On operators preserving inequalities between polynomials

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Abstract

In this survey article, we explore operator aspects in extremal properties of Bernstein-type polynomial inequalities. We shall also see that a linear operator which send polynomials to polynomials and have zero-preserving property naturally preserve Bernstein's inequality.

1   Introduction

Polynomials play an important role in applied sciences in general and are fundamental objects in harmonic and complex analysis in particular. Polynomials find their place everywhere in Mathematics. The extremal properties of polynomials are extensively used in approximation theory and numerical analysis. The problems related to applied mathematics are dependent on polynomials to a greater degree. The study of extremal properties of the derivatives of polynomials started with some scientific investigations by well-known Russian chemist Dmitri Mendeleev (for reference see Milovanovic, Rassias, and Mitrinovic 1994, chapter 6). Mathematically, Mendeleev's problem amounts to estimate $|p'(t)|$ on $-1 \le t \le 1$ for a quadratic polynomial $p(t)$ satisfying $-1 \le p(t) \le 1$ for $-1 \le t \le 1$. Mendeleev was himself able to find the solution. In fact, he found that $-4 \le p'(t) \le 4$ for $t \in [-1, 1]$ and $p(t) = 1 - 2t^2$ is the extremal polynomial. Mendeleev communicated his conclusions to the distinguished Russian Mathematician A. A. Markov 1889, who extended Mendeleev's result to general real polynomials and obtained the following generalization.

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Theorem 1 – *If a polynomial $p(t)$ of degree n with real coefficients satisfy $-1 \le p(t) \le 1$ over $[-1, 1]$, then*

$$|p'(t)| \le n^2 \quad for \quad -1 \le t \le 1.$$

The inequality cannot be improved as equality holds for $p(t) = \cos(n \arccos t)$.

This result of Markov marked the beginning in the research of extremal properties of polynomials. Subsequently, his younger brother V. A. Markov 1916 obtained a best possible estimate for the modulus of sth derivative, $|p^{(s)}(t)|$ where $s \le n$.

It was a few years later, another Russian Mathematician Serge Bernstein 1912 while looking for its analogue for complex polynomials over $|z| \le 1$ obtained the following theorem.

Theorem 2 – *If $f(z)$ is a complex-polynomial having degree at most n, then for $|z| = 1$*

$$|f'(z)| \le n \max_{|w|=1} |f(w)|. \tag{1}$$

The bound is sharp as is shown by $f(z) = az^n$, $a \ne 0$.

Proof. Let M represent the maximum of $|f(z)|$ over $|z| \le 1$. Therefore, $|f(z)| < \tau M |z^n|$ for $\tau \in \mathbb{C}$ with $|\tau| > 1$ and $|z| \le 1$. This implies by invoking Rouché's theorem that the polynomial $F(z) = f(z) + \tau M z^n$ has no zeros outside $|z| < 1$.

To complete the proof, two important properties of the derivative are needed: (1) Linearity (2) zero preserving property for polynomials.

The first property is obvious. The second property is a special case of Gauss-Lucas theorem. According to which for any non-constant polynomial $f(z)$, all the zeros of $f'(z)$ are completely enclosed in the smallest convex polygon formed by the zeros of $f(z)$.

Therefore by utilizing this zero-preserving property along the linearity of the derivative, we conclude that every zero of $F'(z) = f'(z) + \tau n M z^{n-1}$ is in $|z| < 1$. This implies

$$|f'(z)| \le nM|z|^{n-1} \quad for \quad |z| \ge 1. \tag{2}$$

If inequality (2) were not true, then we could find $\delta \in \mathbb{C}$ with $|\delta| \ge 1$ such that $|f'(\delta)| > nM|\delta|^{n-1}$. Now, if we choose $\tau = -f'(\delta)/nM\delta^{n-1}$ then $|\tau| > 1$ and $F'(\delta) = 0$, which is a contradiction since no zero of $F(z)$ lie outside $|z| < 1$. This verifies inequality (2) and consequently the proof is completed. □

In Bernstein's inequality, equality holds iff $f(z) = az^n$ where $a \ne 0$. That is, if every zero of $f(z)$ is at origin and the inequality gets strict when $f(z) = 0$ has a non-zero root. This suggests that if no zero of $f(z)$ is at origin then the upper bound in (2) may be improved. This fact was deeply examined by Paul Erdös which lead him to conjecture that if $f(z)$ does not vanish in open disk $|z| < 1$ then in the upper bound of (2), n can be replaced by $n/2$. Peter D Lax 1944 was the first one to prove this conjecture. In fact, he proved:

Theorem 3 – *For any polynomial $f(z)$ of degree at most n such that $f(z) \neq 0$ for $|z| < 1$ then*

$$|f'(z)| \leq \frac{n}{2} \max_{|z|=1}|f(z)| \quad where \quad |z| = 1.$$

The sharpness of this inequality is shown by $f(z) = az^n + b$ where $|a| = |b| \neq 0$.

Professor Ralph P. Boas suggested a research problem to find analogue of Theorem 3 in the case when $f(z)$ does not vanish for $|z| < R$, $R > 0$. The subcase $R \geq 1$ of this problem was resolved by M.A. Malik 1969 but the actual problem still remains open. The results of Lax and Malik were improved in several ways. In particular, M.A. Qazi 1992 obtained some remarkable improvements by involving the coefficients of polynomials in the bound.

These kind of inequalities are usually called as Markov Bernstein type inequalities. Several authors have penned down monographs on this topic and numerous papers concerning this area have been published.

2 B_n-**operator**

We have seen in Bernstein's inequality that linearity and zero-preserving property of the derivative play an essential role in the proof given above. In fact, it was Rahman and Schmeisser 2002, p. 538 who in their celebrated monograph, Analytic Theory of Polynomials, showed that any operator satisfying these two properties satisfy Bernstein-type inequalities. They named such operators as B_n-operator.

Definition 1 – Let the space of polynomials over the field of complex numbers of degree at most n be denoted by $\mathcal{P}_n$ and $\mathcal{P}_n^0 \subset \mathcal{P}_n$ be set of those polynomials whose all zeros lie in $|z| \leq 1$. A linear operator $T : \mathcal{P}_n \rightarrow \mathcal{P}_n$ is called B_n-operator if $f \in \mathcal{P}_n^0$ then $T[f] \in \mathcal{P}_n^0$.

In the same monograph (Rahman and Schmeisser 2002, p. 539), the following theorem is also presented for B_n-operators.

Theorem 4 – *Let $f(z)$ be a complex-polynomial and have degree at most n. Moreover, if $\varphi_n(z) = z^n$ and T is a B_n-operator, then*

$$|T[f](z)| \leq |T[\varphi_n](z)| \max_{|z|=1}|f(z)| \quad for \quad |z| \geq 1.$$

The inequality cannot be sharpened under this hypothesis as one can easily verify that $f(z) = az^n$, $a \neq 0$ is extremal.

It is pertinent to mention here that the above theorem follows in the similar lines as of the proof of Bernstein's inequality given in the introductory section. To comprehend the similitude in proofs, here we also would like to write-down a proof of above theorem.

Proof (Proof of Theorem 4). Let us denote the maximum of $|f(z)|$ over $|z| = 1$ by M. Then $|f(z)| < \lambda M |z^n|$ for $\lambda \in \mathbb{C}$ with $|\lambda| > 1$ and $|z| \le 1$. Again by employing Rouché's theorem, we obtain that all the zeros of polynomial $F(z) = f(z) + \lambda M z^n$ are in $|z| < 1$.

Here, the linearity and zero-preserving property are carried by B_n-operator. Therefore for any B_n-operator T, every zero of $T[F](z) = T[f](z) + \lambda M T[\varphi_n](z)$ resides in $|z| < 1$, where $\varphi_n(z) = z^n$. This implies

$$|T[f](z)| \le |T[\varphi_n](z)|M \qquad \text{for} \quad |z| \ge 1. \tag{3}$$

If inequality (3) were false, then a complex number say w could be found with $|w| \ge 1$ such that $|T[f](w)| > |T[\varphi_n](w)|M$. Now, if we choose $\lambda = -T[f](w)/T[\varphi_n](w)M$ then $|\lambda| > 1$ and $T[F](w) = 0$. This is straightforward contradiction about the region containing zeros of $T[F]$. Hence, inequality (3) is valid and accordingly the proof gets complete. □

From above theorem, it is evident that the operators which are linear and have zero-preserving property preserve Bernstein-type inequalities for polynomials. Although the original proof of Bernstein's inequality does not use these two properties intrinsically. But a natural question arises: Does there exist non-linear zero-preserving operators that preserve Bernstein-type inequalities? This question has not been studied extensively.

The next result which is an analogue of Theorem 3 for B_n-operators was also obtained by Rahman and Schmeisser 2002, p. 539.

Theorem 5 – *Let $f \in \mathcal{P}_n$ and does not vanish for $|z| < 1$, then for any B_n-operator T,*

$$|T[f](z)| \le \frac{|T[1](z)| + |T[\varphi_n](z)|}{2} \max_{|z|=1}|f(z)| \qquad for \quad |z| \ge 1,$$

where $\varphi_n(z) = z^n$. Indeed, this result is also best possible and $f(z) = uz^n + v$, $|u| = |v| \ne 0$, is a polynomial for which equality holds in this inequality.

3 Inequalities Preserving Operators

It is obvious from Gauss Lucas theorem that $T \equiv \frac{d^{(s)}}{dz^{(s)}}$ represents a B_n-operator and an analogous result of Bernstein's inequality for sth derivative of polynomial $f(z)$ having degree n takes the following form.

$$\left|\frac{d^{(s)}}{dz^{(s)}}f(z)\right| \le \left|\frac{d^{(s)}}{dz^{(s)}}z^n\right| \max_{|z|=1}|f(z)| \qquad for \quad |z| = 1 \ (s < n).$$

It is natural to seek characterization of operators preserving Bernstein-type inequalities. In this direction, we shall refer the readers to some well-known operators

which fall under this characterization. The first result, we ought to present here is due to V.K. Jain 1997. He proved that an operator T on $\mathcal{P}_n$ which sends $f(z)$ to $T[f](z) := zf'(z) + \frac{n\beta}{2} f(z)$ is a B_n-operator if $\beta \in \mathbb{C}$ with $|\beta| \leq 1$. In this case, the corresponding Bernstein-type inequality is;

$$\left| zf'(z) + \frac{n\beta}{2} f(z) \right| \leq n \left| 1 + \frac{\beta}{2} \right| \max_{|z|=1} |f(z)| \qquad \text{for} \quad |z| = 1.$$

The next operator we are going to present here is due to Aziz and N. A. Rather 2012. In this manuscript, they developed a unified method to arrive at various polynomial inequalities including the inequalities of Bernstein and of Jain. They showed that the operator which sends an nth degree polynomial $f(z)$ to $f(Rz) - \beta f(rz) + \alpha \left\{ \left(\frac{R+1}{r+1} \right)^n - |\beta| \right\} f(rz)$ is a B_n-operator, where $R \geq r \geq 1$ and α, β belongs to closed unit circle $|z| \leq 1$. Their analogue of Bernstein's inequality also included results concerning the growth of polynomials.

For given two nth degree polynomials $p(z) = \sum_{j=0}^{n} \binom{n}{j} a_j z^j$ and $g(z) = \sum_{j=0}^{n} \binom{n}{j} b_j z^j$, Schur-Szegö composition is defined by $p * g(z) = \sum_{j=0}^{n} \binom{n}{j} a_j b_j z^j$. By Schur-Szegö composition theorem, if all the zeros of $f(z)$ and $g(z)$ are of modulus at most r and s respectively, then the moduli of the zeros of $f * g$ cannot exceed rs. Now, if we fix a polynomial $h(z)$ having degree n and all zeros in $|z| \leq 1$, then the operator which sends an nth degree polynomial $f(z)$ to $f * h(z)$ is clearly a B_n-operator. In this direction, the next result concerning this operator is due to Gulzar and N. A. Rather 2018.

Theorem 6 – *Let $f(z)$ be a polynomial and have degree n. Further, let $h(z) = \sum_{j=0}^{n} h_j z^j$ be of degree n and have all zeros in $|z| \leq 1$, then*

$$|f * h(z)| \leq |h_n| \max_{|z|=1} |f(z)| \qquad \text{for} \quad |z| = 1.$$

The estimate is sharp and $f(z) = az^n$ is the extremal polynomial.

For the choice $h(z) = \sum_{j=0}^{n} \binom{n}{j} j z^j$ the above inequality condenses to Bernstein's inequality.

Like Schur-Szegö composition, several other types of 'compositions' were studied by Mathematicians. Moris Marden 1996, p. 86 in his comprehensive book also discussed a composition of two polynomials wherein he proved that if the zeros of polynomials $p(z)$ of degree n and $g(z) = \mu_0 + \binom{n}{1}\mu_1 z + \ldots + \binom{n}{n}\mu_n z^n$ respectively are in $|z| \leq r$ and in the circular region: $|z| \leq s|z - \sigma|, s > 0$, then the zeros of

$$f(z) = \mu_0 p(z) + \mu_1 p'(z) \frac{(\sigma z)}{1!} + \ldots + \mu_n p^{(n)}(z) \frac{(\sigma z)^n}{n!}$$

are in the circle $|z| \leq \max(r, rs)$. Recently, N. A. Rather, Dar, and Gulzar 2021 showed in the following result that the hypothesis of Marden's theorem can be relaxed with regards to the degrees of $f(z)$ and $g(z)$.

Theorem 7 – *If every zero of nth degree polynomial $p(z)$ is of modulus at most r and the polynomial*

$$g(z) = \mu_0 + \binom{n}{1}\mu_1 z + \ldots + \binom{n}{m}\mu_m z^m$$

has no zero outside the region $|z| \le s|z - \sigma|$, $s > 0$, then the polynomial

$$f(z) = \mu_0 p(z) + \mu_1 p'(z)\frac{(\sigma z)}{1!} + \ldots + \mu_m p^{(m)}(z)\frac{(\sigma z)^m}{m!}$$

has all its zeros in $|z| \le \max(r, rs)$

By using this Theorem, they N. A. Rather, Dar, and Gulzar 2021 introduced an operator N which takes $f \in \mathcal{P}_n$ into $N[f] \in \mathcal{P}_n$ where $N[f](z) := \sum_{i=0}^{m} \mu_i \left(\frac{nz}{2}\right)^i \frac{f^{(i)}(z)}{i!}$, and μ_i, $i = 0, 1, 2, \ldots, m$ are chosen such that every zero of $\phi(z) = \sum_{i=0}^{m} \binom{n}{i}\mu_i z^i$, $m \le n$ lie in $|z| \le |z - \frac{n}{2}|$. It is obvious from Theorem 7 that the operator N is a B_n-operator. They also obtained the following Bernstein-type inequality for this operator.

Theorem 8 – *If $f(z)$ is a polynomial and have degree at most n, then*

$$|N[f](z)| \le |N[\varphi_n](z)| \max_{|z|=1} |f(z)|, \qquad for \quad |z| \ge 1, \tag{4}$$

where $\varphi_n(z) = z^n$. This bound is also best possible and for $f(z) = e^{i\alpha} \max_{|z|=1} |f(z)|z^n$, $\alpha \in \mathbb{R}$, equality holds in (4).

4   Bernstein's Inequality in L_p-norm

Let $f \in \mathcal{P}_n$, then the Hardy space norm is defined by

$$\|f\|_p = \left(\frac{1}{2\pi} \int_0^{2\pi} |f(e^{i\theta})|^p d\theta\right)^{1/p}, \quad 0 < p < \infty;$$

and the Mahler measure by

$$\|f\|_0 = \exp\left(\frac{1}{2\pi} \int_0^{2\pi} \ln\left(|f(e^{i\theta})|\right) d\theta\right).$$

It is not hard to see that $\lim_{p\to 0+} \|f\|_p = \|f\|_0$. Also note that the supremum norm satisfies $\lim_{p\to\infty} \|f\|_p = \max_{|z|=1} |f(z)|$.
    If $f \in \mathcal{P}_n$ then

$$\|f'\|_p \le n\|f\|_p, \quad 0 \le p \le \infty \tag{5}$$

Inequality (5) is an extension of (1) in the Hardy space norm settings. Zygmund 1932 proved this inequality for $p \geq 1$. It was unknown for quite a long time, whether inequality (5) holds for $0 < p < 1$ or not. Later, this case was settled by Arestov 1982. For $p = 0$, (5) is a consequence of a remarkable inequality of De Bruijn and Springer 1947.

Like the above extension of Bernstein's inequality by Zygmund, most of the Bernstein-type inequalities brought up in the first three sections of this review paper have also been extended in the Hardy space norm (see Aziz and N. Rather 2012; Aziz and N. Rather 1997; Gulzar 2016; Mir, Dewan, and Singh 2010). Those inequalities are usually called as Zygmund-type inequalities. Moreover, all the Bernstein-type inequalities in sup-norm mentioned in this paper can be easily derived from Theorem 4 by choosing the B_n-operator T suitably. Like sup-norm counterparts, from the expected extension of Theorem 4, the Zygmund-type inequalities for different B_n-operators should be derivable in a unified manner.

In one of most cited papers of Arestov 1982, one of his results may be considered as an extension of Theorem 4 in Hardy space norm. For $f(z) = \sum_{j=0}^{n} a_j z^j \in \mathcal{P}_n$ and $\vartheta = (\vartheta_0, \vartheta_1, \ldots, \vartheta_n) \in \mathbb{C}^{n+1}$, Arestov introduced the linear operator $\Lambda_\vartheta f(z) = \sum_{j=0}^{n} \vartheta_j a_j z^j$ in $\mathcal{P}_n$. He called Λ_ϑ admissible if one of the following properties is preserved by it.

(i) $f(z)$ having every zeros in $\{z \in \mathbb{C} : |z| \leq 1\}$,

(ii) $f(z)$ having every zeros in $\{z \in \mathbb{C} : |z| \geq 1\}$.

We end this section with the following result of Arestov.

Theorem 9 – *Let $\Theta : \mathbb{R} \to \mathbb{R}$ be a convex & non-decreasing function and $\Psi(x) = \Theta(\ln x)$ then for each $f \in \mathcal{P}_n$ and every admissible operator Λ_ϑ,*

$$\int_0^{2\pi} \Psi\left(|\Lambda_\vartheta f(e^{i\theta})|\right) d\theta \leq \int_0^{2\pi} \Psi\left(\lambda(\vartheta)|f(e^{i\theta})|\right) d\theta,$$

where $\lambda(\vartheta) = \max(|\vartheta_0|, |\vartheta_n|)$.

One can get (5) from above inequality by taking $\Psi(x) = x^p$, $p \in (0, \infty)$.

5   Concluding Remarks

From the proof of theorem 4, one gets an impression that it is hard to get the proof without putting the linearity of a B_n-operator or its zero-preserving feature in use. There are many operators defined on $\mathcal{P}_n$ which have zero-preserving property like Nagy's generalized derivative (see Cheung et al. 2015) but are not linear. A natural question one can ask here is that, whether theorem 4 holds for those operators which are not linear but does preserve location of zeros. Similarly, to what extent one

can withhold the zero-preserving property and keep linearity to achieve theorem 4. Moreover, when it comes to the question of extending theorem 4 in L_p-norm setting via theorem 9, a representation of a B_n-operator is needed.

These and many other questions concerning Markov Bernstein type inequalities are intended to be taken up in a subsequent work in the future.

Acknowledgments

References

Arestov, V. V. (1982). "On integral inequalities for trigonometric polynomials and their derivatives". *Mathematics of the USSR-Izvestiya* **18** (1), p. 1 (cit. on p. 193).

Aziz, A. and N. A. Rather (2012). "Some new generalizations of Zygmund-type inequalities for polynomials". *Math. Ineq. Appl.* **15**, pp. 469–486 (cit. on p. 191).

Aziz, A. and N. Rather (2012). "Some new generalizations of Zygmund-type inequalities for polynomials". *Math. Ineq. Appl* **15**, pp. 469–486 (cit. on p. 193).

Aziz, A. and N. Rather (1997). "L_p inequalities for polynomials". *Glasnik matematicki* **32** (1), p. 39 (cit. on p. 193).

Bernstein, S. (1912). "On the best approximation of continuous functions by polynomials of a given degree". *Comm. Soc. Math. Kharkow, Ser* **2** (13), pp. 49–194 (cit. on p. 188).

Cheung, P.-L. et al. (2015). "Higher-order, polar and Sz.-Nagy's generalized derivatives of random polynomials with independent and identically distributed zeros on the unit circle". *Comput. Methods Funct. Theory* **15** (1), pp. 159–186 (cit. on p. 193).

De Bruijn, N. and T. A. Springer (1947). "On the zeros of composition-polynomials". *Proceedings of the Section of Sciences of the Koninklijke Nederlandse Akademie van Wetenschappen te Amsterdam* **50** (7-8), pp. 895–903 (cit. on p. 193).

Gulzar, S. (2016). "Some Zygmund type inequalities for the sth derivative of polynomials". *Analysis Mathematica* **42** (4), pp. 339–352 (cit. on p. 193).

Gulzar, S. and N. A. Rather (2018). "On a Composition Preserving Inequalities between Polynomials". *Journal of Contemporary Mathematical Analysis (Armenian Academy of Sciences)* **53** (1), pp. 21–26 (cit. on p. 191).

Jain, V. K. (1997). "Generalization at certain well known inequalities for polynomials". *Glasnik Matematicki* **32**, pp. 45–52 (cit. on p. 191).

Lax, P. D. (1944). "Proof of a conjecture of P. Erdös on the derivative of a polynomial". *Bulletin of the American Mathematical Society* **50** (8), pp. 509–513 (cit. on p. 188).

Malik, M. A. (1969). "On the derivative of a polynomial". *Journal of the London Mathematical Society* **2** (1), pp. 57–60 (cit. on p. 189).

References

Marden, M. (1996). *Geometry of polynomials*. American Mathematical Soc., Providence (cit. on p. 191).

Markov, A. A. (1889). "On a problem of DI Mendeleev". *Zap. Im. Akad. Nauk* **62**, pp. 1–24 (cit. on p. 187).

Markov, V. A. (1916). "Über Polynome, die in einem gegebenen Intervalle moglichst wenig von Null abweichen," *Math. Ann.* **77**, pp. 218–258 (cit. on p. 188).

Milovanovic, G. V., T. M. Rassias, and D. Mitrinovic (1994). *Topics In Polynomials: Extremal Problems, Inequalities, Zeros*. World Scientific (cit. on p. 187).

Mir, A., K. K. Dewan, and N. Singh (2010). "Some L_p inequalities for polynomials". *Funct. Approx. Comment. Math* **42**, pp. 131–143 (cit. on p. 193).

Qazi, M. A. (1992). "On the maximum modulus of polynomials". *Proceedings of the American Mathematical Society* **115** (2), pp. 337–343 (cit. on p. 189).

Rahman, Q. I. and G. Schmeisser (2002). *Analytic theory of polynomials*. Oxford University Press (cit. on pp. 189, 190).

Rather, N. A., I. Dar, and S. Gulzar (2021). "On the zeros of certain composite polynomials and an operator preserving inequalities". *The Ramanujan Journal* **54** (3), pp. 605–612 (cit. on pp. 191, 192).

Zygmund, A. (1932). "A remark on conjugate series". *Proceedings of the London Mathematical Society* **2** (1), pp. 392–400 (cit. on p. 193).

Contents