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# On a moment estimate of sum of weakly dependent random variables using simple random walk on graph

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**Abstract**

We obtained a moment estimate for the sum of Rademacher random variables under condition that they are dependent in the way that their sum is zero.

**Keywords:** random walk, concentration inequality.

MSC: 05C81, 60A10.

## 1 Introduction

Let $X_1, \dots, X_N$ be a sequence of independent real valued random variables and let $\Sigma = \sum_{i=1}^{N} X_i$. The estimate of moments of $\Sigma$, that is of the quantities $\|\Sigma\|_p = \mathbb{E}(\Sigma^p)^{1/p}$, appear often in many areas of mathematics. The growth of moments is closely related to the behavior of the tails of $\Sigma$.

Probabilists have been interested in the moments of sums of random variables since the early part of last century. Khinchine's 1923 paper appears to make the first significant contribution to this problem[2]. It provides inequalities for the moments of a sum of Rademacher random variables. In 1970, Rosenthal generalised Khinchine's result to the case of positive or mean-zero random variables[3]. Further refinements to these bounds have been made by Latala and Hitczenko, Montogomery-Smith and Oleszkiewiez in more recent times[4]. Nowadays, it appears that in the different applications of mathematics, statistics, computer science and engineering similar estimates for the case when random variables are not independent are important

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[2] Khinchine, 1923, "Über dyadische Brüche".

[3] H.P. Rosental, 1980, "On the subspaces of $L_p$, $(p > 2)$ spanned by sequences of independent random variables".

[4] P. Hitczenko, 1997, "Moment inequalities for sums of certain independent symmetric random variables";

R. Latala, 1997, "Estimation of moments of sums of independent real random variables".

(see for example B. Pass (2008) and P. Doukhan (2007)). The main motivation of the current work came from an idea, represented by authors in A.B. Kashlak (2020), on using a concentration inequality for dependent Rademacher random variables to obtained a computation-free approach to permutation testing.

Our aim in the present work is to obtain a concentration inequality for Rademacher random variables under condition that their sum is zero. More precisely, we would like to find a bound on the sum of random variables, $\Sigma = \sum_{i=1}^{2n} X_i$, in the case when $X_i = a_i \varepsilon_i$, where $a \in \mathbb{R}^{2n}$ and $\varepsilon_i, i = 1, \dots, 2n$ are independent Rademacher random variables, that is variables satisfying the following condition: $\mathbb{P}(\varepsilon_i = 1) = \mathbb{P}(\varepsilon_i = -1) = \frac{1}{2}$. As usual, for $\varepsilon \in \{\pm 1\}^{2n}$ by $\varepsilon_1, \dots, \varepsilon_{2n}$ we denote coordinates of $\varepsilon$. We will put an additional assumption on the Rademacher random variables, namely

$$S = \sum_{i=1}^{2n} \varepsilon_i = 0. \tag{1}$$

To shorter notation, by $\mathbb{E}_{S=0}$ we denote an expectation with condition that (1) holds, i.e $\mathbb{E}_{S=0} \left| \sum_{i=1}^{2n} a_i \varepsilon_i \right|^p = \mathbb{E}\left( \left| \sum_{i=1}^{2n} a_i \varepsilon_i \right|^p \mid S = \sum_{i=1}^{2n} \varepsilon_i = 0 \right)$.

Consider the following set

$$\Omega = \left\{ \varepsilon \in \{-1, 1\}^{2n} \;\middle|\; \sum_{i=1}^{2n} \varepsilon_i = 0 \right\} = \left\{ \varepsilon \in \{-1, 1\}^{2n} \mid \operatorname{card}\{i : \varepsilon_i = 1\} = n \right\}. \tag{2}$$

Thus, for $\varepsilon$ taken uniformly at random in $\Omega$ the sequence of its coordinates is a sequence of a weekly dependent Rademacher random variables.

For $\varepsilon \in \Omega$ we put into correspondence a subset of the group $\Pi_{2n}$ of all permutations of set $\{1, \dots, 2n\}$ as

$$\sigma \in \Pi_{2n} \longleftrightarrow A_\sigma = \{ \varepsilon \in \Omega \mid \varepsilon_i = 1 \text{ if } \sigma(i) \le n; \varepsilon_i = -1 \text{ if } \sigma(i) > n \}.$$

It is easy to see that this correspondence brings uniform measure on $\Pi_{2n}$ to the uniform measure on $\Omega$.

Define $f : \Pi_{2n} \longrightarrow \mathbb{R}$ by

$$f(\sigma) := \left| \sum_{i=1}^{n} a_{\sigma(i)} - \sum_{i=n+1}^{2n} a_{\sigma(i)} \right|. \tag{3}$$

Note, that $\mathbb{E}_{S=0} \left| \sum_{i=1}^{n} a_i \varepsilon_i \right|^p = \mathbb{E} f^p$, where $\mathbb{E} f^p$ is an expectation when one takes a permutation $\sigma$ uniformly at random on $\Pi_{2n}$. Thus, it is enough to estimate $p$-th moments of $f$.

In the present paper we obtained the following result.

**Theorem 1 –** *Let $f$ defined as above. Then, for $p > 0$,*

$$(\mathbb{E}f^p)^{1/p} \le \mathbb{E}|f| + Cp\frac{2n-1}{2n}\|a\|_2,$$

*where $C > 0$ is a constant independent of $p$ and $n$.*

The paper is organized as follows. In the next section we provide the necessary known tools and definitions. In Section 3, we will establish the bound on the $p$-th moment of $\sum_{i=1}^{2n} a_i\varepsilon_i$ under condition that $\sum_{i=1}^{2n} \varepsilon_i = 0$.

# 2 Preliminaries

Let $G(V,E)$ be a connected undirected finite graph, where $V$ stays for a set of vertices and $E$ is a set of edges. A *simple random walk* is a sequence of vertices $v_0, v_1, \ldots, v_t$, where $v_i \sim v_{i+1}$ (that is $\{v_i, v_{i+1}\} \in E$) for $i = 0, 1, \ldots, t-1$. That is, given an initial vertex $v_0$, select randomly an adjacent vertex $v_1$, and move to this neighbor. Then, select randomly a neighbor $v_2$ of $v_1$, and move to it, etc. The probability it moves from vertex $v_i$ to $v_{i+1}$ (assuming it sits at $v_i$) is given by

$$p(v_i, v_{i+1}) = \begin{cases} \dfrac{1}{\deg(v_i)}, & \text{if } v_i \sim v_{i+1}, \\ 0, & \text{otherwise}, \end{cases} \tag{4}$$

where $\deg(v_i)$ denotes the degree of vertex $v_i$. This is a walk using a transition probability matrix, $P = (p(v_i, v_{i+1}))_{v_i, v_{i+1} \in V}$. The transition probability (4) (see for example G. Grimmett (2020)) has a reversible equilibrium probability distribution $\mu(v_i)$. That is,

$$\mu(v_i)p(v_i, v_{i+1}) = \mu(v_{i+1})p(v_{i+1}, v_i)$$

and $\mu(v_i)$ is proportional to $\deg(v_i)$.

Let $I$ be the $V \times V$ identity matrix. The discrete Laplacian is the matrix $L = P - I$ with its eigenvalues $0 < \lambda_1 \le \lambda_2 \le \ldots$, ordered in non-increasing order. The smallest eigenvalue, $\lambda_1 > 0$, is called the *spectral gap* of the random walk.

For $f : V \longrightarrow \mathbb{R}$ define

$$|||f|||_\infty^2 = \frac{1}{2}\sup_{v_i \in V} \sum_{v_{i+1} \in V} |f(v_i) - f(v_{i+1})|^2 p(v_i, v_{i+1}). \tag{5}$$

We will use the following concentration inequality (see A. Aida (1994) or M. Ledoux (2001)):

**Theorem 2 –** *Assume that $(p,\mu)$ is reversible on the finite graph $G(V,E)$, and let $\lambda_1 > 0$ be the spectral gap. Then,*

$$\mu\left(f > \int f\,d\mu + t\right) \le 3\exp\left(\frac{-t\sqrt{\lambda_1}}{2|||f|||_\infty^2}\right). \tag{6}$$

For purpose of our work we specialize now to $V = \Pi_{2n}$, the group of all permutations $\sigma$ of the set $\{1,\ldots,2n\}$, and to $E = \{(\sigma,\sigma\tau) \mid \tau \text{ is a transposition on } \Pi_{2n}\}$. We will be using a random walk with positive probability to stay at the same vertex. We will follow P. Diaconis (1981) and consider the identical permutation as transposition. The transition probability $p(\sigma,\sigma\tau)$ on $G = (\Pi_{2n},E)$ is

$$p(\sigma,\sigma\tau) = \frac{2}{(2n)^2}, \tag{7}$$

and reversible equilibrium distribution $\mu$ on $\Pi_{2n}$ is a unique invariant measure for $p$ (see for example S. Chatterjee (2009) for these facts). Also, as proved in P. Diaconis (1981), the spectral gap of the random transposition walk on $\Pi_{2n}$ is $\lambda_1 = \frac{2}{2n} = \frac{1}{n}$. Thus, the concentration inequality (6) for simple random walk on $G(\Pi_{2n},E)$ can be rewritten as

$$\mu(\{\sigma : f(\sigma) - \mathbb{E}f \geq t\}) \leq 3\exp\left(\frac{-t}{2|||f|||_\infty^2\sqrt{n}}\right). \tag{8}$$

# 3 Proof of Theorem 1

We are going to use inequality (8). We calculate first

$$|||f|||_\infty^2 = \frac{1}{2}\sup_{\sigma\in\Pi_{2n}}\sum_{all\,\tau}|f(\sigma)-f(\sigma\tau)|^2 p(\sigma,\sigma\tau),$$

where $p(\sigma,\sigma\tau)$ is defined in (7).

Consider $f(\sigma) := |g(\sigma)| = \left|\sum_{i=1}^n a_{\sigma(i)} - \sum_{i=n+1}^{2n} a_{\sigma(i)}\right|$. Since $\tau(i,j)$ is a random transposition with $i,j$ chosen uniformly from the set $\{1,\ldots,2n\}$, we obtain

$$g(\sigma) - g(\sigma\tau) = 2(a_i - a_j)h(i,j),$$

where

$$h(i,j) = \begin{cases} 1, & \text{if } j \leq n < i \leq 2n, \\ -1, & \text{if } i \leq n < j \leq 2n, \\ 0, & \text{otherwise.} \end{cases}$$

Thus, $|f(\sigma)-f(\sigma\tau)|^2 = 4(a_i - a_j)^2 h^2(i,j)$. And we can calculate

$$\begin{aligned} |||f|||_\infty^2 &= \frac{1}{n^2}\sum_{\tau(i,j)}(a_i-a_j)^2 h^2(i,j) \\ &= \frac{2}{n^2}\sum_{i=1}^{n}\sum_{j=n+1}^{2n}(a_i-a_j)^2 h^2(i,j) = \frac{2}{n^2}\left(n\|a\|_2^2 - 2\sum_{i=1}^{n}\sum_{j=n+1}^{2n} a_i a_j\right). \end{aligned}$$

Since

$$-\sum_{i=1}^{n}\sum_{j=n+1}^{2n} a_i a_j \leq \sum_{i=1}^{n}\sum_{j=n+1}^{2n} \frac{a_i^2 + a_j^2}{2} = \frac{n}{2}\|a\|_2^2,$$

the last equation can be bounded by

$$|||f|||_\infty^2 \leq \frac{4}{n}\|a\|_2^2. \tag{9}$$

Now, using (8), (9) and an upper bound $\Gamma(x) \leq x^{x-1}$, for all $x \geq 1$ (see for example G. D. Anderson (1997)), we obtain

$$\begin{aligned}\mathbb{E}|f - \mathbb{E}f|^p &= \int_0^\infty \mu((f(\sigma) - \mathbb{E}f)^p \geq t^p) dt^p \\ &\leq 6p \int_0^\infty e^{-t/(4\|a\|_2)} t^{p-1} dt = 6\, p\, 4^p \Gamma(p) \|a\|_2^p \\ &\leq 4^p\, 6 p^p \|a\|_2^p.\end{aligned}$$

Hence

$$(\mathbb{E}f^p)^{1/p} \leq \mathbb{E}|f| + 24p\|a\|_2. \tag{10}$$

Let us note now that under condition (1) the random variables are invariant under the shifts. Replacing $a_i$ with $a_i - \frac{1}{2n}\sum_{i=1}^{2n} a_i$ for $i = 1, \dots 2n$ in (10) would give us the desired result.

**Remark –** Note that $\mathbb{E}|f| \leq \left(\mathbb{E}|f|^2\right)^{1/2}$, where $\mathbb{E}|f|^2$ can be directly calculated (see S. Spektor (2016)).

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