{e}_i\right) = \pi_V\left(\sum_{i=1}^{k} a_i \mathbf{f}_i\right)$$

is an isomorphism and $\phi \circ \pi = \pi_V$. Thus

$$|\pi(A)| = |\pi_V(A)| \leq 2k. \tag{22}$$

Note that $\pi(A)$ is $\mathbb{Z}^k$-reduced by Remark 3. Thus, from (22), the assumptions of Lemma 5 are satisfied by $\pi(A)$, and therefore there is a q-partition $\pi(A) = A_1'' \uplus A_2''$. Taking $A_1 := \pi^{-1}(A_1'') \cap A$ and $A_2 := \pi^{-1}(A_2'') \cap A$, Remark 5 ii) implies that $A = A_1 \uplus A_2$ is a q-partition. Set $\delta := \delta(A_1, A_2)$. The definition of q-partition yields that the sets $A_i + q \cdot A_j$ for $i, j \in \{1, 2\}$ are pairwise disjoint. Thus

$$|A + q \cdot A| = |A_1 + q \cdot A_1| + |A_2 + q \cdot A_2| + |A_1 + q \cdot A_2| + |A_2 + q \cdot A_1|. \tag{23}$$

Now Lemma 1 applied to (A_1, A_2) and (A_2, A_1) leads to

$$\begin{aligned} \dim(A_1 + q \cdot A_2) &\geq d - 1 + \delta \\ \dim(A_2 + q \cdot A_1) &\geq d - 1 + \delta. \end{aligned} \tag{24}$$

We apply Corollary 2 to $(A_1, q \cdot A_2)$ and $(A_2, q \cdot A_1)$, and we get by (24) that

$$\begin{aligned} |A_1 + q \cdot A_2| &\geq (\delta + d - d_1)|A_1| + (\delta + d - d_2)|A_2| - (\delta + d - d_1)(\delta + d - d_2) \\ |A_2 + q \cdot A_1| &\geq (\delta + d - d_1)|A_1| + (\delta + d - d_2)|A_2| - (\delta + d - d_1)(\delta + d - d_2). \end{aligned} \tag{25}$$

Thus, from (23) and (25), we get that

$$\begin{aligned} |A + q \cdot A| \geq\, & |A_1 + q \cdot A_1| + |A_2 + q \cdot A_2| + 2\delta|A| \\ & + 2(d - d_1)|A_1| + 2(d - d_2)|A_2| - 2(1 + d - d_1)(1 + d - d_2). \end{aligned}$$

□

Lemma 8 – *Let A be a reduced subset of $\mathbb{Z}^d$ and $q \in \mathbb{Z}$ with $|q| > 1$. Assume that there is a low-dimensional q-weak partition. Also assume that all low-dimensional q-weak partitions of A are not flat. Hence there is a q-weak partition $A = A_1 \uplus A_2$ such that, if $d_1 := \dim A_1$ and $d_2 := \dim A_2$, then*

$$\begin{aligned} |A + q \cdot A| &\geq |A_1 + q \cdot A| + |A_2 + q \cdot A_2| + 2(d - d_2)(|A_2| - 1) \\ |A + q \cdot A| &\geq |A_2 + q \cdot A| + |A_1 + q \cdot A_1| + 2(d - d_1)(|A_1| - 1) \end{aligned}$$

and

$$\begin{aligned} |A + q \cdot A| \geq\, & |A_1 + q \cdot A_1| + |A_2 + q \cdot A_2| + \min\{|A_1|, |A_2|\} \\ & + 2(d - d_1)|A_1| + 2(d - d_2)|A_2| - 2\max\{(d - d_1), (d - d_2)\}. \end{aligned}$$

Proof. For any low-dimensional q-weak partition $A = A_1' \uplus A_2'$, we have by assumption that it is not flat; thus, since $|\pi_{\mathbb{R}^d}(A)| = 1 = 2(d - \dim \mathbb{R}^d) + 1$, we get that that for all q-weak partition $A = A_1' \uplus A_2'$,

$$|\pi_{\operatorname{aff} A_i'}(A)| \geq 2(d - \dim A_i') + 1 \qquad \forall\, i \in \{1, 2\}. \tag{26}$$

From the family of low-dimensional partitions of A (which is not empty by assumption), take $A = A_1' \uplus A_2'$ with $\min\{\dim A_1', \dim A_2'\}$ minimal (in particular $\min\{\dim A_1', \dim A_2'\} < d$). The proof of the statement is divided into two cases.

- Assume that $|\pi_{\mathrm{aff}A_1'}(A)| \geq 2(d - \dim A_1') + 2$ or $|\pi_{\mathrm{aff}A_2'}(A)| \geq 2(d - \dim A_2') + 2$; without loss of generality suppose that

  $$|\pi_{\mathrm{aff}A_1'}(A)| \geq 2(d - \dim A_1') + 2. \tag{27}$$

  Take $A_1 := A_1'$ and $A_2 := A_2'$, and write $\pi_1 := \pi_{\mathrm{aff}A_1}$ and $\pi_2 := \pi_{\mathrm{aff}A_2}$. Since $A = A_1 \uplus A_2$ is q-weak, $A_1 + q \cdot A$ and $A_2 + q \cdot A$ are disjoint. Thus

  $$|A + q \cdot A| = |A_1 + q \cdot A| + |A_2 + q \cdot A|. \tag{28}$$

  Now, for $i \in \{1, 2\}$, we have that $A_i + q \cdot (A \cap \mathrm{aff}A_i)$ and $A_i + q \cdot (A \setminus \mathrm{aff}A_i)$ are disjoint so

  $$\begin{aligned} |A_1 + q \cdot A| &\geq |A_1 + q \cdot A_1| + |A_1 + q \cdot (A \setminus \mathrm{aff}A_1)| \\ |A_2 + q \cdot A| &\geq |A_2 + q \cdot A_2| + |A_2 + q \cdot (A \setminus \mathrm{aff}A_2)|. \end{aligned} \tag{29}$$

  On the one hand,

  $$\begin{aligned} |A_1 + q \cdot (A \setminus \mathrm{aff}A_1)| &\geq |\pi_1(q \cdot (A \setminus \mathrm{aff}A_1))||A_1| && \big(\text{by Cor. 1}\big) \\ &= |\pi_1(A \setminus \mathrm{aff}A_1)||A_1| \\ &\geq (|\pi_1(A)| - 1)|A_1| \\ &\geq (2(d - d_1) + 1)|A_1|. && \big(\text{by (27)}\big) \end{aligned} \tag{30}$$

  On the other hand,

  $$\begin{aligned} |A_2 + q \cdot (A \setminus \mathrm{aff}A_2)| &\geq |\pi_2(q \cdot (A \setminus \mathrm{aff}A_2))||A_2| && \big(\text{by Cor. 1}\big) \\ &= |\pi_2(A \setminus \mathrm{aff}A_2)||A_2| \\ &\geq (|\pi_2(A)| - 1)|A_2| \\ &\geq 2(d - d_2)|A_2|. && \big(\text{by (26)}\big) \end{aligned} \tag{31}$$

  From (28)–(31), note that

  $$\begin{aligned} |A + q \cdot A| &\geq |A_1 + q \cdot A| + |A_2 + q \cdot A_2| + 2(d - d_2)|A_2| \\ |A + q \cdot A| &\geq |A_2 + q \cdot A| + |A_1 + q \cdot A_1| + (2(d - d_1) + 1)|A_1|, \end{aligned}$$

  and

  $$\begin{aligned} |A + q \cdot A| \geq\ & |A_1 + q \cdot A_1| + |A_2 + q \cdot A_2| + |A_1| \\ & + 2(d - d_1)|A_1| + 2(d - d_2)|A_2|, \end{aligned}$$

  and this concludes this case.

- Assume that

$$|\pi_{\mathrm{aff}A_1'}(A)| = 2(d - \dim A_1') + 1$$
$$|\pi_{\mathrm{aff}A_2'}(A)| = 2(d - \dim A_2') + 1. \tag{32}$$

Without loss of generality assume that $\dim A_1' = \min\{\dim A_1', \dim A_2'\}$. Set $V := \mathrm{aff}A_1'$ and $k := d - \dim V$; translating if necessary, we assume that $\mathbf{0} \in A_1'$ so V is a linear subspace of $\mathbb{R}^d$ defined over $\mathbb{Z}$. From Lemma 2, there is an ordered basis $\mathcal{B} := \{\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_d\}$ of $\mathbb{Z}^d$ such that $\{\mathbf{f}_{k+1}, \mathbf{f}_{k+2}, \ldots, \mathbf{f}_d\}$ is an ordered basis of $V \cap \mathbb{Z}^d$. Set $\mathcal{B}' := \{\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_k\}$ and $\pi := \pi_{\mathcal{B},\mathcal{B}'}$. The function

$$\phi : \mathbb{R}^k \to \mathbb{R}^d/V, \qquad \phi\left(\sum_{i=1}^{k} a_i \mathbf{e}_i\right) = \pi_V\left(\sum_{i=1}^{k} a_i \mathbf{f}_i\right)$$

is an isomorphism and $\phi \circ \pi = \pi_V$. Thus

$$|\pi(A)| = |\pi_V(A)| = 2k + 1. \tag{33}$$

Note that $\pi(A)$ is $\mathbb{Z}^k$-reduced by Remark 3. Thus, from (33), the assumptions of Lemma 6 is satisfied by $\pi(A)$, and therefore there is a q-weak partition $\pi(A) = A_1'' \uplus A_2''$ with $|A_1''| = 1$. Taking $A_1 := \pi^{-1}(A_1'') \cap A$ and $A_2 := \pi^{-1}(A_2'') \cap A$, Remark 5 i) implies that $A = A_1 \uplus A_2$ is a q-weak partition. Furthermore, since $|A_1''| = 1$,

$$|\pi_V(A_1)| = |\pi(A_1)| = 1$$
$$|\pi_V(A_2)| = |\pi(A_2)| = |\pi(A)| - 1. \tag{34}$$

Note that $A = A_1 \uplus A_2$ is low-dimensional since A_1 is contained in a translation of V by (34). From the minimality of $\dim A_1' = \min\{\dim A_1', \dim A_2'\}$, we have that $\dim A_1 \geq \dim A_1'$; however, since A_1 is contained in a translation of $V = \mathrm{aff}A_1'$ by (34), we conclude that $\dim A_1 = \dim A_1'$, and therefore $\mathrm{aff}A_1$ is a translation of V. Write $\pi_1 := \pi_{\mathrm{aff}A_1}$ and $\pi_2 := \pi_{\mathrm{aff}A_2}$; thus $\pi_V = \pi_1$. Inasmuch as $\mathrm{aff}A_1$ is a translation of V, we get from (34) that A_2 does not intersect $\mathrm{aff}A_1$; therefore

$$A_2 = A \setminus \mathrm{aff}A_1. \tag{35}$$

By (33) and (34),

$$|\pi_1(A_2)| = |\pi(A)| - 1 = 2(d - \dim A_1). \tag{36}$$

Since $A = A_1 \uplus A_2$ is q-weak, it follows that $A_1 + q \cdot A$ and $A_2 + q \cdot A$ are disjoint. Thus

$$|A + q \cdot A| = |A_1 + q \cdot A| + |A_2 + q \cdot A|. \tag{37}$$

Now, for $i \in \{1,2\}$, we have that $A_i + q\cdot(A \cap \text{aff}A_i)$ and $A_i + q\cdot(A \setminus \text{aff}A_i)$ are disjoint so

$$\begin{aligned}|A_1+q\cdot A| &\geq |A_1+q\cdot A_1| + |A_1+q\cdot(A\setminus \text{aff}A_1)|\\ |A_2+q\cdot A| &\geq |A_2+q\cdot A_2| + |A_2+q\cdot(A\setminus \text{aff}A_2)|.\end{aligned} \tag{38}$$

On the one hand,

$$\begin{aligned}|A_1+q\cdot(A\setminus\text{aff}A_1)| &= |A_1+q\cdot A_2| && \big(\text{by (35)}\big)\\ &\geq |\pi_1(q\cdot A_2)|(|A_1|-1)+|q\cdot A_2| && \big(\text{by Cor. 1}\big)\\ &= |\pi_1(A_2)|(|A_1|-1)+|A_2|\\ &= 2(d-d_1)|A_1|+|A_2|-2(d-d_1). && \big(\text{by (36)}\big)\end{aligned} \tag{39}$$

On the other hand,

$$\begin{aligned}|A_2+q\cdot(A\setminus\text{aff}A_2)| &\geq |\pi_2(q\cdot(A\setminus\text{aff}A_2))||A_2| && \big(\text{by Cor. 1}\big)\\ &= |\pi_2(A\setminus\text{aff}A_2)||A_2|\\ &\geq (|\pi_2(A)|-1)|A_2|\\ &\geq 2(d-d_2)|A_2|. && \big(\text{by (26)}\big)\end{aligned} \tag{40}$$

From (37)–(40), note that

$$\begin{aligned}|A+q\cdot A| &\geq |A_1+q\cdot A| + |A_2+q\cdot A_2| + 2(d-d_2)|A_2|\\ |A+q\cdot A| &\geq |A_2+q\cdot A| + |A_1+q\cdot A_1| + 2(d-d_1)|A_1| + |A_2| - 2(d-d_1),\end{aligned}$$

and

$$\begin{aligned}|A+q\cdot A| \geq\ & |A_1+q\cdot A_1| + |A_2+q\cdot A_2| + |A_2|\\ &+ 2(d-d_1)|A_1| + 2(d-d_2)|A_2| - 2(d-d_1),\end{aligned}$$

and this concludes this case. □

4 Proof of Theorem 2

In this section we complete the proof of Theorem 2.

Proof. (Theorem 2) We start the proof with three reductions. First, translating if necessary, we will assume that $\mathbf{0} \in A$. Second, note that if $\dim A < d$, then there is a basis $\mathcal{B} := \{\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_d\}$ of $\mathbb{Z}^d$ such that $\mathcal{B}' := \{\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_{\dim A}\}$ is a basis of $\text{aff}A \cap \mathbb{Z}^d$. Thus $\pi_{\mathcal{B},\mathcal{B}'}$ satisfies that $\dim \pi_{\mathcal{B},\mathcal{B}'}(A) = \dim A = \dim \mathbb{R}^{\dim A}$, $\pi_{\mathcal{B},\mathcal{B}'}(A) \subseteq$

$\mathbb{Z}^{\dim A}$, $|\pi_{\mathcal{B},\mathcal{B}'}(A)| = |A|$ and $|\pi_{\mathcal{B},\mathcal{B}'}(A+q\cdot A)| = |A+q\cdot A|$; hence we will assume from now on that $\dim A = d$. Third, since $\dim A = d$, Remark 2 allows us to assume that A is reduced.

Recall that for each $i,j\in\mathbb{Z}$, we have that $c_{i,j} = 4^{j^{4^i}}$. The proof of Theorem 2 is done first by induction on d and then on m. First note that if $d=0$, A and $A+q\cdot A$ are singletons and hence

$$|A+q\cdot A| = 1 \geq \frac{m}{|q|-1}|A| - c_{0,m}.$$

Thus, from now on, we assume that the statement is true for all $0\leq d' < d$. Now assume that $m = |q|+2d-1$. Then Theorem 3 leads to

$$|A+q\cdot A| \geq 2|A|-1 \geq \frac{|q|+2d-1}{|q|+2d-1}|A| - c_{d,|q|+2d-1}.$$

From now on, we will assume that the statement holds for all $|q|+2d-1\leq m' < m$. To complete the induction, we will divide the proof into two cases.

- Assume that there is a low-dimensional q-weak partition of A. This case will also be divided into two subcases.

  ⋆ Assume that there is a flat q-weak partition of A. From Lemma 7, there is a q-partition $A = A_1 \uplus A_2$ such that, if $d_1 := \dim A_1$, $d_2 := \dim A_2$ and $c_1 := 2(1+d-d_1)(1+d-d_2)$, then

$$\begin{aligned}|A+q\cdot A| \geq{}& |A_1+q\cdot A_1| + |A_2+q\cdot A_2| + 2\delta(A_1,A_2)|A|\\ &+ 2(d-d_1)|A_1| + 2(d-d_2)|A_2| - c_1.\end{aligned}\tag{41}$$

  If $d_1 < d$ and $d_2 < d$, then the induction hypothesis yields that

$$\begin{aligned}|A_1+q\cdot A_1| &\geq (|q|+2d_1-1)|A_1| - c_{d_1,(|q|+2d_1-1)^2}\\ |A_2+q\cdot A_2| &\geq (|q|+2d_2-1)|A_2| - c_{d_2,(|q|+2d_2-1)^2}\end{aligned}\tag{42}$$

  and thereby

$$\begin{aligned}|A+q\cdot A| &\geq |A_1+q\cdot A_1| + |A_2+q\cdot A_2|\\ &\quad + 2(d-d_1)|A_1| + 2(d-d_2)|A_2| - c_1 && \big(\text{by (41)}\big)\\ &\geq (|q|+2d-1)|A| - c_{d_1,(|q|+2d_1-1)^2}\\ &\quad - c_{d_2,(|q|+2d_2-1)^2} - c_1 && \big(\text{by (42)}\big)\\ &\geq \frac{m}{|q|+2d-1}|A| - c_{d,m}.\end{aligned}$$

If $d_1 = d$ or $d_2 = d$, we have that $\delta(d_1, d_2) = 1$. For any $i \in \{1,2\}$, we have by the hypothesis of induction,

$$|A_i + q \cdot A_i| \geq \begin{cases} (|q| + 2d_i - 1)|A_i| - c_{d_i,(|q|+2d-1)^2} & \text{if } d_i < d; \\ \frac{m-1}{|q|+2d-1}|A_i| - c_{d,m-1}, & \text{if } d_i = d. \end{cases}$$

and in any case

$$|A_i + q \cdot A_i| + 2(d - d_i)|A_i| \geq \frac{m-1}{|q| + 2d - 1}|A_i| - c_{d,m-1}. \tag{43}$$

Thus

$$\begin{aligned} |A + q \cdot A| &\geq |A_1 + q \cdot A_1| + |A_2 + q \cdot A_2| + 2|A| \\ &\quad + 2(d - d_1)|A_1| + 2(d - d_2)|A_2| - c_1 \quad \big(\text{by (41)}\big) \\ &\geq \frac{m-1}{|q| + 2d - 1}|A| + 2|A| - 2c_{d,m-1} - c_1 \quad \big(\text{by (43)}\big) \\ &\geq \frac{m}{|q| + 2d - 1}|A| - c_{d,m}. \end{aligned}$$

★ Assume that all the low-dimensional q-weak partitions of A are not flat. Hence we can apply Lemma 8 to A and it implies that there is a q-weak partition $A = A_1 \uplus A_2$ such that if $d_1 := \dim A_1$, $d_2 := \dim A_2$ and $c_2 := 2\max\{(d - d_1), (d - d_2)\}$, then

$$\begin{aligned} |A + q \cdot A| &\geq |A_1 + q \cdot A| + |A_2 + q \cdot A_2| + 2(d - d_2)(|A_2| - 1) \\ |A + q \cdot A| &\geq |A_2 + q \cdot A| + |A_1 + q \cdot A_1| + 2(d - d_1)(|A_1| - 1) \end{aligned} \tag{44}$$

and

$$\begin{aligned} |A + q \cdot A| \geq\ & |A_1 + q \cdot A_1| + |A_2 + q \cdot A_2| + \min\{|A_1|, |A_2|\} \\ & + 2(d - d_1)|A_1| + 2(d - d_2)|A_2| - c_2; \end{aligned} \tag{45}$$

without loss of generality assume that $|A_1| = \min\{|A_1|, |A_2|\}$. For any $i \in \{1,2\}$, we have by the hypothesis of induction,

$$|A_i + q \cdot A_i| \geq \begin{cases} (|q| + 2d_i - 1)|A_i| - c_{d_i,(|q|+2d-1)^2} & \text{if } d_i < d; \\ \frac{m-1}{|q|+2d-1}|A_i| - c_{d,m-1}, & \text{if } d_i = d. \end{cases}$$

In any case, we obtain that

$$|A_i + q \cdot A_i| + 2(d - d_i)|A_i| \geq \frac{m-1}{|q| + 2d - 1}|A_i| - c_{d,m-1}. \tag{46}$$

First suppose that

$$|A_1| \leq \frac{1}{|q|+2d-1}|A|. \tag{47}$$

Then

$$\begin{aligned}
|A+q\cdot A| &\geq |A_1+q\cdot A|+|A_2+q\cdot A_2| \\
&\quad +2(d-d_2)(|A_2|-1) && \big(\text{by (44)}\big) \\
&\geq |A_1+q\cdot A|+\frac{m-1}{|q|+2d-1}|A_2| \\
&\quad -c_{d,m-1}-2(d-d_2) && \big(\text{by (46)}\big) \\
&\geq |A_1|+|A|-1+\frac{m-1}{|q|+2d-1}|A_2| \\
&\quad -c_{d,m-1}-2(d-d_2) && \big(\text{by Thm. 3}\big) \\
&\geq \frac{m}{|q|+2d-1}|A|-c_{d,m}. && \big(\text{by (47)}\big)
\end{aligned}$$

Now suppose that

$$|A_1| \geq \frac{1}{|q|+2d-1}|A|. \tag{48}$$

Then

$$\begin{aligned}
|A+q\cdot A| &\geq |A_1+q\cdot A_1|+|A_2+q\cdot A_2|+|A_1| \\
&\quad +2(d-d_1)|A_1|+2(d-d_2)|A_2|-c_2 && \big(\text{by (45)}\big) \\
&\geq \frac{m-1}{|q|+2d-1}|A|+|A_1|-2c_{d,m-1}-c_2 && \big(\text{by (46)}\big) \\
&\geq \frac{m}{|q|+2d-1}|A|-c_{d,m}. && \big(\text{by (48)}\big)
\end{aligned}$$

- Assume that there are no low-dimensional q-weak partitions of A. Denote by $\pi : \mathbb{Z}^d \to \mathbb{Z}^d/q\cdot\mathbb{Z}^d$ the canonical projection. Write $\pi(A)=\{z_1,z_2,\ldots,z_n\}$ and $B_i := \pi^{-1}(z_i)\cap A$ for each $i\in\{1,2,\ldots,n\}$; furthermore, for each $i\in\{1,2,\ldots,n\}$, choose $\mathbf{b}_i\in B_i$ and write $B_i' := q^{-1}\cdot(B_i-\{\mathbf{b}_i\})$. We divide the proof into two subcases.
  - ⋆ Assume that there is $i\in\{1,2\ldots,n\}$ such that B_i' is not q-domain; without loss of generality assume that B_1' is not q-domain. For each $j\in\{1,2,\ldots,n\}$, set $A_{1,j} := B_j$ and $A_{2,j} := A\setminus B_j$. Hence, from Remark 4, $A = A_{1,j}\uplus A_{2,j}$ is a q-weak partition; furthermore, the assumption implies that $\dim A_{1,j} =$

$\dim A_{2,j} = d$ for all $j \in \{1,2,\ldots,n\}$. Since $A = A_{1,j} \uplus A_{2,j}$ is q-weak , $A_{1,j}+q\cdot A$ and $A_{2,j}+q\cdot A$ are disjoint so

$$|A+q\cdot A| = |A_{1,j}+q\cdot A| + |A_{2,j}+q\cdot A|. \tag{49}$$

For $i \in \{1,2\}$ and $j \in \{1,2,\ldots,n\}$, by the hypothesis of induction on $A_{i,j}$, we have that

$$|A_{i,j}+q\cdot A_{i,j}| \geq \frac{m-1}{|q|+2d-1}|A_{i,j}| - c_{d,m-1}. \tag{50}$$

First suppose that there is $j \in \{1,2,\ldots,n\}$ such that

$$|A_{1,j}| \leq \frac{1}{|q|+2d-1}|A|. \tag{51}$$

Then

$$\begin{aligned}
|A+q\cdot A| &= |A_{1,j}+q\cdot A| + |A_{2,j}+q\cdot A| && \big(\text{by (49)}\big)\\
&\geq |A_{1,j}+q\cdot A| + |A_{2,j}+q\cdot A_{2,j}| \\
&\geq |A_{1,j}+q\cdot A| + \frac{m-1}{|q|+2d-1}|A_{2,j}| - c_{d,m-1} && \big(\text{by (50)}\big)\\
&\geq |A_{1,j}| + |A| - 1 + \frac{m-1}{|q|+2d-1}|A_{2,j}| - c_{d,m-1} && \big(\text{by Thm. 3}\big)\\
&\geq \frac{m}{|q|+2d-1}|A| - c_{d,m}. && \big(\text{by (51)}\big)
\end{aligned}$$

Now suppose that

$$\min_{1\leq j\leq n} |A_{1,j}| = \min_{1\leq j\leq n} |B_j| \geq \frac{1}{|q|+2d-1}|A|. \tag{52}$$

Since B_1' is not q-domain, Lemma 4 leads to

$$|A_{1,1}+q\cdot A| \geq |A_{1,1}+q\cdot A_{1,1}| + \min_{1\leq j\leq n} |B_j|. \tag{53}$$

Then

$$\begin{aligned}
|A+q\cdot A| &= |A_{1,1}+q\cdot A| + |A_{2,1}+q\cdot A| && \big(\text{by (49)}\big)\\
&\geq |A_{1,1}+q\cdot A_{1,1}| + \min_{1\leq j\leq n} |A_{1,j}| + |A_{2,1}+q\cdot A_{2,1}| && \big(\text{by (53)}\big)\\
&\geq \frac{m-1}{|q|+2d-1}|A| + \min_{1\leq j\leq n} |A_{1,j}| - 2c_{d,m-1} && \big(\text{by (50)}\big)\\
&\geq \frac{m}{|q|+2d-1}|A| - c_{d,m}. && \big(\text{by (52)}\big)
\end{aligned}$$

★ Assume that B'_j is q-domain for all $j \in \{1,2,\ldots,n\}$. Since $B_i + q \cdot A$ an $B_j + q \cdot A$ are disjoint for all $i,j \in \{1,2,\ldots,n\}$ with $i \neq j$, we have that

$$|A + q \cdot A| = \sum_{i=1}^{n} |B_i + q \cdot A| \geq \sum_{i=1}^{n} |B_i + q \cdot B_i|. \tag{54}$$

For all $i \in \{1,2,\ldots,n\}$, Lemma 3 applied to B'_i leads to

$$|B'_i + q' \cdot B'_i| \geq (d + |q|^d)|B'_i| - \frac{d(d+1)}{2}|q|^d. \tag{55}$$

Since $d > 0$ and $|q| > 1$, we get that

$$d + |q|^d \geq |q| + 2d - 1. \tag{56}$$

Finally

$$\begin{aligned}
|A + q \cdot A| &\geq \sum_{i=1}^{n} |B_i + q \cdot B_i| && \big(\text{by (54)}\big)\\
&= \sum_{i=1}^{n} |B'_i + q \cdot B'_i| \\
&\geq \sum_{i=1}^{n} \left((d + |q|^d)|B'_i| - \frac{d(d+1)}{2}|q|^d \right) && \big(\text{by (55)}\big)\\
&= (d + |q|^d) \sum_{i=1}^{n} |B'_i| - n\frac{d(d+1)}{2}|q|^d \\
&= (d + |q|^d) \sum_{i=1}^{n} |B_i| - n\frac{d(d+1)}{2}|q|^d \\
&\geq (d + |q|^d)|A| - \frac{d(d+1)}{2}|q|^{2d} && \big(\text{since } n \leq |q|^d\big)\\
&\geq (|q| + 2d - 1)|A| - \frac{d(d+1)}{2}|q|^{2d} && \big(\text{by (56)}\big)\\
&\geq \frac{m}{|q| + 2d - 1}|A| - c_{d,m},
\end{aligned}$$

and this concludes the induction. □

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