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Finite quotients of symplectic groups vs mapping class groups

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Abstract

We give alternative computations of the Schur multiplier of $Sp(2g,\mathbb{Z}/D\mathbb{Z})$, when D is divisible by 4 and $g \geq 4$: a first one using K-theory arguments based on the work of Barge and Lannes and a second one based on the Weil representations of symplectic groups arising in abelian Chern-Simons theory. We can also retrieve this way Deligne's non-residual finiteness of the universal central extension $\widetilde{Sp(2g,\mathbb{Z})}$. We prove then that the image of the second homology into finite quotients of symplectic groups over a Dedekind domain of arithmetic type are torsion groups of uniformly bounded size. In contrast, quantum representations produce for every prime p, finite quotients of the mapping class group of genus $g \geq 3$ whose second homology image has p-torsion. We further derive that all central extensions of the mapping class group are residually finite and deduce that mapping class groups have Serre's property A_2 for trivial modules, contrary to symplectic groups. Eventually we compute the module of coinvariants $H_2(\mathfrak{sp}_{2g}(2))_{Sp(2g,\mathbb{Z}/2^k\mathbb{Z})} = \mathbb{Z}/2\mathbb{Z}$.

Keywords: Symplectic groups, group homology, mapping class group, central extension, quantum representation, residually finite.

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1 Introduction and statements

Let $\Sigma_{g,k}$ denote a connected oriented surface of genus g with k boundary components and $M_{g,k}$ be its mapping class group, namely the group of isotopy classes of orientation preserving homeomorphisms that fix point-wise the boundary components. If $k = 0$, we simply write M_g for $M_{g,0}$. The action of M_g on the integral homology of Σ_g equipped with some symplectic basis gives a surjective homomorphism $M_g \to Sp(2g,\mathbb{Z})$, and it is a natural and classical problem to compare the

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properties of these two groups. The present paper is concerned with the central extensions and 2-homology groups of these two groups and their finite quotients and is a sequel to Funar 2011 and Funar and Pitsch 2020.

Our first result is:

Theorem 1 – *The second homology group of finite principal congruence quotients of* $Sp(2g,\mathbb{Z})$, $g \geq 4$ *is*

$$H_2(Sp(2g,\mathbb{Z}/D\mathbb{Z})) = \begin{cases} \mathbb{Z}/2\mathbb{Z}, & \text{if } D \equiv 0 \pmod 4, \\ 0, & \text{otherwise.} \end{cases}$$

This result when D is not divisible by 4 is an old theorem of Stein (see Stein 1973, Thm. 2.13 and Prop. 3.3.a) while the case $D \equiv 0 \pmod 4$ remained open for a while, as mentioned in Putman 2012, remarks after Thm. 3.8, because the condition $D \not\equiv 0 \pmod 4$ seemed essential for all results in there. A short proof using geometric group theory was obtained by the authors in Funar and Pitsch 2020 and another proof was independently obtained in Benson et al. 2018 (see also Benson 2019). One of our aims here is to present alternative proofs based on mapping class group representations arising in the $U(1)$ Chern-Simons theory and K-theory, respectively.

The analogous result for special linear groups has long been known. The equality $H_2(SL(2,\mathbb{Z}/D\mathbb{Z})) = \mathbb{Z}/2\mathbb{Z}$, for $D \equiv 0 \pmod 4$ was proved in Beyl 1986 and for large n Dennis and Stein proved using K-theoretic methods that $H_2(SL(n,\mathbb{Z}/D\mathbb{Z})) = \mathbb{Z}/2\mathbb{Z}$, for $D \equiv 0 \pmod 4$, while $H_2(SL(n,\mathbb{Z}/D\mathbb{Z})) = 0$, otherwise, see Dennis and Stein 1973, Cor. 10.2.

Our main motivation for carrying the computation of Theorem 1 was to better understand the (non-)residual finiteness of central extensions. The second result of this paper is the following:

Theorem 2 – *The universal central extension* $\widetilde{Sp(2g,\mathbb{Z})}$ *is not residually finite when* $g \geq 4$ *since the image of the center under any homomorphism into a finite group has order at most two. Moreover, the image of the center has order two under the natural homomorphism of* $\widetilde{Sp(2g,\mathbb{Z})}$ *into the universal central extension of* $Sp(2g,\mathbb{Z}/D\mathbb{Z})$, *where* D *is a multiple of* 4 *and* $g \geq 4$.

The first part of this result is the statement of *Deligne's non-residual finiteness theorem* from Deligne 1978, which was stated for $g \geq 2$. In what concerns the sharpness statement, in Putman 2012, Thm. F the author has previously obtained the existence of finite index subgroups of $\widetilde{Sp(2g,\mathbb{Z})}$ which contain $2\mathbb{Z}$ but not $\mathbb{Z}$. We provide some explicit constructions of such finite index normal subgroups. The relation between Theorems 1 and 2 is somewhat intricate. For instance, the statement $H_2(Sp(2g,\mathbb{Z}/D\mathbb{Z})) \in \{0,\mathbb{Z}/2\mathbb{Z}\}$, for $g \geq 4$ is a consequence of Deligne's theorem. This statement and the second part of Theorem 2 actually imply Theorem 1 and this is

our first proof of the latter. However, we can reverse all implications and using now a different proof of Theorem 1, based on K-theory arguments, we derive from it another proof of Theorem 2. In particular, this provides a new proof of Deligne's theorem, independent of Moore's theory of topological central extensions from Moore 1968.

Remark 1 – Deligne proved that the image of twice the generator of the center of $\widetilde{Sp(2g,\mathbb{Z})}$ under any homomorphism into a finite group is trivial, for any $g \geq 2$. Moreover, the intersection of the finite index subgroups of $\widetilde{Sp(2g,\mathbb{Z})}$ is precisely the subgroup generated by twice the center generator, when $g \geq 4$. Theorem 2 provides explicit morphisms into finite groups for which the generator of the center maps into a nontrivial element.

Remark 2 – Note that $H_2(Sp(6,\mathbb{Z})) = \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, according to Stein 1975, while $Sp(4,\mathbb{Z}/4\mathbb{Z})$ is not perfect. Thus the central extension by $\mathbb{Z}$ considered in Deligne 1978 is not the universal central extension of $Sp(2g,\mathbb{Z})$, when $g \in \{2,3\}$. The computation of the Schur multiplier for small g was completed in Benson et al. 2018: $H_2(Sp(2g,\mathbb{Z}/D\mathbb{Z})) = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, for $g \in \{2,3\}$ and $D \equiv 0 \pmod 4$. This corrects a misprint in Funar and Pitsch 2020, where for $g = 3$ we only proved that the Schur multiplier is nontrivial.

The theorem stated in Deligne 1978 is much more general and covers higher rank Chevalley groups over number fields. The proof of Theorem 2 also shows that the residual finiteness of central extensions is directly related to the torsion of the second homology of finite quotients of the group. Then the general form of Deligne's theorem can be used to obtain bounds for the torsion arising in the second homology of finite quotients of symplectic groups over more general rings. First, we have:

Theorem 3 – *Let $\mathfrak{A}$ be the ring of S-integers of a number field which is not totally imaginary and $g \geq 3$ an integer. Then there is a uniform bound (independent of g and F) for the order of the torsion group $p_*(H_2(Sp(2g,\mathfrak{A}))) \subseteq H_2(F)$ for any surjective homomorphism $p : Sp(2g,\mathfrak{A}) \to F$ onto a finite group. Here $p_* : H_2(Sp(2g,\mathfrak{A})) \to H_2(F)$ denotes the map induced in homology and all homology groups are considered with (trivial) integral coefficients.*

The result is a rather immediate consequence of the general Deligne theorem from Deligne 1978, along with classical results from Borel and Serre 1971 and Bass, Milnor, and Serre 1967 on the congruence subgroup problem.

Remark 3 – With slightly more effort we can show that this holds also when the number field is totally imaginary due to the finiteness of the congruence kernel. Furthermore, the result holds for any Chevalley group instead of the symplectic group, with a similar proof. More generally it holds under the conditions of Deligne

1978, namely for every absolutely simple simply connected algebraic group $\mathbb{G}$ over a number field $\mathbb{K}$, S a finite set of places of $\mathbb{K}$ containing all archimedean ones and such that $\sum_{v\in S} \operatorname{rank} \mathbb{G}(\mathbb{K}_v) \geq 2$ and $\mathfrak{A}$ the associated ring of S-integers. The quasi-split assumption in Deligne 1978 was removed in Raghunathan 1984. However, for the sake of simplicity we will only consider symplectic groups in the sequel.

Theorem 3 contrasts with the abundance of finite quotients of mapping class groups:

Theorem 4 – *For any prime p and $g \geq 3$ there exist surjective homomorphisms $\pi : M_g \to F$ onto finite groups F such that $\pi_*(H_2(M_g)) \subseteq H_2(F)$ has p-torsion elements, and in particular is not trivial.*

We prove this result by exhibiting explicit finite quotients of the universal central extension of a mapping class group that arise from the so-called quantum representations. We refine here the approach in Funar 2011 where the first author proved that central extensions of M_g by $\mathbb{Z}$ are residually finite. In the meantime, it was proved in Funar 2013; Masbaum and Roberts 1995 by more sophisticated tools that the set of quotients of mapping class groups contains arbitrarily large rank finite groups of Lie type. Notice however that the family of quotients obtained in Theorem 4 are different in nature than those obtained in Funar 2013; Masbaum and Reid 2012, although their source is the same (see Proposition 8 for details).

Theorem 4 shows that in the case of non-abelian quantum representations of mapping class groups there is no finite central extension for which all projective representations could be lifted to linear representations, when the genus is $g \geq 2$ (see Corollary 3 for the precise statement).

When G is a discrete group we denote by $\widehat{G}$ its *profinite completion*, i.e. the projective limit of the directed system of all its finite quotients. There is a natural homomorphism $i : G \to \widehat{G}$ which is injective if and only if G is residually finite. A *discrete $\widehat{G}$-module* is an abelian group endowed with a *continuous* action of $\widehat{G}$. We will simply call them $\widehat{G}$-modules in the sequel. We say that a $\widehat{G}$-module is *trivial* if the $\widehat{G}$-action is trivial. Recall following Serre 1994, p. I.2.6 that:

Definition 1 – A discrete group G has property A_n for the finite $\widehat{G}$-module M if the homomorphism $H^k(\widehat{G}, M) \to H^k(G, M)$ is an isomorphism for $k \leq n$ and injective for $k = n+1$. Furthermore G is called *good* if it has property A_n for all n and for all finite $\widehat{G}$-modules.

It is known, for instance, that all groups have property A_1.

Now, Deligne's theorem on the non-residual finiteness of the universal central extension of $Sp(2g, \mathbb{Z})$ actually is equivalent to the fact that $Sp(2g, \mathbb{Z})$ has not property A_2 for the trivial $Sp(2g, \mathbb{Z})$-modules (see also Grunewald, Jaikin-Zapirain, and Zalesskii 2008).

Our next result is:

Theorem 5 – *For $g \geq 4$ the mapping class group M_g has property A_2 for the trivial $\widehat{M_g}$-modules.*

Our proof also yields the following:

Corollary 1 – *Central extensions of M_g, $g \geq 4$, by finite abelian groups are virtually trivial and can be obtained as pull-backs from central extensions of finite quotients of M_g.*

The last part of this article is devoted to a partial extension of the method used by Putman in Putman 2012 to compute $H_2(Sp(2g,\mathbb{Z}/D\mathbb{Z}))$, when $D \equiv 0 \pmod 4$, using induction. One key point is to show that there is a potential $\mathbb{Z}/2\mathbb{Z}$ factor that appears for $H_2(Sp(2g,\mathbb{Z}/4\mathbb{Z}))$. Although we couldn't complete the proof of Theorem 1 this way, our main result in this direction may be of independent interest. Set $\mathfrak{sp}_{2g}(p)$ for the additive group of those $2g$-by-$2g$ matrices M with entries in $\mathbb{Z}/p\mathbb{Z}$ that satisfy the equation $M^\top J_g + J_g M \equiv 0 (\text{mod } p)$, where $J_g = \begin{pmatrix} 0 & \mathbf{1}_g \\ -\mathbf{1}_g & 0 \end{pmatrix}$ is the symplectic form. Then $\mathfrak{sp}_{2g}(p)$ is endowed with a natural $Sp(2g,\mathbb{Z}/p\mathbb{Z})$-action.

Theorem 6 – *For any integers $g \geq 4$, $k \geq 1$ and prime p, the space of co-invariants in homology is:*

$$H_2(\mathfrak{sp}_{2g}(p))_{Sp(2g,\mathbb{Z}/p^k\mathbb{Z})} = \begin{cases} 0, & \text{if } p \text{ is odd,} \\ \mathbb{Z}/2\mathbb{Z}, & \text{if } p = 2. \end{cases}$$

An immediate consequence of this result is the alternative $H_2(Sp(2g,\mathbb{Z}/4\mathbb{Z})) \in \{0, \mathbb{Z}/2\mathbb{Z}\}$, without use of Deligne's theorem.

The plan of this article is the following.

In Section 2 we prove Theorem 1. Although it is easy to show that the groups $H_2(Sp(2g,\mathbb{Z}/2^k\mathbb{Z}))$ are cyclic for $g \geq 4$, their non-triviality is much more involved. That this group is trivial for $k = 1$ is a known fact, for instance by Stein's results Stein 1978. We give two different proofs of the non-triviality, each one of them having its advantages and disadvantages in terms of bounds for detections or sophistication. The first proof is K-theoretical in nature and uses a generalization of Sharpe's exact sequence relating K-theory to symplectic K-theory due to Barge and Lannes Barge and Lannes 2008. Indeed, by the stability results, this $\mathbb{Z}/2\mathbb{Z}$ should correspond to a class in $KSp_2(\mathbb{Z}/4\mathbb{Z})$. There is a natural map from this group to a Witt group of symmetric non-degenerate bilinear forms on free $\mathbb{Z}/4\mathbb{Z}$-modules, and it turns out that the class is detected by the class of the bilinear map of matrix $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$. The second proof uses mapping class groups. We show that there is a perfect candidate to detect this $\mathbb{Z}/2\mathbb{Z}$ that comes from a Weil representation of the symplectic group.

This is, by construction, a representation of $Sp(2g,\mathbb{Z})$ into a projective unitary group that factors through $Sp(2g,\mathbb{Z}/4n\mathbb{Z})$. To show that it detects the factor $\mathbb{Z}/2\mathbb{Z}$ it is enough to show that this representation does not lift to a linear representation. We will show that the pull-back of the representation on the mapping class group M_g does not linearize. This proof relies on deep results from Gervais 1996. The projective representation that we use is related to the theory of theta functions on symplectic groups, this relation is explained in an appendix to this article.

In Section 3 we first state the relation between the torsion in the second homology of a perfect group and residual finiteness of its universal central extension. We then prove Theorem 3 for Dedekind domains by analyzing Deligne's central extension. We further specify our discussion to the group $Sp(2g,\mathbb{Z})$, and show how the result stated in Theorem 1 allows to show that Deligne's result is sharp.

Finally in Section 4 we discuss the case of the mapping class groups and prove Theorem 4 and Theorem 5 using the quantum representations that arise from the $SU(2)$-TQFT's. These representations are the non-abelian counterpart of the Weil representations of symplectic groups, which might be described as the quantum representations that arise from the $U(1)$-TQFT.

Finally, in appendix A we give a small overview of the relation between Weil representations and extensions of the symplectic group.

In all this work, unless otherwise specified, all (co)homology groups are with coefficients in $\mathbb{Z}$, and we drop it from the notation so that for a group G, $H_*(G) = H_*(G;\mathbb{Z})$ and $H^*(G) = H^*(G;\mathbb{Z})$.

2 Proof of Theorem 1

2.1 Preliminaries

Let $D = p_1^{n_1} p_2^{n_2} \cdots p_s^{n_s}$ be the prime decomposition of an integer D. Then, according to Newman and Smart 1964, Thm. 5 we have $Sp(2g,\mathbb{Z}/D\mathbb{Z}) = \oplus_{i=1}^{s} Sp(2g,\mathbb{Z}/p_i^{n_i}\mathbb{Z})$. Since symplectic groups are perfect for $g \geq 3$, see e.g. Putman 2012, Thm. 5.1, from the Künneth formula, we derive:

$$H_2(Sp(2g,\mathbb{Z}/D\mathbb{Z})) = \oplus_{i=1}^{s} H_2(Sp(2g,\mathbb{Z}/p_i^{n_i}\mathbb{Z})).$$

Then, from Stein's computations for $D \not\equiv 0 \pmod 4$, see Stein 1973, 1978, Theorem 1 is equivalent to the statement:

$$H_2(Sp(2g,\mathbb{Z}/2^k\mathbb{Z})) = \mathbb{Z}/2\mathbb{Z}, \text{ for all } g \geq 4, k \geq 2.$$

We will freely use in the sequel two classical results due to Stein. *Stein's isomorphism theorem*, see Stein 1973, Prop. 3.3.(a), states that there is an isomorphism:

$$H_2(Sp(2g,\mathbb{Z}/2^k\mathbb{Z})) \simeq H_2(Sp(2g,\mathbb{Z}/2^{k+1}\mathbb{Z})), \text{ for all } g \geq 3, k \geq 2.$$

Further, *Stein's stability theorem* from Stein 1973, Thm. 2.13 states that the stabilization homomorphism $Sp(2g,\mathbb{Z}/2^k\mathbb{Z}) \hookrightarrow Sp(2g+2,\mathbb{Z}/2^k\mathbb{Z})$ induces an isomorphism:

$$H_2(Sp(2g,\mathbb{Z}/2^k\mathbb{Z})) \simeq H_2(Sp(2g+2,\mathbb{Z}/2^k\mathbb{Z})), \text{ for all } g \geq 4, k \geq 1.$$

Therefore, to prove Theorem 1 it suffices to show that:

$$H_2(Sp(2g,\mathbb{Z}/2^k\mathbb{Z})) = \mathbb{Z}/2\mathbb{Z}, \text{ for some } g \geq 4, k \geq 2.$$

We provide hereafter two different proofs of this statement, each having its own advantage.

The first proof, based on an extension of Sharpe's sequence in symplectic K-theory from Barge and Lannes 2008, gives the result already for $Sp(2g,\mathbb{Z}/4\mathbb{Z})$. Moreover, this proof does not rely on Deligne's theorem.

For the second proof, the starting point is the following intermediary result:

Proposition 1 – *We have* $H_2(Sp(2g,\mathbb{Z}/2^k\mathbb{Z})) \in \{0,\mathbb{Z}/2\mathbb{Z}\}$, *when* $g \geq 4$ *and* $k \geq 2$.

This was obtained in Funar and Pitsch 2020, Prop. 3.1 as an immediate consequence of Deligne's theorem. A direct proof of the alternative $H_2(Sp(2g,\mathbb{Z}/4\mathbb{Z})) \in \{0,\mathbb{Z}/2\mathbb{Z}\}$ when $g \geq 4$ will be given in Section 5.1, under the form of Corollary 4 of Theorem 6. When $g = 3$ our arguments only provide a weaker statement, namely that $H_2(Sp(6,\mathbb{Z}/2^k\mathbb{Z})) \in \{0,\mathbb{Z}/2\mathbb{Z},\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}\}$.

Then it will be enough to find a non-trivial extension of $Sp(2g,\mathbb{Z}/2^k\mathbb{Z})$ by $\mathbb{Z}/2\mathbb{Z}$ for some $g \geq 4$, $k \geq 2$.

Our second proof seems more elementary and provides an explicit non-trivial central extension of $Sp(2g,\mathbb{Z}/4n\mathbb{Z})$ by $\mathbb{Z}/2\mathbb{Z}$, for all integers $n \geq 1$. Moreover, it does not use Stein's isomorphism theorem and relies instead on the study of the Weil representations of symplectic groups, or equivalently abelian quantum representations of mapping class groups. Since these representations come from theta functions this approach is deeply connected to Putman's approach. In fact the proof of the Theorem F in Putman 2012 is based on his Lemma 5.5 whose proof needed the transformation formulas for the classical theta nulls.

2.2 A K-theory computation of $H_2(Sp(2g,\mathbb{Z}/4\mathbb{Z}))$

The proof below uses slightly more sophisticated tools which were developed in Barge and Lannes 2008 and allow us to bypass Deligne's theorem. According to Stein's stability theorem Stein 1973 it is enough to prove that $H_2(Sp(2g,\mathbb{Z}/4\mathbb{Z})) = \mathbb{Z}/2\mathbb{Z}$, for g large. It is well-known that the second homology of the linear and symplectic groups can be interpreted in terms of the K-theory group K_2. Denote by $K_1(\mathfrak{A}), K_2(\mathfrak{A})$ and $KSp_1(\mathfrak{A}), KSp_2(\mathfrak{A})$ the groups of algebraic K-theory of the stable

linear groups and symplectic groups over the commutative ring $\mathfrak{A}$, respectively. See Hahn and O'Meara 1989 for definitions. Our claim is equivalent to the fact that $KSp_2(\mathbb{Z}/4\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$.

For an arbitrary ring R, the group $V(R)$ is defined as follows (see Barge and Lannes 2008, Section 4.5.1). Consider the set of triples $(L; q_0, q_1)$, where L is a free R-module of finite rank and q_0 and q_1 are non-degenerate symmetric bilinear forms. Two such triples $(L; q_0, q_1)$ and $(L'; q_0', q_1')$ are equivalent, if there exists an R-linear isomorphism $a : L \to L'$ such that $a^* \circ q_0' \circ a = q_0$ and $a^* \circ q_1' \circ a = q_1$. Under orthogonal sum these triples form a monoid. The group $V(R)$ is by definition the quotient of the Grothendieck group associated to the monoid of such triples by the subgroup generated by Chasles' relations, that is the subgroup generated by the elements of the form:

$$[L; q_0, q_1] + [L; q_1, q_2] - [L; q_0, q_2].$$

Our key ingredient is the exact sequence from Barge and Lannes 2008, Thm. 5.4.1, which is a generalization of Sharpe's exact sequence (see Hahn and O'Meara 1989, Thm. 5.6.7) in K-theory:

$$K_2(\mathbb{Z}/4\mathbb{Z}) \to KSp_2(\mathbb{Z}/4\mathbb{Z}) \to V(\mathbb{Z}/4\mathbb{Z}) \to K_1(\mathbb{Z}/4\mathbb{Z}) \to 1. \tag{1}$$

We first show:

Lemma 1 – *The homomorphism $K_2(\mathbb{Z}/4\mathbb{Z}) \to KSp_2(\mathbb{Z}/4\mathbb{Z})$ is trivial.*

Proof (Proof of Lemma 1). Recall from Barge and Lannes 2008 that this homomorphism is induced by the hyperbolization inclusion $SL(g, \mathbb{Z}/4\mathbb{Z}) \to Sp(2g, \mathbb{Z}/4\mathbb{Z})$, which sends the matrix A to $A \oplus (A^{-1})^\top$. By stability of the homology groups of the special linear and symplectic groups it is enough to show that the induced map

$$H : H_2(SL(g, \mathbb{Z}/4\mathbb{Z})) \to H_2(Sp(2g, \mathbb{Z}/4\mathbb{Z}))$$

is trivial for $g \geq 5$.

Together with the hyperbolization, mod 4 reduction of the coefficients gives us a commutative diagram:

$$\begin{array}{ccc} H_2(SL(g,\mathbb{Z})) & \longrightarrow & H_2(Sp(2g,\mathbb{Z})) \\ \downarrow & & \downarrow \\ H_2(SL(g,\mathbb{Z}/4\mathbb{Z})) & \longrightarrow & H_2(Sp(2g,\mathbb{Z}/4\mathbb{Z})) \end{array}$$

It is known that $H_2(SL(g, \mathbb{Z})) = \mathbb{Z}/2\mathbb{Z}$, when $g \geq 5$ and $H_2(Sp(2g, \mathbb{Z})) = \mathbb{Z}$, when $g \geq 4$, see e.g. Matsumoto 1969 and Milnor 1971, Thm. 10.1, Thm. 5.10, Remark. Moreover, the homomorphism induced in homology $H_2(SL(g, \mathbb{Z})) \to H_2(SL(g, \mathbb{Z}/4\mathbb{Z}))$ by

the reduction mod 4 of coefficients is surjective, see Milnor 1971, section 10 pp. 92. Alternatively, we can infer it from the proof Funar and Pitsch 2020, Prop. 3.2. Actually, Dennis proved that this map is an isomorphism and hence both groups are isomorphic to $\mathbb{Z}/2\mathbb{Z}$, see Dennis and Stein 1975. It follows that the hiperbolization map h is trivial, and so is H. □

Remark 4 – An alternative argument is as follows. The hyperbolization homomorphism $H : K_2(\mathbb{Z}/4\mathbb{Z}) \to KSp_2(\mathbb{Z}/4\mathbb{Z})$ sends the Dennis-Stein symbol $\{r,s\}$ to the Dennis-Stein symplectic symbol $[r^2,s]$, see e.g. Hahn and O'Meara 1989, Paragraph 5.6.2. According to Stein 1973, Prop. 3.3 (b) the group $K_2(\mathbb{Z}/4\mathbb{Z})$ is generated by $\{-1,-1\}$ and thus its image by H is generated by $[1,-1] = 0$.

Going back to Sharpe's exact sequence, it is known that:

$$K_1(\mathbb{Z}/4\mathbb{Z}) \cong (\mathbb{Z}/4\mathbb{Z})^* \cong \mathbb{Z}/2\mathbb{Z}, \tag{2}$$

and the problem is to compute the discriminant map $V(\mathbb{Z}/4\mathbb{Z}) \to K_1(\mathbb{Z}/4\mathbb{Z})$.

Recall from Barge and Lannes 2008, Section 4.5.1 that there is a canonical map from $V(R)$ to the Grothendieck-Witt group of symmetric non-degenerate bilinear forms over free modules that sends the class $[L;q_0,q_1]$ to $q_1 - q_0$. Since $\mathbb{Z}/4\mathbb{Z}$ is a local ring, we know that $SK_1(\mathbb{Z}/4\mathbb{Z}) = 1$ and hence by Barge and Lannes 2008, Corollary 4.5.1.5 we have a pull-back square of abelian groups:

$$\begin{array}{ccc} V(\mathbb{Z}/4\mathbb{Z}) & \longrightarrow & I(\mathbb{Z}/4\mathbb{Z}) \\ \downarrow & & \downarrow \\ (\mathbb{Z}/4\mathbb{Z})^* & \longrightarrow & (\mathbb{Z}/4\mathbb{Z})^*/((\mathbb{Z}/4\mathbb{Z})^*)^2 \end{array}$$

where $I(\mathbb{Z}/4\mathbb{Z})$ is a the augmentation ideal of the Grothendieck-Witt ring of $\mathbb{Z}/4\mathbb{Z}$. But $(\mathbb{Z}/4\mathbb{Z})^* = \{1,3\}$, and only 1 is a square, hence the bottom arrow in the square is an isomorphism $\mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z}$. Thus $V(\mathbb{Z}/4\mathbb{Z}) \cong I(\mathbb{Z}/4\mathbb{Z})$ and the kernel of $V(\mathbb{Z}/4\mathbb{Z}) \to (\mathbb{Z}/4\mathbb{Z})^* \cong K_1(\mathbb{Z}/4\mathbb{Z})$ is the kernel of the discriminant homomorphism $I(\mathbb{Z}/4\mathbb{Z}) \to (\mathbb{Z}/4\mathbb{Z})^*/((\mathbb{Z}/4\mathbb{Z})^*)^2$. To compute $V(\mathbb{Z}/4\mathbb{Z})$ it is therefore enough to compute the Witt ring $W(\mathbb{Z}/4\mathbb{Z})$. Recall that this is the quotient of the monoid of symmetric non-degenerate bilinear forms over finitely generated projective modules modulo the sub-monoid of *split* forms. A bilinear form is split if the underlying free module contains a direct summand N such that $N = N^\perp$. By a classical result of Kaplansky, finitely generated projective modules over $\mathbb{Z}/4\mathbb{Z}$ are free. Then, from Milnor and Husemoller 1973, Lemma 6.3 any split form can be written in matrix form as:

$$\begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & A \end{pmatrix},$$

for some symmetric matrix A, where $\mathbf{1}$ denotes the identity matrix. Isotropic submodules form an inductive system, and therefore any isotropic submodule is

contained in a maximal one. These have all the same rank. In the case of a split form this rank is necessarily half of the rank of the underlying free module, which is therefore even. The main difficulty in the following computation is due to the fact that 2 is not a unit in $\mathbb{Z}/4\mathbb{Z}$, so that the classical Witt cancellation lemma is not true. As usual, for any invertible element u of $\mathbb{Z}/4\mathbb{Z}$ we denote by $\langle u\rangle$ the non-degenerate symmetric bilinear form on $\mathbb{Z}/4\mathbb{Z}$ of determinant u.

Proposition 2 – *The Witt ring $W(\mathbb{Z}/4\mathbb{Z})$ is isomorphic to $\mathbb{Z}/8\mathbb{Z}$, and it is generated by the class of $\langle -1\rangle$.*

The computation of $W(\mathbb{Z}/4\mathbb{Z})$ was obtained independently in Gurevich and Hadani 2012.

The discriminant of $\omega = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ is -1 and in the proof of Proposition 2 below we show that its class is non-trivial in $W(\mathbb{Z}/4\mathbb{Z})$ and hence it represents a non-trivial element in the kernel of the discriminant map $I(\mathbb{Z}/4\mathbb{Z}) \to (\mathbb{Z}/4\mathbb{Z})^*/(\mathbb{Z}/4\mathbb{Z})^{*2}$. From the Cartesian diagram above we get that it also represents a non-trivial element in the kernel of the leftmost vertical homomorphism $V(\mathbb{Z}/4\mathbb{Z}) \to K_1(\mathbb{Z}/4\mathbb{Z})$. In particular $KSp_2(\mathbb{Z}/4\mathbb{Z})$ is $\mathbb{Z}/2\mathbb{Z}$.

Proof (Proof of Proposition 2). According to Milnor and Husemoller 1973, p. I.3.3 every symmetric bilinear form is equivalent to a direct sum of a diagonal form (with invertible entries) and a bilinear form β on a submodule $N \subseteq V$ such that $\beta(x,x)$ is not a unit, for every $x \in N$. Thus, given a free $\mathbb{Z}/4\mathbb{Z}$-module L, any non-degenerate symmetric bilinear form on L is an orthogonal sum of copies of $\langle 1\rangle$, of $\langle -1\rangle$ and of a bilinear form β on a free summand N such that for all $x \in N$ we have $\beta(x,x) \in \{0,2\}$. Fix a basis $e_1, \cdots, e_n$ of N. Let B denote the matrix of β in this basis. Expanding the determinant of β along the first column we see that there must be an index $i \geq 2$ such that $\beta(e_1, e_i) = \pm 1$, for otherwise the determinant would not be invertible. Without loss of generality we may assume that $i = 2$ and that $\beta(e_1, e_2) = 1$. Replacing if necessary e_j for $j \geq 3$ by $e_j - \frac{\beta(e_1,e_j)}{\beta(e_1,e_2)} e_2$, we may assume that B is of the form:

$$\begin{pmatrix} s & 1 & 0 \\ 1 & t & c \\ 0 & c^\top & A \end{pmatrix}$$

where A and c are a square matrix and a row matrix respectively, of the appropriate sizes, and $s, t \in \{0,2\}$. Since $st = 0$, the form β restricted to the submodule generated by e_1 and e_2 defines a non-singular symmetric bilinear form and therefore $N = \langle e_1, e_2\rangle \oplus \langle e_1, e_2\rangle^\perp$, where on the first summand the bilinear form is either split (if at least one of s or t is 0) or is ω. By induction we have that any symmetric bilinear form is an orthogonal sum of copies of $\langle 1\rangle, \langle -1\rangle$, of

$$\omega = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

and split spaces.

It's a classical fact (see Milnor and Husemoller 1973, Chapter I) that in $W(\mathbb{Z}/4\mathbb{Z})$ one has $\langle 1\rangle = -\langle -1\rangle$. Also $\langle -1\rangle \oplus \langle -1\rangle \oplus \langle -1\rangle \oplus \langle -1\rangle$ is isometric to $-\omega \oplus \langle -1\rangle \oplus \langle 1\rangle$. To see this notice that, if $e_1, \ldots, e_4$ denotes the preferred basis for the former bilinear form, then the matrix in the basis $e_1+e_2, e_1+e_3, e_1-e_2-e_3, e_4$ is precisely $-\omega \oplus \langle -1\rangle \oplus \langle 1\rangle$. Also, in the Witt ring $\langle 1\rangle \oplus \langle -1\rangle = 0$, so $4\langle -1\rangle = -\omega$. Finally, if we denote by e_1, e_2, e_3, e_4 the preferred basis of the orthogonal sum $\omega \oplus \omega$, then a direct computation shows that the subspace generated by $e_1 + e_3$ and $e_2 + e_4$ is isotropic, hence $2\omega = 0$ and in particular ω has order at most 2. All these show that $W(\mathbb{Z}/4\mathbb{Z})$ is generated by $\langle -1\rangle$ and that this form is of order at most 8. It remains to show that ω is a non-trivial element to finish the proof.

Assume the contrary, namely that there is a split form σ such that $\omega \oplus \sigma$ is split. We denote by A the underlying module of ω and by $\{a, b\}$ its preferred basis. Similarly, we denote by S the underlying space of σ of dimension $2n$ and by $\{e_1, \ldots, e_n, f_1, \ldots, f_n\}$ a basis that exhibits it as a split form. Let F denote the submodule generated by $f_1, \ldots, f_n$. By construction $e_1, \ldots, e_n$ generate a totally isotropic submodule E of rank n in $A \oplus S$, and since it is included into a maximal isotropic submodule, we can adjoin to it a new element v such that $v, e_1, \ldots, e_n$ is a totally isotropic submodule of $A \oplus S$, and hence has rank $n+1$. By definition there are unique elements $x, y \in \mathbb{Z}/4\mathbb{Z}$ and elements $\varepsilon \in E$ and $\phi \in F$ such that $v = xa + yb + \varepsilon + \phi$. Since v is isotropic we have:

$$2x^2 + 2xy + 2y^2 + 2\sigma(\varepsilon, \phi) + \sigma(\phi, \phi) \equiv 0 \pmod 4.$$

Since $E \oplus \mathbb{Z}/4\mathbb{Z}v$ is totally isotropic, then $(\omega \oplus \sigma)(e_i, v) = 0$, for every $1 \le i \le n$. Now, $\sigma(e_i, \varepsilon) = 0$, since E is isotropic in S, $(\omega \oplus \sigma)(e_i, a) = 0$ and $(\omega \oplus \sigma)(e_i, b) = 0$, so that $\sigma(e_i, \phi) = 0$. In particular, since ϕ belongs to the dual module to E with respect to σ, $\phi = 0$, so the above equation implies:

$$2x^2 + 2xy + 2y^2 \equiv 0 \pmod 4.$$

But now this can only happen when x and y are multiples of 2 in $\mathbb{Z}/4\mathbb{Z}$. Therefore reducing mod 2, we find that v mod 2 belongs to the $\mathbb{Z}/2\mathbb{Z}$ -vector space generated by the mod 2 reduction of the elements $e_1, \ldots, e_n$, and by Nakayama's lemma this contradicts the fact that the $\mathbb{Z}/4\mathbb{Z}$-module generated by $v, e_1, \ldots, e_n$ has rank $n+1$.□

Remark 5 – In Dennis and Stein 1975 the authors prove that $K_2(\mathbb{Z}/2^k\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$ for any $k \ge 2$. If we had proved directly that the class of the symmetric bilinear form $\begin{pmatrix} 2^{k-1} & 1 \\ 1 & 2^{k-1} \end{pmatrix}$ generates the kernel of the homomorphism $I(\mathbb{Z}/2^k\mathbb{Z}) \to (\mathbb{Z}/2^k\mathbb{Z})^*/(\mathbb{Z}/2^k\mathbb{Z})^{*2}$, which is of order two for all $k \ge 2$, then the Sharpe-type exact sequence of Barge and Lannes would yield $KSp_2(\mathbb{Z}/2^k\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$, for any $k \ge 2$. This would permit us to do without Stein's stability results. However the description

of the Witt group $W^f(\mathbb{Z}/2^k\mathbb{Z})$ seems more involved for $k \geq 3$ and it seems more cumbersome than worthy to fill in all the details.

2.3 Detecting the non-trivial class via Weil representations

Preliminaries on Weil representations

The projective representation that we use is related to the theory of theta functions on symplectic groups, this relation being briefly explained in an appendix to this article. Although the Weil representations of symplectic groups over finite fields of characteristic different from 2 is a classical subject present in many textbooks, the slightly more general Weil representations associated to finite rings of the form $\mathbb{Z}/k\mathbb{Z}$ received less consideration until recently. They first appeared in print as representations associated to finite abelian groups in Kloosterman 1946 for genus $g = 1$ and were extended to locally compact abelian groups in Weil 1964, Chapter I, and independently in the work of Igusa and Shimura on theta functions Hano 1969; Igusa 1964; Shimura 1966 and in the physics literature Hannay and Berry 1980. They were rediscovered as monodromies of generalized theta functions arising in the $U(1)$ Chern-Simons theory in Funar 1993, 2000; Gocho 1992 and then in finite-time frequency analysis, see Kaiblinger and Neuhauser 2009 and references from there. In Funar 1993, 2000; Gocho 1992 these are projective representations of the symplectic group factorizing through the finite congruence quotients $Sp(2g, \mathbb{Z}/2k\mathbb{Z})$, which are only defined for even $k \geq 2$. However, for odd k the monodromy of theta functions leads to representations of the theta subgroup of $Sp(2g, \mathbb{Z})$. These also factor through the image of the theta group into the finite congruence quotients $Sp(2g, \mathbb{Z}/2k\mathbb{Z})$. Notice however that the original Weil construction works as well for $\mathbb{Z}/k\mathbb{Z}$ with odd k, see e.g. Gurevich, Hadani, and Howe 2010; Kaiblinger and Neuhauser 2009.

It is well-known, see Weil 1964, sections 43, 44 or Ranga Rao 1993, Prop. 5.8, that these projective Weil representations lift to linear representations of the integral metaplectic group, which is the pull-back of the symplectic group in a double cover of $Sp(2g, \mathbb{R})$. The usual way to resolve the projective ambiguities is to use the Maslov cocycle (see e.g. Turaev 1994). Moreover, it is known that the Weil representations over finite fields of odd characteristic and over $\mathbb{C}$ actually are linear representations. In fact the vanishing of the second power of the augmentation ideal of the Witt ring of such fields (see e.g. Lam 2004; Suslin 1987) implies that the corresponding metaplectic extension splits. This contrasts with the fact that Weil representations over $\mathbb{R}$ (or any local field different from $\mathbb{C}$) are true representations of the real metaplectic group and cannot be linearized (see e.g. Lam 2004). The Weil representations over local fields of characteristic 2 are subtler as they are rather representations of a double cover of the so-called pseudo-symplectic group (see Weil 1964 and Gurevich and Hadani 2012 for recent work).

Let $k \geq 2$ be an integer, and denote by $\langle , \rangle$ the standard bilinear form on $(\mathbb{Z}/k\mathbb{Z})^g \times (\mathbb{Z}/k\mathbb{Z})^g \to \mathbb{Z}/k\mathbb{Z}$. The Weil representation we consider is a representation in the unitary group of the complex vector space $\mathbb{C}^{(\mathbb{Z}/k\mathbb{Z})^g}$ endowed with its standard Hermitian form. Notice that the canonical basis of this vector space is canonically labeled by elements in $(\mathbb{Z}/k\mathbb{Z})^g$.

It is well-known (see e.g. Igusa 1972) that $Sp(2g, \mathbb{Z})$ is generated by the matrices having one of the following forms: $\begin{pmatrix} \mathbf{1}_g & B \\ 0 & \mathbf{1}_g \end{pmatrix}$ where $B = B^\top$ has integer entries, $\begin{pmatrix} A & 0 \\ 0 & (A^\top)^{-1} \end{pmatrix}$ where $A \in GL(g, \mathbb{Z})$ and $\begin{pmatrix} 0 & -\mathbf{1}_g \\ \mathbf{1}_g & 0 \end{pmatrix}$.

We can now define the Weil representations (see the Appendix for more details) on these generating matrices as follows:

$$\rho_{g,k}\begin{pmatrix} \mathbf{1}_g & B \\ 0 & \mathbf{1}_g \end{pmatrix} = \mathrm{diag}\left(\exp\left(\frac{\pi\sqrt{-1}}{k}\langle m, Bm\rangle\right)\right)_{m\in(\mathbb{Z}/k\mathbb{Z})^g}, \tag{3}$$

where diag stands for diagonal matrix with given entries;

$$\rho_{g,k}\begin{pmatrix} A & 0 \\ 0 & (A^\top)^{-1} \end{pmatrix} = (\delta_{A^\top m,n})_{m,n\in(\mathbb{Z}/k\mathbb{Z})^g}, \tag{4}$$

where δ stands for the Kronecker symbol;

$$\rho_{g,k}\begin{pmatrix} 0 & -\mathbf{1}_g \\ \mathbf{1}_g & 0 \end{pmatrix} = k^{-g/2}\exp\left(-\frac{2\pi\sqrt{-1}\langle m, n\rangle}{k}\right)_{m,n\in(\mathbb{Z}/k\mathbb{Z})^g}. \tag{5}$$

It is proved in Funar 2000; Gocho 1992, that for even k these formulas define a unitary representation $\rho_{g,k}$ of $Sp(2g, \mathbb{Z})$ in $U(\mathbb{C}^{(\mathbb{Z}/k\mathbb{Z})^g})/R_8$. Here $U(\mathbb{C}^N) = U(N)$ denotes the unitary group of dimension N and $R_8 \subset U(1) \subset U(\mathbb{C}^N)$ is the subgroup of scalar matrices whose entries are roots of unity of order 8. For odd k the same formulas define representations of the theta subgroup $Sp(2g, 1, 2)$ (see Funar 2000; Igusa 1964, 1972). Notice that by construction $\rho_{g,k}$ factors through $Sp(2g, \mathbb{Z}/2k\mathbb{Z})$ for even k and through the image of the theta subgroup in $Sp(2g, \mathbb{Z}/k\mathbb{Z})$ for odd k.

Our definition gives us a map $\rho_{g,k} : Sp(2g, \mathbb{Z}) \to U(\mathbb{C}^{(\mathbb{Z}/k\mathbb{Z})^g})$ satisfying the cocycle condition:

$$\rho_{g,k}(AB) = \eta(A, B)\rho_{g,k}(A)\rho_{g,k}(B)$$

for all $A, B \in Sp(2g, \mathbb{Z})$ and some $\eta(A, B) \in R_8$.

Outline of the proof of Theorem 1

The key step is to establish the following:

Proposition 3 – *The projective Weil representation $\rho_{g,k}$ of $Sp(2g,\mathbb{Z})$, for $g \geq 3$ and even k does not lift to linear representations of $Sp(2g,\mathbb{Z})$, namely it determines a generator of the group $H^2(Sp(2g,\mathbb{Z}/2k\mathbb{Z});\mathbb{Z}/2\mathbb{Z})$.*

Remark 6 – For odd k it was already known that Weil representations did not detect any non-trivial element, i.e. that the projective representation $\rho_{g,k}$ lifts to a linear representation Andler and Ramanan 1996. We will give a very short outline of this at the end of the Appendix.

Proposition 1 states that the Schur multiplier $H_2(Sp(2g,\mathbb{Z}/D\mathbb{Z}))$ is either trivial or $\mathbb{Z}/2\mathbb{Z}$, while Proposition 3 provides an explicit nontrivial central extension of $Sp(2g,\mathbb{Z}/D\mathbb{Z})$ by $\mathbb{Z}/2\mathbb{Z}$, when $D \equiv 0 (\mathrm{mod}\ 4)$. Therefore $H_2(Sp(2g,\mathbb{Z}/D\mathbb{Z}))$ is nontrivial, and hence isomorphic to $\mathbb{Z}/2\mathbb{Z}$, thereby proving Theorem 1.

To prove Proposition 3 we first note that the projective Weil representation $\rho_{g,k}$ determines a central extension of $Sp(2g,\mathbb{Z}/2k\mathbb{Z})$ by $\mathbb{Z}/2\mathbb{Z}$, since it factors through the integral metaplectic group, by Weil 1964. We will prove that this central extension is non-trivial thereby proving the claim. The pull-back of this central extension along the homomorphism $Sp(2g,\mathbb{Z}) \to Sp(2g,\mathbb{Z}/2k\mathbb{Z})$ is a central extension of $Sp(2g,\mathbb{Z})$ by $\mathbb{Z}/2\mathbb{Z}$ and it is enough to prove that this last extension is non-trivial.

It turns out to be easier to describe the pull-back of this central extension over the mapping class group M_g of the genus g closed orientable surface.

Definition 2 – Let $\widetilde{M}_g$ be the pull-back of the above central extension, associated to the projective Weil representation $\rho_{g,k}$, along the homomorphism $M_g \to Sp(2g,\mathbb{Z})$.

By the stability results of Harer 1983 for $g \geq 5$, and the low dimensional computations in Pitsch 1999 and Korkmaz and Stipsicz 2003 for $g \geq 4$, the natural homomorphism $M_g \to Sp(2g,\mathbb{Z})$, obtained by choosing a symplectic basis in the surface homology, induces isomorphisms $H_2(M_g;\mathbb{Z}) \to H_2(Sp(2g,\mathbb{Z});\mathbb{Z})$ and $H^2(Sp(2g,\mathbb{Z});\mathbb{Z}) \to H^2(M_g;\mathbb{Z})$ for $g \geq 4$. In particular in this range the class of the central extension $\widetilde{M}_g$ is a generator of $H^2(M_g;\mathbb{Z}/2\mathbb{Z})$.

In contrast for $g = 3$, there is an element of infinite order in $H^2(M_3;\mathbb{Z})$ such that its reduction mod 2 is the class of the central extension $\widetilde{M}_3$, however $H^2(M_3;\mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. This follows from the computation $H_2(M_3;\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, see e.g. Sakasai 2012, Thm. 4.9, Cor. 4.10. Note that $H_2(Sp(6,\mathbb{Z});\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, according to Stein 1975.

Therefore, we can reformulate Proposition 3 at least for $g \geq 4$ in equivalent form in terms of the mapping class group:

Proposition 4 – *If $g \geq 4$ then the class of the central extension $\widetilde{M}_g$ is a generator of the group $H^2(M_g; \mathbb{Z}/2\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z}$.*

The proof is an explicit computation, which turns out to be also valid when $g = 3$. This proves that the central extension coming from the Weil representation (and hence its cohomology class with $\mathbb{Z}/2\mathbb{Z}$ coefficients) for $g = 3$ is non-trivial. This implies that $H_2(Sp(6, \mathbb{Z}/D\mathbb{Z})) \neq 0$, when $D \equiv 0 \pmod 4$. In fact in Benson et al. 2018 the authors proved that $H_2(Sp(6, \mathbb{Z}/4\mathbb{Z}); \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

The rest of this section is devoted to the proof of Proposition 3.

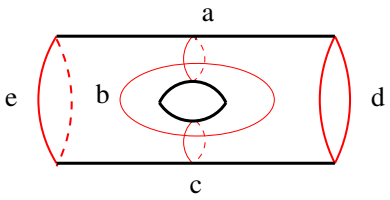
A presentation of $\widetilde{M}_g$ and the proof of Proposition 3

The method we use is from Gervais 1996 and was already used in Funar and Kashaev 2014 for computing central extensions arising in quantum Teichmüller space. We start with a number of notations and definitions. Recall that $\Sigma_{g,r}$ denotes the orientable surface of genus g with r boundary components. If γ is a curve on a surface, then D_γ denotes the right Dehn twist along the curve γ.

Definition 3 – A chain relation C on the surface $\Sigma_{g,r}$ is given by an embedding $\Sigma_{1,2} \subseteq \Sigma_{g,r}$ and the standard chain relation on this 2-holed torus, namely

$$(D_a D_b D_c)^4 = D_e D_d,$$

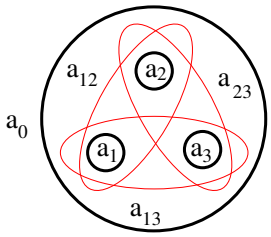
where a, b, c, d, e are the following curves of the embedded 2-holed torus:



Definition 4 – A lantern relation L on the surface $\Sigma_{g,r}$ is given by an embedding $\Sigma_{0,4} \subseteq \Sigma_{g,r}$ and the standard lantern relation on this 4-holed sphere, namely

$$D_{a_0} D_{a_1} D_{a_2} D_{a_3} = D_{a_{12}} D_{a_{13}} D_{a_{23}}, \tag{6}$$

where $a_0, a_1, a_2, a_3, a_{12}, a_{13}, a_{23}$ are the following curves of the embedded 4-holed sphere:



The following lemma is a simple consequence of a deep result of Gervais 1996, Thm. B:

Lemma 2 – *Let $g \geq 3$, then the group M_g has the following presentation:*

1. *Generators are all Dehn twists D_a along all non-separating simple closed curves a on Σ_g.*
2. *Relations:*
   (a) *Braid type 0 relations:*
   $$D_a D_b = D_b D_a,$$
   for each pair of disjoint non-separating simple closed curves a and b;
   (b) *Braid type 1 relations:*
   $$D_a D_b D_a = D_b D_a D_b,$$
   for each pair of non-separating simple closed curves a and b which intersect transversely in one point;
   (c) *One lantern relation for a 4-holed sphere embedded in Σ_g so that all boundary curves are non-separating;*
   (d) *One chain relation for a 2-holed torus embedded in Σ_g so that all boundary curves are non-separating.*

The key step in proving Proposition 4 and hence Proposition 3 is to find an explicit presentation for the central extension $\widetilde{M}_g$ from Definition 2. If we choose arbitrary lifts $\widetilde{D}_a \in \widetilde{M}_g$ for each of the Dehn twists $D_a \in M_g$, then $\widetilde{M}_g$ is generated by the elements $\widetilde{D}_a$ plus a central element z of order at most 2.

Proposition 5 – *Suppose that $g \geq 3$. Then the group $\widetilde{M}_g$ has the following presentation.*

1. *Generators:*
   (a) *With each non-separating simple closed curve a in Σ_g is associated a generator $\widetilde{D}_a$;*

(b) *One (central) element* z, $z \neq 1$.

2. *Relations:*

(a) *Centrality:*

$$z\widetilde{D}_a = \widetilde{D}_a z, \tag{7}$$

for any non-separating simple closed curve a *on* Σ_g*;*

(b) *Braid type* 0 *relations:*

$$\widetilde{D}_a\widetilde{D}_b = \widetilde{D}_b\widetilde{D}_a, \tag{8}$$

for each pair of disjoint non-separating simple closed curves a *and* b*;*

(c) *Braid type* 1 *relations:*

$$\widetilde{D}_a\widetilde{D}_b\widetilde{D}_a = \widetilde{D}_b\widetilde{D}_a\widetilde{D}_b, \tag{9}$$

for each pair of non-separating simple closed curves a *and* b *which intersect transversely at one point;*

(d) *One lantern relation for a* 4*-holed sphere embedded in* Σ_g *so that all boundary curves are non-separating:*

$$\widetilde{D}_{a_0}\widetilde{D}_{a_1}\widetilde{D}_{a_2}\widetilde{D}_{a_3} = \widetilde{D}_{a_{12}}\widetilde{D}_{a_{13}}\widetilde{D}_{a_{23}}, \tag{10}$$

(e) *One chain relation for a 2-holed torus embedded in* Σ_g *so that all boundary curves are non-separating:*

$$(\widetilde{D}_a\widetilde{D}_b\widetilde{D}_c)^4 = z\widetilde{D}_e\widetilde{D}_d. \tag{11}$$

(f) *Scalar equation:*

$$z^2 = 1. \tag{12}$$

In Gervais 1996, Thm. C and Cor. 4.3 the author proved that the universal central extension of the mapping class group has the presentation given in Proposition 5 except for the relation (f) reading $z^2 = 1$. Therefore our group $\widetilde{M}_g$ from Definition 2 will be the non-trivial central extension of M_g by $\mathbb{Z}/2\mathbb{Z}$ obtained from the universal central extension of M_g by reducing mod 2 its kernel.

This will prove Proposition 4 and hence Proposition 3.

Proof of Proposition 5

By a slight abuse of language, we still denote $\rho_{g,k}$ the projective representation of M_g obtained from the Weil representation $\rho_{g,k}$ by the composition with the projection $M_g \to Sp(2g,\mathbb{Z})$. Let then $\widetilde{\rho_{g,k}(M_g)} \subset U(\mathbb{C}^{(\mathbb{Z}/k\mathbb{Z})^g})$ be the pull-back of $\rho_{g,k}(M_g) \subset U(\mathbb{C}^{(\mathbb{Z}/k\mathbb{Z})^g})/R_8$ along this composition.

By definition $\widetilde{M}_g$ fits into a commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & \widetilde{M}_g & \longrightarrow & M_g & \longrightarrow & 1 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & \widetilde{\rho_{g,k}(M_g)} & \longrightarrow & \rho_{g,k}(M_g) & \longrightarrow & 1 \end{array}$$

This presents $\widetilde{M}_g$ as a pull-back and therefore the relations claimed in Proposition 5 will be satisfied if and only if they are satisfied when we project them both into M_g and $\widetilde{\rho_{g,k}(M_g)} \subset U(\mathbb{C}^{(\mathbb{Z}/k\mathbb{Z})^g})$. If this is the case then $\widetilde{M}_g$ will be a quotient of the group obtained from the universal central extension by reducing mod 2 the center and that surjects onto M_g. But, as the mapping class group is Hopfian there are only two such groups: first, $M_g \times \mathbb{Z}/2\mathbb{Z}$ with the obvious projection on M_g and second, the mod 2 reduction of the universal central extension. Then relation (e) shows that we are in the latter case.
The projection on M_g is obtained by killing the center z, and by construction the projected relations are satisfied in M_g and we only need to check them in the unitary group.

Lemma 3 – *For any lifts $\widetilde{D}_a$ of the Dehn twists D_a we have $\widetilde{D}_a\widetilde{D}_b = \widetilde{D}_b\widetilde{D}_a$ and thus the braid type* 0 *relations (b) are satisfied.*

Proof. The commutativity relations are satisfied for particular lifts and hence for arbitrary lifts. □

Lemma 4 – *There are lifts $\widetilde{D}_a$ of the Dehn twists D_a, for each non-separating simple closed curve a such that we have*

$$\widetilde{D}_a\widetilde{D}_b\widetilde{D}_a = \widetilde{D}_b\widetilde{D}_a\widetilde{D}_b$$

for any simple closed curves a, b with one intersection point and thus the braid type 1 *relations (c) are satisfied.*

Proof. Consider an arbitrary lift of one braid type 1 relation (to be called the fundamental one), which has the form $\widetilde{D}_a\widetilde{D}_b\widetilde{D}_a = z^k\widetilde{D}_b\widetilde{D}_a\widetilde{D}_b$. Change then the lift $\widetilde{D}_b$ to $z^k\widetilde{D}_b$. With the new lift the relation above becomes $\widetilde{D}_a\widetilde{D}_b\widetilde{D}_a = \widetilde{D}_b\widetilde{D}_a\widetilde{D}_b$.

Choose now an arbitrary braid type 1 relation of M_g, say $D_xD_yD_x = D_yD_xD_y$. There exists a 1-holed torus $\Sigma_{1,1} \subset \Sigma_g$ containing x, y, namely a neighborhood of $x \cup y$. Let

T be the similar torus containing a, b. Since a, b and x, y are non-separating there exists a homeomorphism $\varphi : \Sigma_{g,r} \to \Sigma_{g,r}$ such that $\varphi(a) = x$ and $\varphi(b) = y$. We have then

$$D_x = \varphi D_a \varphi^{-1}, \quad D_y = \varphi D_b \varphi^{-1}.$$

Let us consider now an arbitrary lift $\widetilde{\varphi} \in \widetilde{M}_g$ of φ, which is well-defined only up to a central element, and set

$$\widetilde{D}_x = \widetilde{\varphi} \widetilde{D}_a \widetilde{\varphi}^{-1}, \quad \widetilde{D}_y = \widetilde{\varphi} \widetilde{D}_b \widetilde{\varphi}^{-1}.$$

These lifts are well-defined since they do not depend on the choice of $\widetilde{\varphi}$ (the central elements coming from $\widetilde{\varphi}$ and $\widetilde{\varphi}^{-1}$ mutually cancel). Moreover, we have then

$$\widetilde{D}_x \widetilde{D}_y \widetilde{D}_x = \widetilde{D}_y \widetilde{D}_x \widetilde{D}_y$$

and so the braid type 1 relations (c) are all satisfied. □

Lemma 5 – *The choice of lifts of all $\widetilde{D}_x$, with x non-separating, satisfying the requirements of Lemma 4 is uniquely defined by fixing the lift $\widetilde{D}_a$ of one particular Dehn twist.*

Proof. In fact the choice of $\widetilde{D}_a$ fixes the choice of $\widetilde{D}_b$. If x is a non-separating simple closed curve on Σ_g, then there exists another non-separating curve y which intersects it in one point. Thus, by Lemma 4, the choice of $\widetilde{D}_x$ is unique. □

Lemma 6 – *One can choose the lifts of Dehn twists in $\widetilde{M}_g$ so that all braid type relations are satisfied and the lift of the lantern relation is trivial, namely*

$$\widetilde{D}_a \widetilde{D}_b \widetilde{D}_c \widetilde{D}_d = \widetilde{D}_u \widetilde{D}_v \widetilde{D}_w,$$

for the non-separating curves in the fixed embedded $\Sigma_{0,4} \subset \Sigma_g$.

Proof. An arbitrary lift of that lantern relation is of the form $\widetilde{D}_a \widetilde{D}_b \widetilde{D}_c \widetilde{D}_d = z^k \widetilde{D}_u \widetilde{D}_v \widetilde{D}_w$. In this case, we change the lift $\widetilde{D}_a$ to $z^{-k} \widetilde{D}_a$ and adjust the lifts of all other Dehn twists along non-separating curves in the unique way such that all braid type 1-relations are satisfied. Then, the required form of the lantern relation is satisfied too. □

We say that the lifts of the Dehn twists are *normalized* if all braid type relations and one lantern relation are lifted in a trivial way.

Now Proposition 5 follows from the following lemma, whose proof is rather calculatory and is postponed to the next subsection.

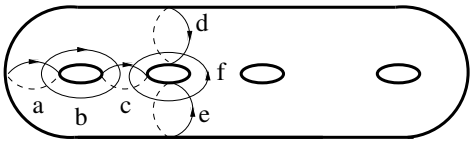
Lemma 7 – *If all lifts of the Dehn twist generators are normalized then $(\widetilde{D}_a \widetilde{D}_b \widetilde{D}_c)^4 = z \widetilde{D}_e \widetilde{D}_d$, where $z^2 = 1$ and $z \neq 1$.*

Proof of Lemma 7

We denote by T_γ the action of D_γ in homology. Moreover we denote by R_γ the matrix in $U(\mathbb{C}^{(\mathbb{Z}/k\mathbb{Z})^g})$ corresponding to the prescribed lift $\rho_{g,k}(T_\gamma)$ of the projective representation. The level k is fixed through this section and we drop the subscript k from now on.

Our strategy is as follows. We show that the braid relations are satisfied by the matrices R_γ. It remains to compute the defect of the chain relation in the matrices R_γ.

Consider an embedding of $\Sigma_{1,2} \subset \Sigma_g$ such that all curves from the chain relation are non-separating, and thus like in the figure below:



By construction, the action of the subgroup generated by D_a, D_b, D_c, D_d, D_e and D_f on the homology of the surface Σ_g preserves the symplectic subspace generated by the homology classes of b, f, a, e and acts trivially on its orthogonal complement. Now the Weil representation behaves well with respect to the direct sum of symplectic matrices and this enables us to focus our attention on the action of this subgroup on the 4-dimensional symplectic subspace generated by b, f, a, e and to use the representation ρ_2. In this basis the symplectic matrices associated to the above Dehn twists are:

$$T_a = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \; T_b = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \; T_c = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{pmatrix},$$

$$T_d = T_e = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}, \; T_f = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Notice that $T_b = J^{-1} T_a J$, where J is the matrix of the standard symplectic structure.

Set $q = \exp\left(\frac{\pi i}{k}\right)$, which is a $2k$-th root of unity. We will change slightly the basis $\{\theta_m, m \in (\mathbb{Z}/k\mathbb{Z})^g\}$ of our representation vector space in order to exchange the two obvious parabolic subgroups of $Sp(2g, \mathbb{Z})$.

The element $\rho_g\begin{pmatrix} 0 & -\mathbf{1}_g \\ \mathbf{1}_g & 0 \end{pmatrix}$ will be central in our argument, we will denote it by S. Specifically we fix the basis given by $-S\theta_m$, with $m \in (\mathbb{Z}/k\mathbb{Z})^g$. We have then:

$$R_a = \mathrm{diag}(q^{\langle L_a x, x\rangle})_{x\in(\mathbb{Z}/k\mathbb{Z})^2}, \text{ where } L_a = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

$$R_c = \mathrm{diag}(q^{\langle L_c x, x\rangle})_{x\in(\mathbb{Z}/k\mathbb{Z})^2}, \text{ where } L_c = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$

and

$$R_e = R_d = \mathrm{diag}(q^{\langle L_e x, x\rangle})_{x\in(\mathbb{Z}/k\mathbb{Z})^2}, \text{ where } L_e = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

We set now:

$$R_b = S^3 R_a S \text{ and } R_f = S^3 R_e S.$$

Lemma 8 – *The matrices R_a, R_b, R_c, R_f, R_e are normalized lifts, namely the braid relations are satisfied.*

We postpone the proof of this lemma a few lines.

For $u, v \in \mathbb{N}$, let us denote by $G(u,v)$ the Gauss sum:

$$G(u,v) = \sum_{x\in\mathbb{Z}/v\mathbb{Z}} \exp\left(\frac{2\pi\sqrt{-1}ux^2}{v}\right).$$

According to Lang 1970, pp. 85-91 the value of the Gauss sum is

$$G(u,v) = dG\left(\frac{u}{d}, \frac{v}{d}\right), \text{ if g.c.d.}(u,v) = d,$$

and for g.c.d.$(u,v) = 1$ we have:

$$G(u,v) = \begin{cases} \varepsilon(v)\left(\frac{u}{v}\right)\sqrt{v}, & \text{for odd } v, \\ 0, & \text{for } v = 2(\mathrm{mod}\ 4), \\ \overline{\varepsilon(u)}\left(\frac{v}{u}\right)\left(\frac{1+\sqrt{-1}}{\sqrt{2}}\right)\sqrt{2v}, & \text{for } v = 0(\mathrm{mod}\ 4), \end{cases}$$

where $\left(\frac{u}{v}\right)$ is the Jacobi symbol and

$$\varepsilon(a) = \begin{cases} 1, & \text{if } a = 1(\mathrm{mod}\ 4), \\ \sqrt{-1}, & \text{if } a = 3(\mathrm{mod}\ 4). \end{cases}$$

Remember that the Jacobi (or the quadratic) symbol $\left(\frac{P}{Q}\right)$ is defined only for odd Q by the formula:

$$\left(\frac{P}{Q}\right)=\prod_{i=1}^{s}\left(\frac{P}{q_i}\right)$$

where $Q=q_1q_2...q_s$ is the prime decomposition of Q, and for prime q the quadratic symbol (also called the Legendre symbol in this setting) is given by:

$$\left(\frac{P}{q}\right)=\begin{cases}0, & \text{if } P\equiv 0(\text{mod } q)\\ 1, & \text{if } P=x^2(\text{mod } q) \text{ and } P\not\equiv 0(\text{mod } q),\\ -1, & \text{otherwise.}\end{cases}$$

while

$$\left(\frac{P}{1}\right)=1.$$

The quadratic symbol satisfies the following reciprocity law

$$\left(\frac{P}{Q}\right)\left(\frac{Q}{P}\right)=(-1)^{\frac{P-1}{2}\frac{Q-1}{2}},$$

when both P and Q are odd.

Denote by $\omega=\frac{1}{2}G(1,2k)$. The lift of the chain relation is of the form:

$$(R_aR_bR_c)^4=\mu R_eR_d,$$

for some $\mu\in U(1)$. Our aim now is to compute the value of μ. Set $X=R_aR_bR_c$, $Y=X^2$ and $Z=X^4$. Let $m,n\in(\mathbb{Z}/k\mathbb{Z})^2$, $m=\begin{pmatrix}m_1\\ m_2\end{pmatrix}$ and $n=\begin{pmatrix}n_1\\ n_2\end{pmatrix}$.

The entry $X_{m,n}$ of the matrix X is given by:

$$X_{m,n}=k^{-1}\omega\delta_{n_2,m_2}q^{-(n_1-m_1)^2+m_1^2+(n_1+n_2)^2}.$$

This implies $Y_{m,n}=0$ if $\delta_{m_2,n_2}=0$. If $m_2=n_2$ then:

$$Y_{m,n}=k^{-2}\omega^2\sum_{r_1\in\mathbb{Z}/k\mathbb{Z}}q^{-(m_1-r_1)^2+m_1^2+(r_1+n_2)^2-(n_1-r_1)^2+r_1^2+(n_1+n_2)^2}=$$

$$=k^{-2}\omega^2\sum_{r_1\in\mathbb{Z}/k\mathbb{Z}}q^{m_2^2+n_2^2+2n_1n_2+2r_1(m_1+m_2+n_1)}.$$

Therefore $Y_{m,n}=0$, unless $m_1+m_2+n_1=0$. Assume that $m_1+m_2+n_1=0$. Then:

$$Y_{m,n}=k^{-1}\omega^2q^{m_2^2+n_2^2+2n_1n_2}=k^{-1}\omega^2q^{-2m_1m_2}.$$

It follows that: $Z_{m,n} = \sum_{r\in(\mathbb{Z}/k\mathbb{Z})^2} Y_{m,r}Y_{r,n}$ vanishes, except when $m_2 = r_2 = n_2$ and $r_1 = -(m_1+m_2)$, $n_1 = -(r_1+r_2) = m_1$. Thus Z is a diagonal matrix. If $m = n$ then:

$$Z_{m,n} = Y_{m,r}Y_{r,n} = k^{-2}\omega^4 q^{-2m_1m_2-2r_1r_2} = k^{-2}\omega^4 q^{2m_2^2}.$$

We have therefore obtained:

$$(R_aR_bR_c)^4 = k^{-2}\omega^4 R_e^2$$

and thus $\mu = k^{-2}\omega^4 = \frac{1}{k^2}\left(\frac{G(1,2k)}{2}\right)^4$. This proves that whenever k is even we have $\mu = -1$. Since this computes the action of the central element z, it follows that $z \neq 1$. This ends the proof of Lemma 7.

Proof (Proof of Lemma 8). We know that R_b is S^3R_aS, where S is the S-matrix, up to an eighth root of unity. The normalization of this root of unity is given by the braid relation:

$$R_aR_bR_a = R_bR_aR_b.$$

We have therefore:

$$(R_b)_{m,n} = k^{-2}\sum_{x\in(\mathbb{Z}/k\mathbb{Z})^2} q^{\langle L_ax,x\rangle+2\langle n-m,x\rangle}.$$

This entry vanishes except when $m_2 = n_2$. Assume that $n_2 = m_2$. Then:

$$(R_b)_{m,n} = k^{-1}\sum_{x_1\in\mathbb{Z}/k\mathbb{Z}} q^{x_1^2+2(n_1-m_1)x_1} = k^{-1}q^{-(n_1-m_1)^2}\sum_{x_1\in\mathbb{Z}/k\mathbb{Z}} q^{(x_1+n_1-m_1)^2} = k^{-1}q^{-(n_1-m_1)^2}\omega$$

where $\omega = \frac{1}{2}\sum_{x\in\mathbb{Z}/2k\mathbb{Z}} q^{x^2}$ is a Gauss sum. We have first:

$$(R_aR_bR_a)_{m,n} = k^{-1}\omega\delta_{m_2,n_2}q^{-(n_1-m_1)^2+m_1^2+n_1^2} = k^{-1}\omega\delta_{m_2,n_2}q^{2n_1m_1}.$$

Further

$$(R_bR_a)_{m,n} = k^{-1}\omega\delta_{m_2,n_2}q^{-(n_1-m_1)^2+n_1^2}$$

so that:

$$(R_bR_aR_b)_{m,n} = k^{-2}\omega^2\sum_{r\in(\mathbb{Z}/k\mathbb{Z})^2}\delta_{m_2,r_2}\delta_{n_2,r_2}q^{-(m_1-r_1)^2+r_1^2-(r_1-n_1)^2} =$$

$$= k^{-1}\omega^2\delta_{m_2,n_2}q^{2m_1n_1}\sum_{r_1\in\mathbb{Z}/k\mathbb{Z}} q^{-(r_1-m_1+n_1)^2} = k^{-1}\omega\delta_{m_2,n_2}q^{2n_1m_1}.$$

Similar computations hold for the other pairs of non-commuting matrices in the set R_b, R_c, R_f, R_e. This ends the proof of Lemma 8. □

3 Residual finiteness, finite quotients and their second homology

3.1 Residual finiteness for perfect groups

Perfect groups have a universal central extension with kernel canonically isomorphic to their second integral homology group. In this section we show how to translate the residual finiteness problem for the universal central extension for a perfect group Γ into a homological problem about $H_2(\Gamma)$. We will need the following lemmas from Funar and Pitsch 2020, Lemma 2.1 & Lemma 2.2:

Lemma 9 – *Let Γ and F be perfect groups, $\widetilde{\Gamma}$ and $\widetilde{F}$ their universal central extensions and $p : \Gamma \to F$ be a group homomorphism. Then there exists a unique homomorphism $\widetilde{p} : \widetilde{\Gamma} \to \widetilde{F}$ lifting p such that the following diagram is commutative:*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & H_2(\Gamma) & \longrightarrow & \widetilde{\Gamma} & \longrightarrow & \Gamma & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle p_*} & & \downarrow{\scriptstyle \widetilde{p}} & & \downarrow{\scriptstyle p} & & \\ 1 & \longrightarrow & H_2(F) & \longrightarrow & \widetilde{F} & \longrightarrow & F & \longrightarrow & 1 \end{array}$$

where $p_* = H_2(p)$ is the map induced by p in homology.

Lemma 10 – *Let Γ be a finitely generated perfect group and $\widetilde{\Gamma}$ be its universal central extension. We denote by C the central group* $\ker(\widetilde{\Gamma} \to \Gamma)$ *of $\widetilde{\Gamma}$.*

1. *Suppose that the finite index (normal) subgroup $H \subseteq \Gamma$ has the property that the image of $H_2(H)$ into $H_2(\Gamma)$ contains dC, for some $d \in \mathbb{Z}$. Let $F = \Gamma/H$ be the corresponding finite quotient of Γ and $p : \Gamma \to F$ the quotient map. Then $d \cdot p_*(H_2(\Gamma)) = 0$, where $p_* : H_2(\Gamma) \to H_2(F)$ is the homomorphism induced by p. In particular, if $p_* : H_2(\Gamma) \to H_2(F)$ is surjective, then $d \cdot H_2(F) = 0$.*

2. *Assume that F is a finite quotient of Γ and let $d \in \mathbb{Z}$ such that $d \cdot p_*(H_2(\Gamma)) = 0$. For instance, this is satisfied when $d \cdot H_2(F) = 0$. Let $\widetilde{F}$ denote the universal central extension of F. Then the homomorphism $p : \Gamma \to F$ has a unique lift $\widetilde{p} : \widetilde{\Gamma} \to \widetilde{F}$ and the kernel of $\widetilde{p}$ contains $d \cdot C$. Observe that since F is finite, $H_2(F)$ is also finite, hence we can take $d = |H_2(F)|$.*

3.2 Proof of Theorem 3

Let $\mathbb{K}$ be a *number field*, $\mathcal{R}$ be the set of inequivalent valuations of $\mathbb{K}$ and $S \subset \mathcal{R}$ be a finite set of valuations of $\mathbb{K}$ including all the Archimedean (infinite) ones. Let

$$O(S) = \{x \in \mathbb{K} : \ v(x) \leq 1, \text{ for all } v \in \mathcal{R} \setminus S\}$$

be the ring of S-integers in $\mathbb{K}$ and $\mathfrak{q} \subset O(S)$ be a nonzero ideal. By $\mathbb{K}_v$ we denote the completion of $\mathbb{K}$ with respect to $v \in \mathcal{R}$. Following Bass, Milnor, and Serre 1967, we call a domain $\mathfrak{A}$ which arises as $O(S)$ above a *Dedekind domain of arithmetic type*.

Let $\mathfrak{A} = O(S)$ be a Dedekind domain of arithmetic type and $\mathfrak{q}$ be an ideal of $\mathfrak{A}$. Denote by $Sp(2g, \mathfrak{A}, \mathfrak{q})$ the kernel of the surjective homomorphism $p : Sp(2g, \mathfrak{A}) \to Sp(2g, \mathfrak{A}/\mathfrak{q})$. The surjectivity is not a purely formal fact and follows from the fact that in these cases the symplectic group coincides with the so-called "elementary symplectic group", and that it is trivial to lift elementary generators of $Sp(2g, \mathfrak{A}/\mathfrak{q})$ to $Sp(2g, \mathfrak{A})$; for a proof of this fact when $\mathfrak{A} = \mathbb{Z}$ see Hahn and O'Meara 1989, Thm. 9.2.5. The restriction to $\mathbb{K}$ which are not totally imaginary comes from the result of Bass, Milnor, and Serre 1967 which states that symplectic groups $Sp(2g, \mathfrak{A})$, for $g \geq 3$ have the congruence subgroup property.

Consider the central extension of $Sp(2g, \mathfrak{A})$ constructed by Deligne in Deligne 1978, as follows. Let μ (respectively μ_v for a non-complex place v) be the group of roots of unity in $\mathbb{K}$ (and respectively $\mathbb{K}_v$). By convention one sets $\mu_v = 1$ for a complex place v. Moore showed in Moore 1968 that there exists a universal topological central extension $\widetilde{Sp(2g, \mathbb{K}_v)}$ of $Sp(2g, \mathbb{K}_v)$ by a discrete group $\pi_1(Sp(2g, \mathbb{K}_v))$. When v is a non real place Moore proved that there is an isomorphism between $\pi_1(Sp(2g, \mathbb{K}_v))$ and μ_v. For a real place v, it is well-known that $\pi_1(Sp(2g, \mathbb{K}_v)) = \mathbb{Z}$. Set also $Sp(2g)_S = \prod_{v \in S} Sp(2g, \mathbb{K}_v)$ and recall that $Sp(2g, \mathfrak{A})$ is a subgroup of $Sp(2g)_S$. Then the universal topological central extension $\widetilde{Sp(2g)_S}$ of $Sp(2g)_S$ is isomorphic to the universal covering $\prod_{v \in S} \widetilde{Sp(2g, \mathbb{K}_v)}$ by the abelian group $\pi_1(Sp(2g)_S) = \prod_{v \in S} \pi_1(Sp(2g, \mathbb{K}_v))$. Denote then by $\widetilde{Sp(2g, \mathfrak{A})}^D$ the inverse image of $Sp(2g, \mathfrak{A})$ in the universal covering $\widetilde{Sp(2g)_S}$ of $Sp(2g)_S$. Then $\widetilde{Sp(2g, \mathfrak{A})}^D$ is a central extension of $Sp(2g, \mathfrak{A})$ which fits in an exact sequence:

$$1 \to \pi_1(Sp(2g)_S) \to \widetilde{Sp(2g, \mathfrak{A})}^D \to Sp(2g, \mathfrak{A}) \to 1. \tag{13}$$

There is a natural surjective homomorphism $\pi_1(Sp(2g, \mathbb{K}_v)) \to \mu_v$, for all places v. When composed with the map $\mu_v \to \mu$ sending x to $x^{[\mu_v : \mu]}$ we obtain a homomorphism:

$$R_S : \pi_1(Sp(2g)_S) \to \mu. \tag{14}$$

Deligne's theorem from Deligne 1978 states that the intersection of all finite index subgroups of $\widetilde{Sp(2g, \mathfrak{A})}^D$ coincides with $\ker R_S$, when $g \geq 2$. Then, Lemmas 9 and 10 would prove the statement of Theorem 3 if we knew that $\widetilde{Sp(2g, \mathfrak{A})}^D$ is the universal central extension of $Sp(2g, \mathfrak{A})$. This is, for instance, the case when $\mathfrak{A} = \mathbb{Z}$ and $g \geq 4$, but not true in full generality. In order to circumvent this difficulty, we drop out the torsion part of the kernels of the two central extensions.

The key step in proving Theorem 3 is a result stating the equivalence between the non-residual finiteness of Deligne's central extension and the existence of a

uniform bound for the 2-homology of the finite congruence quotients, whose proof is postponed a few lines later.

Proposition 6 – *The following statements are equivalent:*

1. *There exists a homomorphism* $R : \prod_{v\in S} \pi_1(Sp(2g,\mathbb{K}_v)) \to G$, *where* G *is a finite group such that every finite index subgroup of the Deligne central extension* $\widetilde{Sp(2g,\mathfrak{A})}^D$, *for* $g \geq 3$, *contains* $\ker R$.

2. *For fixed* $\mathfrak{A}$ *and* $g \geq 3$ *there exists some uniform (independent on* g *and* $\mathfrak{q}$*) bound for the size of the finite torsion groups* $H_2(Sp(2g,\mathfrak{A}/\mathfrak{q}))$, *for any nontrivial ideal* $\mathfrak{q}$ *of* $\mathfrak{A}$.

Proof (Proof of Theorem 3). Deligne's result from Deligne 1978 yields an effective uniform bound for the size of the torsion group of $H_2(Sp(2g,\mathfrak{A}/\mathfrak{q}))$, since the first statement of Proposition 6 holds for $R = R_S$. Eventually, $Sp(2g,\mathfrak{A})$ has the congruence subgroup property, for $g \geq 2$, according to Bass, Milnor, and Serre 1967, namely any surjective homomorphism $Sp(2g,\mathfrak{A}) \to F$ onto a finite group F factors through some finite congruence quotient $Sp(2g,\mathfrak{A}/\mathfrak{q})$. This proves our claim. □

The main steps in the proof of Proposition 6 are the following. First, the natural homomorphism $H_2(Sp(2g,\mathfrak{A})) \to H_2(Sp(2g,\mathfrak{A}/\mathfrak{q}))$ is surjective, for any $g \geq 3$. Further, the groups $H_2(Sp(2g,\mathfrak{A}))$ stabilize for large enough g (depending on $\mathfrak{A}$) so that there are upper bounds (independent on g) on the number of its generators and on the size of its torsion part. Therefore, the finite abelian group $H_2(Sp(2g,\mathfrak{A}/\mathfrak{q}))$ has uniformy bounded size if and only if there is some uniform bound for the orders of the images of the generators of the free part of $H_2(Sp(2g,\mathfrak{A}))$. The homomorphism between the universal central extension of $Sp(2g,\mathfrak{A})$ and the Deligne extension induces a map between their kernels $H_2(Sp(2g,\mathfrak{A})) \to \prod_{v\in S} \pi_1(Sp(2g,\mathbb{K}_v))$. This map is an isomorphism at the level of their free parts. Therefore $H_2(Sp(2g,\mathfrak{A}/\mathfrak{q}))$ has uniformly bounded size if and only if the image of $\prod_{v\in S} \pi_1(Sp(2g,\mathbb{K}_v))$ through homomorphisms of the Deligne central extension into finite groups has uniformly bounded size.

Before proceeding, we collect some of the results involved in the proof of Proposition 6. We first need the following technical proposition, whose proof can be found in Funar and Pitsch 2020:

Proposition 7 – *The homomorphism* $p_* : H_2(Sp(2g,\mathfrak{A})) \to H_2(Sp(2g,\mathfrak{A}/\mathfrak{q}))$ *is surjective for any ideal* $\mathfrak{q} \subseteq \mathfrak{A}$ *and* $g \geq 3$.

Although the Deligne central extension is not in general the universal central extension, it is not far from it. The free part of the kernel of the universal central extension can be determined by means of:

Lemma 11 – *We have:*

$$H_2(Sp(2g,\mathfrak{A});\mathbb{R}) = \mathbb{R}\otimes_{\mathbb{Z}} \prod_{v\in S\cap\mathbb{R}(S)} \pi_1(Sp(2g,\mathbb{K}_v)),$$

where $\mathbb{R}(S)$ *denotes the real Archimedean places in* S.

Proof. In the case where $\mathfrak{A}$ is the ring of integers of a number field this is basically the result of Borel computing the stable cohomology of arithmetic groups from Borel 1974, pp. 276. For the general case see Borel 1972; Borel and Serre 1971. □

Furthermore, we have the following general statement:

Lemma 12 – *The group* $H_2(Sp(2g,\mathfrak{A}))$ *is finitely generated.*

Proof. This follows from the existence of the Borel-Serre compactification Borel and Serre 1971 associated to an arithmetic group. □

For a finitely generated abelian group A we denote by $F(A)$ its free part and by $T(A)$ its torsion subgroup, so that $A = F(A)\oplus T(A)$.

It is known that $Sp(2g,\mathfrak{A})$ is perfect, when $g\geq 3$. It has then a universal central extension $\widetilde{Sp(2g,\mathfrak{A})}$ by $H_2(Sp(2g,\mathfrak{A}))$. Consider the quotients $E = \widetilde{Sp(2g,\mathfrak{A})}/T(H_2(Sp(2g,\mathfrak{A})))$ and $D = \widetilde{Sp(2g,\mathfrak{A})}^D/T(\pi_1(Sp(2g)_S))$. Then E and D are central extensions of $Sp(2g,\mathfrak{A})$ by torsion-free groups.

Lemma 13 – *There is a natural embedding of central extensions* $E\to D$ *which lifts the identity of* $Sp(2g,\mathfrak{A})$ *and identifies* E *with a finite index normal subgroup of* D.

Proof. From Lemma 11 and the above description of fundamental groups $\pi_1(Sp(2g,\mathbb{K}_v))$ due to Moore 1968, both $F(H_2(Sp(2g,\mathfrak{A})))$ and $F(\pi_1(Sp(2g)_S))$ are isomorphic to the abelian group $\prod_{v\in S\cap\mathbb{R}(S)}\pi_1(Sp(2g,\mathbb{K}_v))$, because the non real places only provide torsion factors of $\pi_1(Sp(2g)_S)$.

By universality of the central extension $\widetilde{Sp(2g,\mathfrak{A})}$, there is a homomorphism $\widetilde{Sp(2g,\mathfrak{A})}\to\widetilde{Sp(2g,\mathfrak{A})}^D$ lifting the identity of $Sp(2g,\mathfrak{A})$; it induces a homomorphism between the quotients $\iota : E\to D$ sending the kernel of the first extension into the kernel of the second one, namely such that $\iota(F(H_2(Sp(2g,\mathfrak{A}))))\subseteq F(\pi_1(Sp(2g)_S))$. The linear map induced between the associated real vector spaces $\iota\otimes 1_{\mathbb{R}} : H_2(Sp(2g,\mathfrak{A}))\otimes_{\mathbb{Z}}\mathbb{R}\to\pi_1(Sp(2g)_S)\otimes_{\mathbb{Z}}\mathbb{R}$ can be identified with the map induced in homology by the inclusion $Sp(2g,\mathfrak{A})\to Sp(2g)_S$. Recall that $Sp(2g,\mathfrak{A})$ is a discrete subgroup of the locally compact group $Sp(2g)_S$. A consequence of the Garland-Matsushima vanishing theorem from Borel 1981, Thm. 6.4 states that the map induced by the inclusion at the level of H_2 is an isomorphism for large enough g. This implies that the restriction $\iota : F(H_2(Sp(2g,\mathfrak{A})))\to F(\pi_1(Sp(2g)_S))$ is injective. Thus ι should be injective, as well, because it is a lift of the identity of $Sp(2g,\mathfrak{A})$. □

Lemma 14 – *The orders of the groups $T(H_2(Sp(2g,\mathfrak{A})))$, $g \geq 2$, are bounded from above by a constant which only depends on $\mathfrak{A}$.*

Proof. From Stein's surjective stability Stein 1978 the embeddings $Sp(2g,\mathfrak{A}) \to Sp(2g+2,\mathfrak{A})$ induce surjective maps $H_2(Sp(2g,\mathfrak{A})) \to H_2(Sp(2g+2,\mathfrak{A}))$, for g larger than a constant depending on $\mathfrak{A}$. The claim follows now, because the abelian groups $H_2(Sp(2g,\mathfrak{A}))$ are finitely generated, by Lemma 12. □

Proof (Proof of Proposition 6). Recall that $p : Sp(2g,\mathfrak{A}) \to Sp(2g,\mathfrak{A}/q)$ is the surjective homomorphism induced by the reduction mod q. Let $p_* : H_2(Sp(2g,\mathfrak{A})) \to H_2(Sp(2g,\mathfrak{A}/q))$ be the corresponding homorphism in homology, which is surjective according to Proposition 7. By Lemma 9 there exists a lift $\tilde{p} : \widetilde{Sp(2g,\mathfrak{A})} \to \widetilde{Sp(2g,\mathfrak{A}/q)}$ of p, whose restriction to the kernel is precisely p_*.

Bounding the size of $H_2(Sp(2g,\mathfrak{A}/q))$ is equivalent to bounding the size of the finite group $F = H_2(Sp(2g,\mathfrak{A}/q))/p_*(T(H_2(Sp(2g,\mathfrak{A}))))$, according to Lemma 14. Note that $\tilde{p}$ induces a surjective homomorphism $\tilde{p} : E \to F$.

Assume now that the Deligne central extension has the property from the first statement of the proposition. Lemma 13 identifies E with a finite index subgroup of D. Consider the representation $\mathrm{Ind}_E^D \tilde{p} : D \to \tilde{F}$ induced from $\tilde{p} : E \to F$. Its image is a finite group $\tilde{F}$, which is a quotient of the Deligne extension $\widetilde{Sp(2g,\mathfrak{A})}^D$. By our hypothesis, we have $\ker \mathrm{Ind}_E^D \tilde{p} \supseteq \ker R$, and hence there is some surjective homomorphism $\lambda : G \to \tilde{F}$ so that:

$$\mathrm{Ind}_E^D \tilde{p}|_{\pi_1(Sp(2g)_S)} = \lambda \circ R.$$

In particular the image $\tilde{p}(H_2(Sp(2g,\mathfrak{A}))) \subseteq \tilde{F}$ is covered by G and hence has uniformly bounded size.

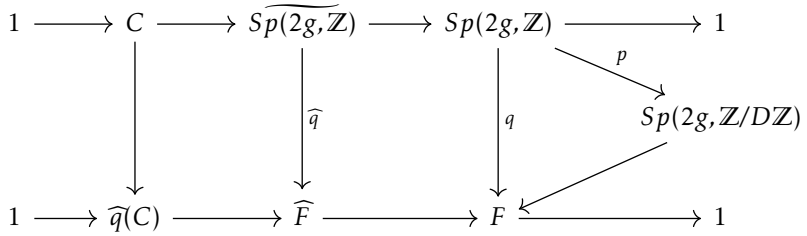
Conversely, assume that there exists some $k(\mathfrak{A}) \in \mathbb{Z} \setminus \{0\}$ such that $k(\mathfrak{A}) \cdot H_2(Sp(2g,\mathfrak{A}/\mathfrak{q})) = 0$, for every ideal $\mathfrak{q} \subset \mathfrak{A}$. Then the surjectivity of p_* implies that $p_*(k(\mathfrak{A}) \cdot c) = 0$, for every ideal $\mathfrak{q}$ and $c \in H_2(Sp(2g,\mathfrak{A}))$.

This implies that $f_*(|\mu| \cdot k(\mathfrak{A}) \cdot c) = 0$, for every morphism $f : Sp(2g,\mathfrak{A}) \to F$ onto a finite group F, where $|\mu|$ is the cardinal of the finite group μ. In fact, by Bass, Milnor, and Serre 1967 the congruence subgroup kernel is the finite cyclic group μ, when $\mathbb{K}$ is totally imaginary (i.e. it has no non-complex places), and is trivial if $\mathbb{K}$ has at least one non-complex place. This means that for any finite index normal subgroup H (e.g. $\ker f$) there exists an elementary subgroup $ESp(2g,\mathfrak{A},\mathfrak{q})$ contained in H, where $ESp(2g,\mathfrak{A},\mathfrak{q}) \subseteq Sp(2g,\mathfrak{A},\mathfrak{q})$ is a normal subgroup of finite index dividing $|\mu|$. Therefore $f(-)$ factors through the quotient $Sp(2g,\mathfrak{A})/ESp(2g,\mathfrak{A},\mathfrak{q})$ and the composition $f(|\mu| \cdot -)$ factors through $Sp(2g,\mathfrak{A}/\mathfrak{q})$. Since $H_2(Sp(2g,\mathfrak{A}/\mathfrak{q}))$ is $k(\mathfrak{A})$-torsion we obtain $f_*(|\mu| \cdot k(\mathfrak{A}) \cdot c) = 0$, as claimed.

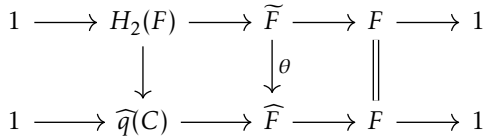
In particular, we can apply this equality to the morphism f between the universal central extensions of $Sp(2g,\mathfrak{A})$ and F. The restriction of f_* to the free part of $H_2(Sp(2g,\mathfrak{A}))$ is then trivial on multiples of $|\mu|\cdot k(\mathfrak{A})$. Thus these multiple elements lie in the kernel of any homomorphism of $\widetilde{Sp(2g,\mathfrak{A})}$ into a finite group. This proves Proposition 6. □

3.3 Proof of Theorem 2

Consider an arbitrary surjective homomorphism $\widehat{q}\colon \widetilde{Sp(2g,\mathbb{Z})} \to \widehat{F}$ onto some finite group $\widehat{F}$. We set $F = \widehat{F}/\widehat{q}(C)$, where $C = \ker(\widetilde{Sp(2g,\mathbb{Z})} \to Sp(2g,\mathbb{Z}))$. Then there is an induced homomorphism $q : Sp(2g,\mathbb{Z}) \to F$. Since F is finite the congruence subgroup property for the symplectic groups (see Bass, Milnor, and Serre 1967; Mennicke 1965a,b) implies that there is some D such that q factors as $s \circ p$, where $p : Sp(2g,\mathbb{Z}) \to Sp(2g,\mathbb{Z}/D\mathbb{Z})$ is the reduction mod D and $s : Sp(2g,\mathbb{Z}/D\mathbb{Z}) \to F$ is a surjective homomorphism.



Since F, a quotient of a symplectic group with $g \geq 4$, is perfect it has a universal central extension $\widetilde{F}$. By Lemma 9, there exists then unique lifts $\widetilde{p}\colon \widetilde{Sp(2g,\mathbb{Z})} \to \widetilde{Sp(2g,\mathbb{Z}/D\mathbb{Z})}$, $\widetilde{q}\colon \widetilde{Sp(2g,\mathbb{Z})} \to \widetilde{F}$ and $\widetilde{s}\colon \widetilde{Sp(2g,\mathbb{Z}/D\mathbb{Z})} \to \widetilde{F}$ of the homomorphisms p, q and s, respectively such that $\widetilde{q} = \widetilde{s} \circ \widetilde{p}$.



Since $\widetilde{F}$ is universal there is a unique homomorphism $\theta : \widetilde{F} \to \widehat{F}$ which lifts the identity of F. We claim that $\theta \circ \widetilde{q} = \widehat{q}$. Both homomorphisms are lifts of q and $\widetilde{q} = \widetilde{s} \circ \widetilde{p}$, by the uniqueness claim of Lemma 9. Thus $\theta \circ \widetilde{q} = \alpha \cdot \widehat{q}$, where α is a homomorphism on $\widetilde{Sp(2g,\mathbb{Z})}$ with target $\widehat{q}(C) = \ker(\widehat{F} \to F)$, which is central in $\widehat{F}$. Since $\widetilde{Sp(2g,\mathbb{Z})}$ is universal we have $H_1(\widetilde{Sp(2g,\mathbb{Z})}) = 0$ and thus α is trivial, as claimed. Summing up, we have $\widehat{q} = \theta \circ \widetilde{q} = \theta \circ \widetilde{s} \circ \widetilde{p}$.

We know that $H_2(Sp(2g,\mathbb{Z}/D\mathbb{Z})) \in \{0, \mathbb{Z}/2\mathbb{Z}\}$, when $g \geq 4$, and from Lemma 9 we obtain that $\widetilde{p}(2c) = 2 \cdot p_*(c) = 0$, where c is the generator of $H_2(Sp(2g,\mathbb{Z}))$. In particular, $2 \cdot \widehat{q}(c) = 0 \in \widehat{F}$, as claimed.

Moreover, Lemma 9 along with Proposition 7 provide a surjective homomorphism onto the universal central extension of $Sp(2g,\mathbb{Z}/D\mathbb{Z})$, when $D \equiv 0 \pmod 4$. Thus, by Theorem 1 the image of the center of $\widetilde{Sp(2g,\mathbb{Z})}$ has order two, as claimed.

Remark 7 – Note that $H_2(Sp(6,\mathbb{Z}/D\mathbb{Z})) \in \{0, \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}\}$, extending Funar and Pitsch 2020, Prop. 3.1. This is a consequence of Proposition 7 and the fact that $H_2(Sp(6,\mathbb{Z});\mathbb{Z}) = \mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$, according to Stein 1975. Then Proposition 3 shows that the image of some element of the center of $\widetilde{Sp(6,\mathbb{Z})}$ in $\widetilde{Sp(6,\mathbb{Z}/D\mathbb{Z})}$ has order two, when $D \equiv 0 \pmod 4$.

Remark 8 – Using the results in Brownstein and Lee 1992, 1994 we can obtain that $H_2(Sp(4,\mathbb{Z}/D\mathbb{Z})) = 0$, if $D \neq 2$ is prime and $H_2(Sp(4,\mathbb{Z}/2\mathbb{Z})) = \mathbb{Z}/2\mathbb{Z}$. Notice that $Sp(4,\mathbb{Z}/2\mathbb{Z})$ is not perfect, but the extension $\widetilde{Sp(4,\mathbb{Z})}$ still makes sense.

An immediate corollary of Theorem 1 is the following K-theory result:

Corollary 2 – *For $g \geq 4$ we have*

$$KSp_{2,2g}(\mathbb{Z}/D\mathbb{Z}) = \begin{cases} \mathbb{Z}/2\mathbb{Z}, & \text{if } D \equiv 0 (\text{mod } 4), \\ 0, & \text{otherwise.} \end{cases}$$

Proof. According to Stein's stability results (Stein 1973, Thm 2.13, and Stein 1978) we have

$$KSp_{2,2g}(\mathbb{Z}/D\mathbb{Z}) \cong KSp_2(\mathbb{Z}/D\mathbb{Z}), \text{ for } g \geq 4, \tag{15}$$

and

$$KSp_{2,2g}(\mathbb{Z}/D\mathbb{Z}) \cong H_2(Sp(2g,\mathbb{Z}/D\mathbb{Z})), \text{ for } g \geq 4. \tag{16}$$

□

Remark 9 – The analogous result that $K_{2,n}(\mathbb{Z}/D\mathbb{Z}) \in \{0, \mathbb{Z}/2\mathbb{Z}\}$, if $n \geq 3$ was proved a long time ago (see Dennis and Stein 1975).

4 Mapping class group quotients

4.1 Preliminaries on quantum representations

The results of this section are the counterpart of those obtained in Section 2.3, by considering $SU(2)$ instead of abelian quantum representations. The first author proved in Funar 2011 that central extensions of the mapping class group M_g by $\mathbb{Z}$ are residually finite. The same method actually can be used to show the abundance of finite quotients with large torsion in their second homology and to prove that the mapping class group has property A_2 trivial modules.

A quantum representation is a projective representation, depending on an integer k, which lifts to a linear representation $\widetilde{\rho}_k : M_g(12) \to U(N(k,g))$ of the central extension $M_g(12)$ of the mapping class group M_g by $\mathbb{Z}$. The latter representation corresponds to invariants of 3-manifolds with a p_1-structure. In Masbaum and Roberts 1995 and Gervais 1996 the authors gave a precise description of this extension. Namely, the cohomology class $c_{M_g(12)} \in H^2(M_g, \mathbb{Z})$ associated to this extension equals 12 times the signature class χ. It is known (see Korkmaz and Stipsicz 2003) that the group $H^2(M_g)$ is generated by χ, when $g \geq 3$. Recall that χ is one fourth of the Meyer signature class. We denote more generally by $M_g(n)$ the central extension by $\mathbb{Z}$ whose class is $c_{M_g(n)} = n\chi$.

It is known that M_g is perfect and $H_2(M_g) = \mathbb{Z}$, when $g \geq 4$ (see Pitsch 1999, for instance). Thus, for $g \geq 4$, M_g has a universal central extension by $\mathbb{Z}$, which can be identified with the central extension $M_g(1)$. This central extension makes sense for M_3, as well, although it is not the universal central extension since $H_2(M_3) = \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. However, using its explicit presentation for $g = 3$ we derive that $M_g(1)$ is perfect for $g \geq 3$.

Let c be the generator of the center of $M_g(1)$, which is 12 times the generator of the center of $M_g(12)$. Denote by $M_g(1)_n$ the quotient of $M_g(1)$ obtained by imposing $c^n = 1$; this is a non-trivial central extension of the mapping class group by $\mathbb{Z}/n\mathbb{Z}$. We will say that a quantum representation $\widetilde{\rho}_p$ *detects* the center of $M_g(1)_n$ if it factors through $M_g(1)_n$ and is injective on its center.

Consider the SO(3)-TQFT with parameter $A = -\zeta_p^{(p+1)/2}$, where ζ_p is a primitive p-th root of unity, so that A is a primitive $2p$ root of unity with $A^2 = \zeta_p$. This data leads to a quantum representation $\widetilde{\rho}_p$ and it was computed in Masbaum and Roberts 1995 that $\widetilde{\rho}_p(c) = A^{-12-p(p+1)}$.

Lemma 15 – *For each prime power q^s there exists some quantum representation $\widetilde{\rho}_p$ which detects the center of $M_g(1)_{q^s}$.*

Proof. We noted that $\widetilde{\rho}_p(c) = \zeta_{2p}^{-12-p(p+1)}$, where ζ_{2p} is a $2p$-root of unity.

1. If q is a prime number $q \geq 5$ we let $p = q^s$. Then $2p$ divides $p(p+1)$ and $\widetilde{\rho}_p(c) = \zeta_{2p}^{-12} = \zeta_p^{-6}$ is of order $p = q^s$. Thus the representation $\widetilde{\rho}_p$ detects the center of $M_g(1)_{q^s}$.

2. If $q = 2$, we set $p = 2$. Then $\widetilde{\rho}_2(c) = \zeta_2$, and $\widetilde{\rho}_2$ detects the center of $M_g(1)_2$.

3. Set now $p = 12r$ for some integer $r > 1$ to be fixed later. Then $\widetilde{\rho}_p(c) = \zeta_{24r}^{-12-12r(12r+1)} = \zeta_{2r}^{-1-r(12r+1)} = \zeta_{2r}^{-1-r}$. This $2r$-th root of the unit has order $\text{l.c.m.}(1+r, 2r)/(1+r) = 2r/\text{g.c.d.}(1+r, 2r)$. An elementary computation shows that $\text{g.c.d.}(1+r, 2r) = 1$ or 2 depending on whether r is even or odd.

- If $q = 2^{s+1}$ with $s \geq 1$, we set $r = 2^s$. Then $\zeta_{2\cdot 2^s}^{-1-2^s}$ is of order 2^{s+1}, the representation $\widetilde{\rho}_p$ detects the center of $M_g(1)_{2^{s+1}}$.
- If $g = 3^s$ with $s \geq 1$, we set $r = 3^s$. Then $\zeta_{2\cdot 3^s}^{-1-3^s}$ is of order 3^s, the representation $\widetilde{\rho}_p$ detects the center of $M_g(1)_{3^s}$. □

4.2 Proof of Theorem 4

We have first the following lemma from (Funar and Pitsch 2020, Lemma 2.3):

Lemma 16 – *Let G be a perfect finitely presented group, $\widetilde{G}$ denote its universal central extension, $p : G \to F$ be a surjective homomorphism onto a finite group F and $\widehat{p} : \widetilde{G} \to \Gamma$ be some lift of p to a finite central extension Γ of F. Assume that the image $C = \widehat{p}(Z(\widetilde{G}))$ of the center $Z(\widetilde{G})$ of $\widetilde{G}$ contains an element of order q. Then there exists an element of $p_*(H_2(G)) \subseteq H_2(F)$ of order q.*

Remark 10 – A cautionary remark is in order. Assume that $H_2(G) = \mathbb{Z}$ in the lemma above and that let $H = \ker p$. If the image of $H_2(H)$ in $H_2(G)$ is $d\mathbb{Z}$ then we can only assert that the image $p_*(H_2(G))$ in $H_2(F)$ is of the form $\mathbb{Z}/d'\mathbb{Z}$, for some divisor d' of d, which might be proper divisor. For instance taking $G = Sp(2g, \mathbb{Z})$ and $F = Sp(2g, \mathbb{Z}/D\mathbb{Z})$, where D is even and not multiple of 4, then $d = 2$ by Putman 2012, Thm. F, while $d' = 1$ as $H_2(Sp(2g, \mathbb{Z}/D\mathbb{Z})) = 0$. The apparent contradiction with Lemma 16 comes from the fact that we required the existence of a lift $\widehat{p} : \widetilde{G} \to \Gamma$ for which the image of the center has order d. If H were perfect, as it is the case when $F = Sp(2g, \mathbb{Z}/2\mathbb{Z})$ then the universal central extension $\widetilde{H}$ would come with a homomorphism $\widetilde{H} \to \widetilde{G}$. Although the image of $\widetilde{H}$ is a finite index subgroup, it is not, in general, a normal subgroup of $\widetilde{G}$. Passing to a finite index normal subgroup of $\widetilde{H}$ would amount to change H into a smaller subgroup H' and hence F is replaced by a larger quotient F'.

Proof (Proof of Theorem 4). By a classical result of Malcev 1940, finitely generated subgroups of linear groups over a commutative unital ring are residually finite. This applies to the images of quantum representations. Hence there are finite quotients $\widetilde{F}$ of these for which the image of the generator of the center is not trivial. By Lemma 15 we may find quantum representations for which the order of the image of the center can have arbitrary prime power order p. Hence, for any prime p there are finite quotients $\widetilde{F}$ of $M_g(1)$ in which the image of the center has an element of order p. We apply Lemma 16 to the quotient F of $\widetilde{F}$ by the image of the center to get finite quotients of the mapping class group with arbitrary prime order elements in the second homology. □

4.3 Some finite quotients of mapping class groups

Concrete finite quotients with arbitrary torsion in their homology can be constructed from mapping class groups as follows. Let p be a prime different from 2 and 3. Recall that we have a linear representation $\widetilde{\rho}_p : M_g(12) \to U(N(p,g))$ which lifts a projective representation $\rho_p : M_g(12) \to PU(N(p,g))$. Moreover, $M_g(1)$ is naturally embedded in $M_g(12)$, by sending the generator of the center into 12 times the generator of the center of $M_g(12)$, see Funar 2011. According to Gilmer and Masbaum 2007 for prime p, we have that $\widetilde{\rho}_p(M_g(1)) \subseteq U(N(p,g)) \cap GL(\mathcal{O}_p)$, where $\mathcal{O}_p$ is the following ring of cyclotomic integers

$$\mathcal{O}_p = \begin{cases} \mathbb{Z}[\zeta_p], & \text{if } p \equiv -1 (\mathrm{mod}\, 4) \\ \mathbb{Z}[\zeta_{4p}], & \text{if } p \equiv 1 (\mathrm{mod}\, 4). \end{cases}$$

Let us then consider the principal ideal $\mathfrak{m} = (1 - \zeta_p)$ which is a prime ideal of $\mathcal{O}_p$. It is known that prime ideals of $\mathcal{O}_p$ are maximal and then $\mathcal{O}_p/\mathfrak{m}^n$ is a finite ring for every n. Let then $\Gamma_{p,\mathfrak{m},n}$ be the image of $\widetilde{\rho}_p(M_g(1))$ into the finite group $GL(N(p,g),\mathcal{O}_p/\mathfrak{m}^n)$ and $F_{p,\mathfrak{m},n}$ be the quotient $\Gamma_{p,\mathfrak{m},n}/\langle\widetilde{\rho}_p(c)\rangle$ by the image of the center of $M_g(1)$. We derive a surjective homomorphism $\psi_{p,\mathfrak{m},n} : M_g \to F_{p,\mathfrak{m},n}$ and a lift $\widetilde{\psi}_{p,\mathfrak{m},n} : M_g(1) \to \Gamma_{p,\mathfrak{m},n}$.

The image $\widetilde{\rho}_p(c)$ of the generator c into $\Gamma_{p,\mathfrak{m},n}$ is the scalar root of unity ζ_p^{-6}, which is a non-trivial element of order p in the ring $\mathcal{O}_p/\mathfrak{m}^n$ and hence an element of order p into $GL(N(p,g),\mathcal{O}_p/\mathfrak{m}^n)$. Notice that this is a rather exceptional situation, which does not occur for other prime ideals in unequal characteristic (see Proposition 8).

Lemma 16 implies then that the image $(\psi_{p,\mathfrak{m},n})_*(H_2(M_g))$ within $H_2(F_{p,\mathfrak{m},n})$ contains an element of order p. This result also shows the contrast between mapping class group representations and Weil representations:

Corollary 3 – *If $g \geq 3$, p is prime and $p \notin \{2,3\}$ (or more generally, p does not divide 12 and not necessarily prime), then $\widetilde{\rho}_p(M_g(1))$ is a non-trivial central extension of $\rho_p(M_g)$. Furthermore, under the same hypotheses on g and p, if $\mathfrak{m} = (1 - \zeta_p)$, then the extension $\Gamma_{p,\mathfrak{m},n}$ of the finite quotient $F_{p,\mathfrak{m},n}$ is non-trivial.*

Proof. The kernel of the homomorphism $\widetilde{\rho}_p(\widetilde{M_g(1)}) \to \rho_p(M_g)$ is a finite cyclic group of $2p$-th roots of unity. Therefore $\ker(\Gamma_{p,\mathfrak{m},n} \to F_{p,\mathfrak{m},n})$ is also some finite cyclic group ν. If the latter extension were trivial then we could find an isomorphism $\Gamma_{p,\mathfrak{m},n} \to F_{p,\mathfrak{m},n} \times \nu$. Since $g \geq 3$ the group M_g is perfect and thus $M_g(1)$ and hence $\Gamma_{p,\mathfrak{m},n}$ are also perfect. Thus the projection on the second factor $\Gamma_{p,\mathfrak{m},n} \to \nu$ must be trivial. This contradicts the fact that the image of the generator c of $\ker(M_g(1) \to M_g)$ is an element of order p in $\Gamma_{p,\mathfrak{m},n}$ and hence the group of roots of unity ν has at least order p. Thus the extension $\Gamma_{p,\mathfrak{m},n} \to F_{p,\mathfrak{m},n}$ is nontrivial. This argument implies the first claim, as well. □

Remark 11 – Although the group M_2 is not perfect, because $H_1(M_2) = \mathbb{Z}/10\mathbb{Z}$, it still makes sense to consider the central extension $\widetilde{M_2}$ arising from the TQFT. Then the computations above show that the results of Theorem 4 and Corollary 3 hold for $g = 2$ if p is a prime and $p \notin \{2, 3, 5\}$.

Remark 12 – The finite quotients $F_{p,\mathfrak{m},n}$ associated to the ramified principal ideal $\mathfrak{m} = (1 - \zeta_p)$ were previously considered by Masbaum in Masbaum 2008.

When $p \equiv -1 (\mathrm{mod}\, 4)$ the authors of Funar 2013; Masbaum and Reid 2012 found many finite quotients of M_g by using more sophisticated means. However, the results of Funar 2013; Masbaum and Reid 2012 and the present ones are of a rather different nature. Assume that $\mathfrak{n}$ is a prime ideal of $\mathcal{O}_p$ such that $\mathcal{O}_p/\mathfrak{n}$ is the finite field $\mathbb{F}_q$ with q elements. In fact the case of equal characteristics $\mathfrak{n} = \mathfrak{m} = (1 - \zeta_p)$ is the only case where non-trivial torsion can arise, according to:

Proposition 8 – *If $\mathfrak{n}$ is a prime ideal of unequal characteristic (i.e. such that* g.c.d.$(p, q) = 1$*) and $p, q \geq 5$ then the image $(\psi_{p,\mathfrak{m},n})_*(H_2(M_g))$ within $H_2(F_{p,\mathfrak{m},n})$ is trivial. Moreover, for all but finitely many prime ideals $\mathfrak{n}$ of unequal characteristic both groups $\Gamma_{p,\mathfrak{n},1}$ and $F_{p,\mathfrak{n},1}$ coincide with $SL(N(p,g), \mathbb{F}_q)$ and hence $H_2(F_{p,\mathfrak{n},1}) = 0$.*

Proof. The image of a p-th root of unity scalar in $SL(N(p,g), \mathbb{F}_q)$ is trivial, as soon as g.c.d.$(p, q) = 1$. Thus $\Gamma_{p,\mathfrak{n},n} \to F_{p,\mathfrak{n},n}$ is an isomorphism and hence the image of $H_2(M_g)$ into $H_2(F_{p,\mathfrak{n},n})$ must be trivial. A priori this does not mean that $H_2(F_{p,\mathfrak{n},n}) = 0$. However, Masbaum and Reid proved in Masbaum and Reid 2012 that for all but finitely many prime ideals $\mathfrak{n}$ in $\mathcal{O}_p$ the image $\Gamma_{p,\mathfrak{n},1} \subseteq GL(N(p,g), \mathbb{F}_q)$ is the whole group $SL(N(p,g), \mathbb{F}_q)$. It follows that the projection homomorphism $M_g(1) \to SL(N(p,g), \mathbb{F}_q)$ factors through $M_g \to SL(N(p,g), \mathbb{F}_q)$. But $H_2(SL(N, \mathbb{F}_q)) = 0$, for $N \geq 4, q \geq 5$, as $SL(N, \mathbb{F}_q)$ itself is the universal central extension of $PSL(N, \mathbb{F}_q)$. □

4.4 Property A_2 and the proof of Theorem 5

Recall that an equivalent formulation of Serre's property A_2 is

Definition 5 – Let G be a discrete group and $\widehat{G}$ its profinite completion. Then G has property A_2 for the finite $\widehat{G}$-module M if the homomorphism $H^k(\widehat{G}, M) \to H^k(G, M)$ is an isomorphism for $k \leq 2$ and injective for $k = 3$.

Proposition 9 – *Let $g \geq 4$ be an integer. For any finitely generated abelian group A and any central extension*

$$1 \to A \to E \to M_g \to 1$$

the group E is residually finite.

The key result that interlocks between Proposition 9 and property A_2 is:

Proposition 10 – *1. A residually finite group G has property A_2 for all finite $\widehat{G}$-modules M if and only if any extension by a finite abelian group is residually finite.*

2. Moreover for trivial $\widehat{G}$-modules it is enough to consider central extensions of G.

Then Theorem 5 is a consequence of the two propositions above.

Proof of Proposition 9

First we treat the following special case:

Proposition 11 – *For any integer $n \geq 2$, the group $M_g(1)_n$ obtained by reducing mod n a generator of the center of $M_g(1)$ is residually finite.*

Proof. Write $\mathbb{Z}/n\mathbb{Z}$ as a finite product of cyclic groups of prime power order $\mathbb{Z}/n\mathbb{Z} = \mathbb{Z}/p_1^{r_1}\mathbb{Z} \times \cdots \times \mathbb{Z}/p_s^{r_s}\mathbb{Z}$. Then this isomorphism induces an embedding $M_g(1)_n \hookrightarrow M_g(1)_{p_1^{r_1}} \times \cdots \times M_g(1)_{p_s^{r_s}}$, and it suffices to prove Proposition 11 when n is a prime power. Since M_g is known to be residually finite Grossman 1974, by Malcev's result on the residual finiteness of finitely generated linear groups, it is enough to find for each prime power q^s a linear representation of the universal central extension $M_g(1)$ that factors through $M_g(1)_{q^s}$ and detects its center, and this is what Lemma 15 provides.

Proof (Proof of Proposition 9). We will use below that a group is residually finite if and only if finite index subgroups are residually finite. Given our central extension

$$1 \to A \to E \to M_g \to 1$$

the five term exact sequence in homology reduces to:

$$H_2(M_g;\mathbb{Z}) \to A \to H_1(E,\mathbb{Z}) \to 0$$

because the mapping class group is perfect. Therefore, any element $f \in E$ that is not in A projects non-trivially in the mapping class group and is therefore detected by a finite quotient of this group. If $f \in A$ but is not in the image of $H_2(M_g;\mathbb{Z})$, then it projects non-trivially into the finitely generated abelian group $H_1(E;\mathbb{Z})$, and is therefore detected by a finite abelian quotient of E. It remains to detect the elements in the image of $H_2(M_g;\mathbb{Z})$. Recall the following result:

Lemma 17 – *Let A be a finitely generated abelian group, B a subgroup of A. Then there exists a direct factor C of A that contains B as a subgroup of finite index.*

Apply this lemma to the image B of $H_2(M_g;\mathbb{Z})$ into A, let p_C be the projection onto the subgroup C and consider the push-out diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & A & \longrightarrow & E & \longrightarrow & M_g & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle p_C} & & \downarrow & & \| & & \\ 1 & \longrightarrow & C & \longrightarrow & E_C & \longrightarrow & M_g & \longrightarrow & 1 \end{array}$$

Then it is sufficient to prove that E_C is residually finite in order to show that E is residually finite.

Now, the mapping class group M_g is perfect, and therefore we have a push-out diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & H_2(M_g;\mathbb{Z}) & \longrightarrow & M_g(1) & \longrightarrow & M_g & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & C & \longrightarrow & E_C & \longrightarrow & M_g & \longrightarrow & 1 \end{array}$$

where the first row is the universal central extension, and the arrow $H_2(M_g;\mathbb{Z}) \to C$ is the one appearing in the five term exact sequence of the bottom extension. Recall that for $g \geq 4$, $H_2(M_g;\mathbb{Z}) = \mathbb{Z}$. Two cases could occur:

1. Either $H_2(M_g;\mathbb{Z}) \to C$ is injective and in this case E_C contains the residually finite group $M_g(1)$ as a subgroup of finite index, and this is known to be residually finite (see Funar 2011).
2. Or the image of $H_2(M_g;\mathbb{Z}) \to C$ is a cyclic group $\mathbb{Z}/k\mathbb{Z}$ and E_C contains as a finite index subgroup the reduction mod k of the universal central extension, and we conclude by applying Proposition 11. □

Proof of Proposition 10 (1)

Assume that every extension of G by a finite abelian group is residually finite. Let $x \in H^2(G;A)$ be represented by the extension:

$$1 \to A \to E \to G \to 1.$$

By the equivalent property D_2 (see the next subsection), it is enough to find a finite index subgroup $H \subseteq G$ such that x is zero in $H^2(H;A)$. Observe that property A_1, and thus D_1, is automatic. Since E is residually finite, for each non-trivial element $a \in A$ choose a finite quotient F_a of E in which the image of a is not identity. Let B_a be the image of A in F_a, and $Q_a = F_a/B_a$. Denote by F_A, B_A and Q_A the products of these finitely many groups over the non-trivial elements in A. Then the diagonal map $E \to F_A$ fits into a commutative diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & A & \longrightarrow & E & \longrightarrow & G & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & B_A & \longrightarrow & F_A & \longrightarrow & Q_A & \longrightarrow & 1 \end{array}$$

Let K be the kernel $\ker(G \to Q_A)$. Then K is a finite index normal subgroup and the pull back of x to $H^2(K;A)$ is trivial.

Conversely, assume that the residually finite group G has property A_2 and let

$$1 \to A \to E \to G \to 1$$

be an extension of G by the finite abelian group A. Then, by Serre 1994, Ex. 2 Ch. I.2.6, we have a natural short exact sequence of profinite completions:

$$1 \to \widehat{A} \to \widehat{E} \to \widehat{G} \to 1$$

that fits into a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & A & \longrightarrow & E & \longrightarrow & G & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \widehat{A} & \longrightarrow & \widehat{E} & \longrightarrow & \widehat{G} & \longrightarrow & 1 \end{array}$$

Since A is finite $A \simeq \widehat{A}$, and since G is residually finite the rightmost downward arrow is an injection. By the five lemma the homomorphism $E \hookrightarrow \widehat{E}$ is also injective, and hence E is residually finite as it is a subgroup of a profinite group.

Proof of proposition 10 (2) and property E_n

It would be nice, but probably difficult, to understand under which assumptions property A_2 for a residually finite, finitely presented group G and all finite trivial $\widehat{G}$-modules implies property A_2. However, there exists a stronger related condition on groups for which this kind of statement will hold. Serre introduced several properties in Serre 1994, Ex. 1, I.2.6 as follows. One says that a residually finite group G has property:

1. (A_n) if $H^j(\widehat{G};M) \to H^j(G;M)$ is bijective for $j \leq n$ and injective for $j = n+1$, for all finite $\widehat{G}$-modules M.
2. (B_n) if $H^j(\widehat{G};M) \to H^j(G;M)$ is surjective for $j \leq n$ and for all finite $\widehat{G}$-modules M.
3. (C_n) if for each finite $\widehat{G}$-module M and $x \in H^j(G;M)$, $1 \leq j \leq n$, there exists a discrete $\widehat{G}$-module M' containing M such that the image of x in $H^j(G;M')$ is zero.
4. (D_n) if for each finite $\widehat{G}$-module M and $x \in H^j(G;M)$, $1 \leq j \leq n$, there exists a subgroup $H \subseteq G$ of finite index in G such that the image of x in $H^j(H;M)$ is zero.
5. (E_n) if $\widehat{H}^j(G;M) = 0$, for $1 \leq j \leq n$ and for all finite $\widehat{G}$-modules M.

Then Serre stated that properties A_n, B_n, C_n and D_n are equivalent. It is easy to see that these properties are also equivalent when we fix the $\widehat{G}$-module M, or we let it run over the finite trivial $\widehat{G}$-modules.

Denote by $\widehat{H}^n(G;M) = \lim_{H\subset_f G} H^n(H;M)$, where the *direct* limit is taken with respect with the directed set of $H \subset_f G$, meaning that H is a finite index subgroup of G. The directed set of inclusions homomorphisms induces a homomorphism

$$H^n(G;M) \to \widehat{H}^n(G;M)$$

Note that if the homomorphisms $H^j(G;M) \to \widehat{H}^j(G;M)$ have zero image, for $1 \leq j \leq n$, then condition (D_n) is satisfied. Conversely, if condition (D_n) is verified, and $H^j(G;M)$ are finite for $1 \leq j \leq n$, then there exists some finite index subgroup $H \subset_f G$ such that the image of the restriction homomorphism $H^j(G;M) \to H^j(H;M)$ is trivial. By the universality of the direct limit the homomorphism $H^j(G;M) \to \widehat{H}^j(G;M)$ factors through $H^j(H;M)$ and hence its image is zero. In any case property (D_n) is a consequence of property (E_n). An interesting fact concerning the latter is the following:

Proposition 12 – *If a residually finite group G has property E_n for all finite trivial $\widehat{G}$-modules then it has property E_n.*

Proof. First we can easily step from $\mathbb{F}_p$-coefficients to any trivial G-module.

Lemma 18 – *Condition (D_n) for G and all finite trivial $\widehat{G}$-modules $\mathbb{F}_p$ implies (D_n) for G and all finite trivial $\widehat{G}$-modules M.*

The second ingredient allows us to pass from all trivial coefficients to arbitrary coefficients:

Lemma 19 – *Condition (E_n) for G and all finite trivial $\widehat{G}$-modules M implies (E_n) for G and all finite trivial $\widehat{G}$-modules M.*

This proves the claim of Proposition 12. □

In the proof of Lemma 19 we will make use of the following rather well-known result:

Lemma 20 – *If $J \subseteq I$ are two directed sets such that J is cofinal in I, then for any direct system of abelian groups (A_i, f_{ij}) indexed by I we have*

$$\lim_{\alpha,\beta\in J}(A_\alpha, f_{\alpha\beta}) = 0, \quad \text{if and only if} \quad \lim_{i,j\in I}(A_i, f_{ij}) = 0.$$

Proof (Proof of Lemma 18). This follows from decomposing the finite trivial $\widehat{G}$-module, i.e. a finite abelian group A, into p-primary components and then use induction on the rank of the composition series of A and the 5-lemma. □

Proof (Proof of Lemma 19). Let now M be an arbitrary finite G-module. Let K_M be the kernel of the G-action on M. We denote by A the trivial G-module which is isomorphic as an abelian group to M. By hypothesis condition (E_n) is satisfied for the group G and the trivial module A, so that $\widehat{H}^j(G;A) = 0$, $1 \leq j \leq n$. Since M is finite K_M is of finite index in G. Consider the set $J_M = \{H \subset_f K_M\}$ of finite index subgroups of $K_M \subseteq G$. This is a subset of the directed set I_G of finite index subgroups of G. Further J_M is cofinal in I_G with respect to the inclusion as any subgroup $H \in I_G$ contains the subgroup $H \cap K_M \subset_f K_M$. According to Lemma 20 we have then $\lim_{H \subset_f K_M} H^j(H;A) = 0$. Furthermore, for each $H \subseteq K_M$ the H-modules A and M are isomorphic as both are trivial. Thus there exists a canonical family of isomorphisms $i_H : H^j(H;A) \cong H^j(H;M)$ which is compatible with the direct structures on the cohomology groups indexed by $J_M = \{H \subset_f K_M\}$. We have therefore $\lim_{H \subset_f K_M} H^j(H;M) = 0$. However using again Lemma 20 for the sets J_M and I_G in the reverse direction we obtain $\widehat{H}^j(G;M) = 0$, $1 \leq j \leq n$. □

Proof (End of proof of Proposition 10 (2)). Let G be a residually finite group. Every $x \in H^2(G;\mathbb{F}_p)$ is represented by a central extension E of G by $\mathbb{Z}/p\mathbb{Z}$. By the proof of Proposition 10 (1) E is residually finite if and only if there exists a finite index subgroup $H \subseteq G$ such that the image of x in $H^2(H;\mathbb{F}_p)$ is zero. By Lemma 18 this is equivalent to the group G having property D_2 for all trivial $\widehat{G}$-modules M. Therefore, G has property A_2 for all trivial $\widehat{G}$-modules M if all central extensions by $\mathbb{Z}/p\mathbb{Z}$ are residually finite, for all primes p. □

Remark 13 – The analog of Lemma 18 holds also for property A_n, with the same proof. Hovewer, this is not clear for Lemma 19.

Remark 14 – The discussion about property E_n clarifies some of the statements in Grunewald, Jaikin-Zapirain, and Zalesskii 2008. Specifically, Lemmas 3.1. and 3.2. from Grunewald, Jaikin-Zapirain, and Zalesskii 2008 concern only property E_n instead of property A_n. Nevertheless, the main result of Grunewald, Jaikin-Zapirain, and Zalesskii 2008 is valid with the same proof.

4.5 Proof of Corollary 1

Lemma 21 – *An extension E of the residually finite group G by a finite group A is residually finite if and only if it is virtually trivial.*

Proof. If E is residually finite then there is a finite index subgroup $\Delta \subset E$ with $\Delta \cap A = \{1\}$. Then the projection $E \to G$ restricts to an isomorphism on Δ and hence the extension splits over the image of Δ in G. Conversely, if the extension E splits over a finite index subgroup $\Delta \subseteq G$ then $\Delta \subset E$ is residually finite and of finite index in E and hence E must be residually finite, as well. □

Lemma 22 – *An extension E of the group G by a finite group A is the pull-back of some extension of a finite group F by some surjective homomorphism $f : G \to F$ if and only if it is virtually trivial.*

Proof. If E splits over the finite index subgroup $\Delta \subseteq G$, then the image $\Delta \subseteq E$ of a section intersects A trivially. By passing to a finite index subgroup of Δ we can assume that Δ is a normal subgroup of E. Then the extension E is the pull-back of the extension

$$1 \to A \to E/\Delta \to G/\Delta \to 1$$

In the reverse direction, a pull-back of an extension by $f : G \to F$ is split over the finite index subgroup $\ker f$. □

Now, Proposition 9 along with Lemmas 21 and 22 imply Corollary 1.

5 Towards an inductive proof of Theorem 1

5.1 Motivation

For a prime p, an integer $k \geq 1$ we have two fundamental extensions:

$$1 \to Sp(2g, \mathbb{Z}, p^k) \to Sp(2g, \mathbb{Z}) \to Sp(2g, \mathbb{Z}/p^k\mathbb{Z}) \to 1,$$

and

$$1 \to \mathfrak{sp}_{2g}(p) \to Sp(2g, \mathbb{Z}/p^{k+1}\mathbb{Z}) \to Sp(2g, \mathbb{Z}/p^k\mathbb{Z}) \to 1. \tag{17}$$

In particular, every element in $Sp(2g, \mathbb{Z}, p^k)$ can be written as $\mathbf{1}_{2g} + p^k A$, for some matrix A with integer entries. If the symplectic form is written as $J_g = \begin{pmatrix} 0 & \mathbf{1}_g \\ -\mathbf{1}_g & 0 \end{pmatrix}$ then the matrix A satisfies the equation $A^\top J_g + J_g A \equiv 0 (\mathrm{mod}\ p)$. Then we set $\mathfrak{sp}_{2g}(p)$ for the additive group of those matrices with entries in $\mathbb{Z}/p\mathbb{Z}$ that satisfy the equation $M^\top J_g + J_g M \equiv 0 (\mathrm{mod}\ p)$. In particular this subgroup is independent of the integer k.

The homomorphism $j_q : Sp(2g, \mathbb{Z}, p^k) \to \mathfrak{sp}_{2g}(p)$ sending $\mathbf{1}_{2g} + p^k A$ onto $A \ (\mathrm{mod}\ p)$ is surjective (see Sato 2010).

The different actions of the symplectic group $Sp(2g,\mathbb{Z})$ that $\mathfrak{sp}_{2g}(p)$ inherits from these descriptions coincide. We will use in this text the action that is induced by the conjugation action on $Sp(2g,\mathbb{Z},p)$ via the surjective map j_q. Notice that clearly this action factors through $Sp(2g,\mathbb{Z}/p\mathbb{Z})$.

The second page of the Hochschild-Serre spectral sequence associated to the exact sequence (17) in low degrees is as follows:

$H_2(\mathfrak{sp}_{2g}(p))_{Sp(2g,\mathbb{Z}/p^k\mathbb{Z})}$		
0	$H_1(Sp(2g,\mathbb{Z}/p^k\mathbb{Z});\mathfrak{sp}_{2g}(p))$	
$\mathbb{Z}$	0	$H_2(Sp(2g,\mathbb{Z}/p^k\mathbb{Z}))$

(18)

In fact from Funar and Pitsch 2020, Lemma 3.5 we know that

$$H_1(\mathfrak{sp}_{2g}(p))_{Sp(2g,\mathbb{Z}/p^k\mathbb{Z})} = H_1(\mathfrak{sp}_{2g}(p))_{Sp(2g,\mathbb{Z}/p\mathbb{Z})} = 0.$$

as the action of $Sp(2g,\mathbb{Z}/p^k\mathbb{Z})$ on $\mathfrak{sp}_{2g}(p)$ factors through the action of $Sp(2g,\mathbb{Z}/p\mathbb{Z})$.

Thus the calculations needed for an inductive computation of $H_2(Sp(2g,\mathbb{Z}/2^k\mathbb{Z}))$ are the result in Theorem 6 and the following theorem, see Putman 2012, Thm. G:

Theorem 7 – *For any odd prime p, any integer $k \geq 1$ and any $g \geq 3$ we have:*

$$H_1(Sp(2g,\mathbb{Z}/p^k\mathbb{Z}),\mathfrak{sp}_{2g}(p)) = 0. \tag{19}$$

Moreover, this holds true also when $p = 2$ and $k = 1$.

Unfortunately we do not know whether $H_1(Sp(2g,\mathbb{Z}/2^k\mathbb{Z});\mathfrak{sp}_{2g}(2)) = 0$ or not for $k \geq 2$. However this is true when $k = 1$ and we derive:

Corollary 4 – *Assume Theorem 6 holds. Then $H_2(Sp(2g,\mathbb{Z}/4\mathbb{Z})) \in \{0,\mathbb{Z}/2\mathbb{Z}\}$, for all $g \geq 4$.*

Proof. Proposition 7 implies that $H_2(Sp(2g,\mathbb{Z}/4\mathbb{Z}))$ is cyclic, for $g \geq 4$. Since $H_2(Sp(2g,\mathbb{Z}/2\mathbb{Z})) = 0$, and $H_1(Sp(2g,\mathbb{Z}/2\mathbb{Z}),\mathfrak{sp}_{2g}(2)) = 0$ from Putman's theorem 7, the only non-zero term of the second page of the Hochschild-Serre spectral sequence above computing the cohomology of $Sp(2g,\mathbb{Z}/4\mathbb{Z})$ is $H_2(\mathfrak{sp}_{2g}(2))_{Sp(2g,\mathbb{Z}/2\mathbb{Z})}$. Then, by Theorem 6 the rank of $H_2(Sp(2g,\mathbb{Z}/4\mathbb{Z}))$ is at most 1, which proves the claim. □

5.2 Generators for the module $\mathfrak{sp}_{2g}(p)$

We describe first a small set of generators of $\mathfrak{sp}_{2g}(p)$ as an $Sp(2g,\mathbb{Z})$-module. Denote by $\mathfrak{M}_g(R)$ the R-module of g-by-g matrices with entries from the ring R. A direct computation shows that a matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathfrak{M}_{2g}(\mathbb{Z}/p\mathbb{Z})$ written by blocks is in

$\mathfrak{sp}_{2g}(p)$ if and only if $A + D^\top \equiv 0 \pmod p$ and both B and C are symmetric matrices. It will be important for our future computations to keep in mind that the subgroup $GL(g,\mathbb{Z}) \subset Sp(2g,\mathbb{Z})$ preserves this decomposition into blocks. From this description we immediately get a set of generators of $\mathfrak{sp}_{2g}(p)$ as an additive group. Recall that $e_{ij} \in \mathfrak{M}_g(R)$ denotes the elementary matrix whose only non-zero coefficient is 1 at the place ij. Define now the following matrices in $\mathfrak{sp}_{2g}(p)$ for $i,j \in \{1,2,\dots,g\}$:

$$u_{ij} = \begin{pmatrix} 0 & e_{ij}+e_{ji} \\ 0 & 0 \end{pmatrix}, \quad u_{ii} = \begin{pmatrix} 0 & e_{ii} \\ 0 & 0 \end{pmatrix}, \quad l_{ij} = \begin{pmatrix} 0 & 0 \\ e_{ij}+e_{ji} & 0 \end{pmatrix}, \quad l_{ii} = \begin{pmatrix} 0 & 0 \\ e_{ii} & 0 \end{pmatrix}, \quad (20)$$

$$r_{ij} = \begin{pmatrix} e_{ij} & 0 \\ 0 & -e_{ji} \end{pmatrix}, \quad n_{ii} = \begin{pmatrix} e_{ii} & 0 \\ 0 & -e_{ii} \end{pmatrix}. \quad (21)$$

Therefore we have:

Proposition 13 – *As an $Sp(2g,\mathbb{Z}/p\mathbb{Z})$-module, $\mathfrak{sp}_{2g}(p)$ is generated by $r_{ij}, n_{ii}, u_{ij}, u_{ii}$, l_{ij} and l_{ii}, where $i,j \in \{1,2,\dots,g\}$.*

And as $GL(g,\mathbb{Z}/p\mathbb{Z})$-module we have:

Lemma 23 – *Let $\mathrm{Sym}_g(\mathbb{Z}/p\mathbb{Z}) \subset \mathfrak{M}_g(\mathbb{Z}/p\mathbb{Z})$ denote the submodule of symmetric matrices. We have an identification of $GL(g,\mathbb{Z})$-modules:*

$$\mathfrak{sp}_{2g}(p) = \mathfrak{M}_g(\mathbb{Z}/p\mathbb{Z}) \oplus \mathrm{Sym}_g(\mathbb{Z}/p\mathbb{Z}) \oplus \mathrm{Sym}_g(\mathbb{Z}/p\mathbb{Z}).$$

The action of $GL(g,\mathbb{Z}/p\mathbb{Z})$ on $\mathfrak{M}_g(\mathbb{Z}/p\mathbb{Z})$ is by conjugation, the action on the first copy $\mathrm{Sym}_g(\mathbb{Z}/p\mathbb{Z})$ is given by $x \cdot A = xAx^\top$ and on the second copy $\mathrm{Sym}_g(\mathbb{Z}/p\mathbb{Z})$ is given by $x \cdot A = (x^\top)^{-1}Ax^{-1}$.

A set of generators for $\mathfrak{M}_g(\mathbb{Z}/p\mathbb{Z})$ is given by the set of elements r_{ij} and n_{ii}, $1 \le i,j \le g$, $i \neq j$. The two copies of $\mathrm{Sym}_g(\mathbb{Z}/p\mathbb{Z})$ are generated by the matrices l_{ij} and u_{ij} respectively, where $1 \le i,j \le g$.

5.3 Proof of Theorem 6

Notice first that as the action of $Sp(2g,\mathbb{Z}/p^k\mathbb{Z})$ factors through $Sp(2g,\mathbb{Z}/p\mathbb{Z})$ via the mod p^{k-1} reduction map, for any $k \ge 2$ we have $H_2(\mathfrak{sp}_{2g}(p);\mathbb{Z})_{Sp(2g,\mathbb{Z}/p^k\mathbb{Z})} \simeq H_2(\mathfrak{sp}_{2g}(p))_{Sp(2g,\mathbb{Z}/p\mathbb{Z})}$. Also, as $\mathfrak{sp}_{2g}(p)$ is an abelian group there is a canonical isomorphism:

$$H_2(\mathfrak{sp}_{2g}(p)) = \wedge^2 \mathfrak{sp}_{2g}(p). \quad (22)$$

Let M denote the set of elements r_{ij} and n_{ii}, $1 \leq i,j \leq g$, $i \neq j$ and S denote the set of elements l_{ij} and u_{ij}, where $1 \leq i,j \leq g$. The group $\wedge^2\mathfrak{sp}_{2g}(p)$ is generated by the set of exterior powers of pairs of generators given in Proposition 13, which we split naturally into three disjoint subsets:

1. The subset $S \wedge S$ of exterior powers $u_{ij} \wedge l_{kl}$, $l_{ij} \wedge l_{kl}$, $u_{ij} \wedge u_{kl}$.
2. The subset $S \wedge M$ of exterior power $l_{ij} \wedge n_{kk}$, $u_{ij} \wedge n_{kk}$, $l_{ij} \wedge r_{kl}$ and $u_{ij} \wedge r_{kl}$.
3. The subset $M \wedge M$ of exterior powers $r_{ij} \wedge r_{kl}$, $r_{ij} \wedge n_{kk}$ and $n_{jj} \wedge n_{ii}$.

We will first show that the image of $S \wedge S$ and $S \wedge M$ is 0 in $\wedge^2\mathfrak{sp}_{2g}(p)$, and in a second time we will show how the $\mathbb{Z}/2\mathbb{Z}$ factor appears in the image of $M \wedge M$. It will also be clear from the proof why there is no such non-trivial element in odd characteristic.

We use constantly the trivial fact that the action of $Sp(2g,\mathbb{Z}/p\mathbb{Z})$ is trivial on the coinvariants module. So to show the nullity of the image of a generator it is enough to show that in each orbit of a generating element of $S \wedge S$ or $S \wedge M$ there is the 0 element. We will in particular heavily use the fact that the symmetric group $\mathfrak{S}_g \subset GL(g,\mathbb{Z}/p\mathbb{Z}) \subset Sp(2g,\mathbb{Z}/p\mathbb{Z})$ acts on the basis elements by permuting the indices. Notice that the symmetric group action respects the above partition into three sets of elements. To emphasize when we use such a permutation to identify two elements in the co-invariant module we will use the notation $\doteq$ instead of $=$.

Nullity of the generators in $S \wedge S$. Picking one representative in each $\mathfrak{S}_g$-orbit we are left with the following elements. Here "Type" refers to the number of distinct indexes that appear in the wedge, as this is the only thing that really matters. Of course type IV elements appear only for $g \geq 4$.

1. Type I: $u_{11} \wedge l_{11}$.
2. Type II: $u_{11} \wedge u_{22}$, $l_{11} \wedge l_{22}$, $u_{11} \wedge l_{22}$, $u_{11} \wedge u_{12}$, $l_{11} \wedge l_{12}$, $u_{12} \wedge l_{12}$, $u_{12} \wedge l_{22}$, $u_{11} \wedge l_{12}$.
3. Type III: $u_{11} \wedge u_{23}$, $l_{11} \wedge l_{23}$, $u_{12} \wedge u_{23}$, $l_{12} \wedge l_{23}$, $u_{11} \wedge l_{23}$, $l_{11} \wedge u_{23}$, $u_{12} \wedge l_{23}$.
4. Type IV: $u_{12} \wedge u_{34}$, $l_{12} \wedge l_{34}$, $u_{12} \wedge l_{34}$.

Using the fact that $J_g \cdot u_{ij} = -l_{ij}$, one can identify some generators in $S \wedge S$, for instance $u_{11} \wedge u_{22} = l_{11} \wedge l_{22}$, and we are left with:

1. Type I: $u_{11} \wedge l_{11}$.
2. Type II: $u_{11} \wedge u_{22}$, $u_{11} \wedge l_{22}$, $u_{11} \wedge u_{12}$, $u_{11} \wedge l_{12}$, $u_{12} \wedge l_{12}$, $u_{12} \wedge l_{22}$.

3. Type III: $u_{11}\wedge u_{23}$, , $u_{12}\wedge u_{23}$, $u_{11}\wedge l_{23}$, $u_{12}\wedge l_{23}$.
4. Type IV: $u_{12}\wedge u_{34}$, $u_{12}\wedge l_{34}$.

We consider now two families of elements in the symplectic group, for $1\leq i\neq j\leq g$:

$$\tau^u_{ij}=\begin{pmatrix}1_g & e_{ii}+e_{jj}\\ 0 & 1_g\end{pmatrix},\quad \tau^l_{ij}=\begin{pmatrix}1_g & 0\\ e_{ii}+e_{jj} & 1_g\end{pmatrix}.$$

A direct computation shows that:

$$\tau^u_{ij}\cdot u_{k\ell}=u_{k\ell},\quad \tau^u_{ij}\cdot r_{ij}=r_{ij}-u_{ij},\text{ for all } i,j,k,\ell \tag{23}$$

$$\tau^l_{ij}\cdot l_{k\ell}=l_{k\ell},\quad \tau^l_{ij}\cdot r_{ij}=r_{ij}+l_{ij},\text{ for all } i,j,k,\ell \tag{24}$$

In particular, we obtain:

$$\tau^u_{ij}\cdot(u_{k\ell}\wedge r_{ij})=u_{k\ell}\wedge r_{ij}-u_{k\ell}\wedge u_{ij} \tag{25}$$

and

$$\tau^l_{ij}\cdot(l_{k\ell}\wedge r_{ij})=l_{k\ell}\wedge r_{ij}+l_{k\ell}\wedge l_{ij} \tag{26}$$

This shows that all elements of the form $u\wedge u$ or $l\wedge l$ vanish in the module of coinvariants, except possibly for $u_{11}\wedge u_{22}$ and $l_{11}\wedge l_{22}$. We are left then with:

1. Type I: $u_{11}\wedge l_{11}$.
2. Type II: $u_{11}\wedge u_{22}$, $u_{11}\wedge l_{22}$, $u_{11}\wedge l_{12}$, $u_{12}\wedge l_{12}$, $u_{12}\wedge l_{22}$.
3. Type III: $u_{11}\wedge l_{23}$, $u_{12}\wedge l_{23}$.
4. Type IV: $u_{12}\wedge l_{34}$.

Consider the exchange map E_{ij}, $i\neq j$, defined by $a_i\to -b_i$, $b_i\to a_i$, $a_j\to -b_j$, $b_j\to a_j$, and all other basis elements fixed. By construction, these maps act as follows: $E_{ij}\cdot u_{ij}=-l_{ij}$, $E_{ij}\cdot u_{ii}=-l_{ii}$, $E_{ij}\cdot u_{jj}=-l_{jj}$ and $E_{ij}\cdot u_{kl}=u_{kl}$ if $\{i,j\}\cap\{k,l\}=\emptyset$.

Denoting in the same way the (trivial!) action on the quotient module of coinvariants $(\wedge^2\mathfrak{sp}_{2g}(p))_{Sp_2(2g,\mathbb{Z}/p\mathbb{Z})}$, we find:

1. $u_{11}\wedge l_{23}=E_{23}\cdot(u_{11}\wedge l_{23})=-u_{11}\wedge u_{23}=0$.
2. $u_{12}\wedge l_{22}=E_{12}\cdot(u_{12}\wedge l_{22})=-l_{12}\wedge -u_{22}=u_{11}\wedge l_{12}$.

3. $u_{11} \wedge u_{22} = E_{23} \cdot (u_{11} \wedge u_{22}) = -u_{11} \wedge l_{22}$.

4. $u_{12} \wedge l_{34} = E_{34} \cdot (u_{12} \wedge l_{34}) = u_{12} \wedge -u_{34} = 0$.

Thus the coinvariants module is generated by the classes of the following elements:

1. Type I: $u_{11} \wedge l_{11}$.

2. Type II: $u_{11} \wedge l_{22}$, $u_{11} \wedge l_{12}$, $u_{12} \wedge l_{12}$.

3. Type III: $u_{12} \wedge l_{23}$.

We now use the action of the following symplectic maps, for $1 \leq i \neq j \leq g$:

$$A_{ij} = \begin{pmatrix} 1_g - e_{ji} & 0 \\ 0 & 1_g + e_{ij} \end{pmatrix}.$$

By direct computation we obtain:

$$A_{ij} \cdot u_{ii} = u_{ii} + u_{jj} - u_{ij},\ A_{ij} \cdot u_{jj} = u_{jj},\ A_{ik} \cdot u_{ij} = u_{ij} - u_{jk},\ A_{ki} \cdot u_{ij} = u_{ij}$$

and

$$A_{ij} \cdot l_{ii} = l_{ii} + l_{jj} - l_{ij},\ A_{ij} \cdot l_{jj} = l_{jj},\ A_{ik} \cdot l_{ij} = l_{ij} - l_{jk},\ A_{ki} \cdot l_{ij} = l_{ij}$$

for pairwise distinct values of the indices i, j, k. Further, we derive the following equalities that hold in the quotient module of coinvariants:

1. $u_{12} \wedge l_{23} = A_{13} \cdot (u_{12} \wedge l_{23}) = (u_{12} - u_{23}) \wedge l_{23}$, so $0 = u_{23} \wedge l_{23} \doteq u_{12} \wedge l_{12}$.

2. $u_{22} \wedge l_{32} = A_{13} \cdot (u_{22} \wedge l_{32}) = u_{22} \wedge (l_{32} + l_{12})$, so $u_{11} \wedge l_{12} \doteq u_{22} \wedge l_{12} = 0$.

3. $u_{12} \wedge l_{34} = A_{23} \cdot (u_{12} \wedge l_{34}) = u_{12} \wedge (l_{34} + l_{24})$, so $u_{12} \wedge l_{23} \doteq u_{12} \wedge l_{24} = 0$.

4. $u_{11} \wedge l_{11} = A_{12} \cdot (u_{11} \wedge l_{11}) = (u_{11} + u_{22} - u_{12}) \wedge l_{11}$, so
   $u_{22} \wedge l_{11} = u_{12} \wedge l_{11} = E_{12} \cdot (l_{12} \wedge u_{11}) = 0$.

5. $u_{22} \wedge l_{11} = A_{12} \cdot (u_{22} \wedge l_{11}) = u_{22} \wedge (l_{11} + l_{12} + l_{22})$ so
   $u_{11} \wedge l_{11} \doteq u_{22} \wedge l_{22} = -u_{22} \wedge l_{12} \doteq -u_{11} \wedge l_{12} = 0$.

This finishes the computation.

Nullity of the generators in $S \wedge M$. Here we separate between two types of generators:

1. Generators of the form $u_{ks} \wedge n_{ii}$ and $l_{ks} \wedge n_{ii}$ for arbitray k, s, i.
   We let the matrix τ^{ℓ}_{ij} of the previous section act on these generators. A direct computation shows that, for arbitrary values of k, s, i, j with $i \neq j$, we have:
   $$\tau^{\ell}_{ij} \cdot l_{ks} = l_{ks} \text{ and } \tau^{\ell}_{ij} \cdot u_{jj} = u_{jj} - l_{jj} - n_{jj}.$$
   Therefore, relying on our previous computations:
   $$0 = \tau^{\ell}_{ij} \cdot (\ell_{ks} \wedge u_{jj}) = l_{ks} \wedge (u_{jj} - l_{jj} - n_j) = -l_{ks} \wedge n_{jj}$$
   To get the nullity for elements of the form $u \wedge n$, we apply J_g to the previous element, and use the fact that, up to a sign, J_g exchanges the elements u_{ij} and l_{ij} while fixing n_{ii}.
2. For the elements of the form $u_{ks} \wedge r_{ij}$ and $l_{ks} \wedge r_{ij}$ for arbitrary k, s, i, j and $i \neq j$, we apply the element
   $$\tau^{\ell}_{i} = \begin{pmatrix} 1_g & 0 \\ e_{ii} & 1_g \end{pmatrix}.$$
   By direct computation we obtain:
   $$\tau^{\ell}_{i} \cdot u_{ij} = u_{ij} - r_{ij} + l_{ii} \text{ and } \tau^{\ell}_{i} \cdot l_{ks} = l_{ks}.$$
   Therefore, for $i \neq j$ and k, s arbitrary we find:
   $$0 = \tau^{\ell}_{i} \cdot (u_{ij} \wedge l_{ks}) = (u_{ij} - r_{ij} + l_{ii}) \wedge l_{ks} = -r_{ij} \wedge l_{ks}.$$
   By applying J_g and using that $J_g \cdot r_{ij} = -r_{ji}$ we get the nullity for $u_{ks} \wedge r_{ij}$.

Image of $M \wedge M$. First we will do a small detour through bilinear forms on matrices. Until the very end we work on an arbitrary field $\mathbb{K}$. Recall that $\mathfrak{M}_n(\mathbb{K})$ denotes the $\mathbb{K}$-vector space of n-by-n matrices with entries in $\mathbb{K}$ and $\mathbf{1}_n \in \mathfrak{M}_n(\mathbb{K})$ the identity matrix. Note that, if $i \neq j$, the inverse of $\mathbf{1}_n + e_{ij}$ is $\mathbf{1}_n - e_{ij}$ and that elementary matrices multiply according to the rule $e_{ij}e_{st} = \delta_{js}e_{it}$. We start by a very classical result:

Lemma 24 – *Let* tr *denote the trace map. Then for any field* $\mathbb{K}$ *and any integer* n *the homomorphism:*

$$\begin{array}{rcl} \mathfrak{M}_n(\mathbb{K}) & \to & \mathrm{Hom}(\mathfrak{M}_n(\mathbb{K}), \mathbb{K}) \\ A & \mapsto & B \rightsquigarrow \mathrm{tr}(AB) \end{array}$$

is an isomorphism.

A little less classical is:

Theorem 8 – *Let $n \geq 2$. The $\mathbb{K}$-vector space $Hom_{GL(n,\mathbb{K})}(\mathfrak{M}_n(\mathbb{K}), \mathfrak{M}_n(\mathbb{K}))$ has dimension 2. It is generated by the identity map $Id_{\mathfrak{M}_n(\mathbb{K})}$ and by the map $\Psi(M) = tr(M)\mathbf{1}_n$.*

Proof. It is easy to check that the two equivariant maps $Id_{\mathfrak{M}_n(\mathbb{K})}$ and Ψ are linearly independent. Indeed, evaluating a linear dependence relation $\alpha Id_{\mathfrak{M}_n(\mathbb{K})} + \beta\Psi = 0$ on e_{12} one gets $\alpha = 0 = \beta$.

Fix an arbitrary $\phi \in \mathrm{Hom}_{GL(n,\mathbb{K})}(\mathfrak{M}_n(\mathbb{K}), \mathfrak{M}_n(\mathbb{K}))$. Denote by $A = (a_{ij})$ the matrix $\phi(e_{11})$. Consider two integers $1 < s \neq t \leq n$. From the equality $(\mathbf{1}_n + e_{st})e_{11}(\mathbf{1}_n - e_{st}) = e_{11}$ we deduce that:

$$\phi(e_{11}) = (\mathbf{1}_n + e_{st})\phi(e_{11})(\mathbf{1}_n - e_{st}) \tag{27}$$
$$= \phi(e_{11}) + e_{st}\phi(e_{11}) - \phi(e_{11})e_{st} - e_{st}\phi(e_{11})e_{st} \tag{28}$$
$$= \phi(e_{11}) + \sum_{1\leq j\leq n} a_{tj}e_{sj} - \sum_{1\leq i\leq n} a_{is}e_{it} - a_{ts}e_{st} \tag{29}$$

Therefore, for $1 < s \neq t \leq n$ we have:

$$\sum_{1\leq j\leq n} a_{tj}e_{sj} - \sum_{1\leq i\leq n} a_{is}e_{it} - a_{ts}e_{st} = 0 \tag{30}$$

The first term in this sum is a matrix with only one non-zero row, the second a matrix with only one non-zero column and the third a matrix with a single (possibly) non-zero entry. The only common entry for this three matrices appears for $j = t$ and $i = s$, where we get the equation $a_{tt} - a_{ss} - a_{ts} = 0$. Otherwise, $a_{tj} = 0$, for all $j \neq t$, and $a_{is} = 0$, for all $i \neq s$. Observe that, in particular, $a_{ts} = 0$. Summing up, either in the column s or in the row t of the matrix A, the only possible non-zero elements are those that appear in the equation $a_{tt} - a_{ss} = 0$.

Letting s and t vary, one deduces that $a_{ts} = 0$, and $A = \phi(e_{11})$ is of the form $\lambda e_{11} + \mu\sum_{i=2}^{n} e_{ii}$ for two scalars $\lambda, \mu \in \mathbb{K}$.

Let T_{ij} be the invertible matrix that interchanges the basis vectors i and j. Then $T_{ij}e_{ii}T_{ij} = e_{jj}$, $T_{ij}e_{jj}T_{ij} = e_{ii}$ and $T_{ij}e_{kk}T_{ij} = e_{kk}$ for $k \neq i$ and $k \neq j$. Therefore, $\phi(e_{jj}) = \phi(T_{1j}e_{11}T_{1j}) = T_{1j}\phi(e_{11})T_{1j}$. And from the description of $\phi(e_{11})$ one gets:

$$\phi(e_{jj}) = \lambda e_{jj} + \sum_{i\neq j} \mu e_{ii}, \text{ for all } 1 \leq j \leq n.$$

From the relation $(\mathbf{1}_n + e_{ij})e_{ii}(\mathbf{1}_n - e_{ij}) = e_{ii} - e_{ij}$ we get:

$$\phi((\mathbf{1}_n + e_{ij})e_{ii}(\mathbf{1}_n - e_{ij})) = \phi(e_{ii}) - \phi(e_{ij}) \tag{31}$$
(Cont. next page)
$$= (\mathbf{1}_n + e_{ij})\phi(e_{ii})(\mathbf{1}_n - e_{ij}) \tag{32}$$

$$= \phi(e_{ii}) + e_{ij}\phi(e_{ii}) - \phi(e_{ii})e_{ij} - e_{ij}\phi(e_{ii})e_{ij}, \tag{33}$$

and in particular:

$$\phi(e_{ij}) = -e_{ij}\phi(e_{ii}) + \phi(e_{ii})e_{ij} + e_{ij}\phi(e_{ii})e_{ij} \tag{34}$$
$$= -\mu e_{ij} + \lambda e_{ij} \tag{35}$$
$$= (\lambda - \mu)e_{ij} \tag{36}$$

This shows that ϕ is completely determined by $\phi(e_{ii})$ and that the $\mathbb{K}$-vector space $\mathrm{Hom}_{GL_n(\mathbb{K})}(\mathfrak{M}_n(\mathbb{K}), \mathfrak{M}_n(\mathbb{K}))$ has dimension at most 2. Observe that the identity map $Id_{\mathfrak{M}_n(\mathbb{K})}$ corresponds to $\lambda = 1, \mu = 0$ and that Ψ corresponds to $\lambda = \mu = 1$. □

From these two results we deduce the result that will save us from lengthy computations:

Proposition 14 – *Let $n \geq 2$. For any field $\mathbb{K}$ the vector space of bilinear forms on $\mathfrak{M}_n(\mathbb{K})$ invariant under conjugation by $GL(n, \mathbb{K})$ has as basis the bilinear maps $(A, B) \mapsto tr(A)tr(B)$ and $(A, B) \mapsto tr(AB)$. If $char(\mathbb{K}) \neq 2$, the subspace of alternating bilinear forms is trivial. If $char(\mathbb{K}) = 2$ the space of bilinear alternating forms is generated by the form $(A, B) \mapsto tr(A)tr(B) + tr(AB)$.*

Consider now the canonical map

$$\wedge^2(\mathfrak{M}_g(\mathbb{Z}/p\mathbb{Z})) \to \wedge^2(\mathfrak{M}_g(\mathbb{Z}/p\mathbb{Z}))_{GL_g(\mathbb{Z}/p\mathbb{Z})} \to \wedge^2(\mathfrak{sp}_{2g}(p))_{GL(g,\mathbb{Z}/p\mathbb{Z})} \to$$
$$\to \wedge^2(\mathfrak{sp}_{2g}(p))_{Sp(2g,\mathbb{Z}/p\mathbb{Z})}. \tag{37}$$

By construction its image is the span of the image of $M \wedge M$ in $\wedge^2(\mathfrak{sp}_{2g}(p))_{Sp(2g,\mathbb{Z}/p\mathbb{Z})}$. By Proposition 14, it is 0 if p is odd and it is at most $\mathbb{Z}/2\mathbb{Z}$ if $p = 2$.

Further, the unique $GL(g, \mathbb{Z}/2\mathbb{Z})$-invariant alternating form on $\mathfrak{M}_g(\mathbb{Z}/2\mathbb{Z})$ does not vanish on the element $e_{11} \wedge e_{22}$ and hence we have a nontrivial element of $\wedge^2(\mathfrak{M}_g(\mathbb{Z}/2\mathbb{Z}))_{GL(g,\mathbb{Z}/2\mathbb{Z})}$, which is a submodule of $\wedge^2(\mathfrak{sp}_{2g}(2))_{GL(g,\mathbb{Z}/2\mathbb{Z})}$. Note that the image of $e_{11} \wedge e_{22}$ in $\wedge^2(\mathfrak{sp}_{2g}(2))_{GL(g,\mathbb{Z}/2\mathbb{Z})}$ is the class of the element $n_{11} \wedge n_{22}$.

Furthermore, fix a symplectic basis $\{a_i, b_i\}_{1 \leq i \leq g}$ of $\mathbb{Z}^{2g}$. Then, the $2g+1$ transvections along the elements $a_i, b_j - b_{j+1}$ for $1 \leq i \leq g$ and $1 \leq j \leq g-1$, b_{g-1} and b_g generate $Sp(2g, \mathbb{Z}/2\mathbb{Z})$, for instance because they are the canonical images of the set of Dehn twists generators of the mapping class group considered by Humphries in Humphries 1979. One checks directly, using all elements we know they vanish in $\wedge^2(\mathfrak{sp}_{2g}(2))_{Sp(2g,\mathbb{Z}/2\mathbb{Z})}$, that the action of these generators on $n_{11} \wedge n_{22}$ is trivial and hence it defines an element of $\wedge^2(\mathfrak{sp}_{2g}(2))_{Sp(2g,\mathbb{Z}/2\mathbb{Z})}$. This finishes our proof.

Remark 15 – It is clear from the proof that the copy $\mathbb{Z}/2\mathbb{Z}$ we have detected is stable, in the sense that the homomorphism

$$\wedge^2(\mathfrak{sp}_{2g}(2))_{Sp(2g,\mathbb{Z}/2\mathbb{Z})} \to \wedge^2(\mathfrak{sp}_{2g+2}(2))_{Sp(2g+2,\mathbb{Z}/2\mathbb{Z})}$$

is an isomorphism, for all $g \geq 3$, since both are detected by the obvious stable element $n_{11} \wedge n_{22}$.

A Appendix: Weil representations using theta functions

A.1 Weil representations at level k, for even k

Let $\mathcal{S}_g$ be the Siegel space of $g \times g$ symmetric matrices Ω of complex entries having the imaginary part $\mathrm{Im}\,\Omega$ positive defined. We represent any element $\gamma \in Sp(2g,\mathbb{Z})$ as $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ where A, B, C, D are $g \times g$ matrices. There is a natural $Sp(2g,\mathbb{Z})$ action on $\mathbb{C}^g \times \mathcal{S}_g$ given by

$$\gamma \cdot (z,\Omega) = ((((C\Omega + D)^\top)^{-1})z, (A\Omega + B)(C\Omega + D)^{-1}). \tag{38}$$

The dependence of the classical theta function $\theta(z,\Omega)$ on Ω is expressed by a functional equation which describes its behavior under the action of $Sp(2g,\mathbb{Z})$. Let $\Gamma(1,2)$ be the so-called theta group consisting of elements $\gamma \in Sp(2g,\mathbb{Z})$ which preserve the quadratic form

$$Q(n_1, n_2, ..., n_{2g}) = \sum_{i=1}^{g} n_i n_{i+g} \in \mathbb{Z}/2\mathbb{Z},$$

which means that $Q(\gamma(x)) = Q(x)(\mathrm{mod}\ 2)$. Then $\Gamma(1,2)$ may be alternatively described as the set of those elements γ having the property that the diagonals of $A^\top C$ and $B^\top D$ are even. Let $\langle , \rangle$ denote the standard hermitian product on $\mathbb{C}^{2g}$. The functional equation, as stated in Mumford 1984 is:

$$\theta(\gamma \cdot (z,\Omega)) = \zeta_\gamma \det(C\Omega + D)^{1/2} \exp(\pi\sqrt{-1}\langle z, (C\Omega + D)^{-1}Cz\rangle)\theta(z,\Omega),$$

for $\gamma \in \Gamma(1,2)$, where ζ_γ is a certain 8^{th} root of unity.

If $g = 1$ we may suppose that $C > 0$ or $C = 0$ and $D > 0$ so we have $\mathrm{Im}(C\Omega + D) \geq 0$ for Ω in the upper half plane. Then we will choose the square root $(C\Omega + D)^{1/2}$ in the first quadrant. Now we can express the dependence of ζ_γ on γ as follows:

1. for even C and odd D, $\zeta_\gamma = \sqrt{-1}^{(D-1)/2}(\frac{C}{|D|})$,

2. for odd C and even D, $\zeta_\gamma = \exp(-\pi\sqrt{-1}C/4)(\frac{D}{C})$,

where $(\frac{x}{y})$ is the usual Jacobi symbol, see Hardy and Wright 1979.

For $g > 1$ it is less obvious to describe this dependence. We fix first the choice of the square root of $\det(C\Omega + D)$ in the following manner: let $\det^{\frac{1}{2}}\left(\frac{Z}{\sqrt{-1}}\right)$ be the unique holomorphic function on $\mathcal{S}_g$ satisfying

$$\left(\det^{\frac{1}{2}}\left(\frac{Z}{\sqrt{-1}}\right)\right)^2 = \det\left(\frac{Z}{\sqrt{-1}}\right),$$

and taking in $\sqrt{-1}\mathbf{1}_g$ the value 1. Next define

$$\det^{\frac{1}{2}}(C\Omega + D) = \det^{\frac{1}{2}}(D)\det^{\frac{1}{2}}\left(\frac{\Omega}{\sqrt{-1}}\right)\det^{\frac{1}{2}}\left(\frac{-\Omega^{-1} - D^{-1}C}{\sqrt{-1}}\right),$$

where the square root of $\det(D)$ is taken to lie in the first quadrant. Using this convention we may express ζ_γ as a Gauss sum for invertible D, see Freitag 1987, pp. 26-27:

$$\zeta_\gamma = \det^{-\frac{1}{2}}(D) \sum_{\ell \in \mathbb{Z}^g/D\mathbb{Z}^g} \exp(\pi\sqrt{-1}\langle \ell, BD^{-1}\ell\rangle), \tag{39}$$

and in particular we recover the formula from above for $g = 1$. On the other hand for $\gamma = \begin{pmatrix} A & 0 \\ 0 & (A^\top)^{-1} \end{pmatrix}$ we have $\zeta_\gamma = (\det A)^{-1/2}$. We recall that a multiplier system (Freitag 1987) for a subgroup $\Gamma \subset \mathrm{Sp}(2g, \mathbb{R})$ is a map $m : \Gamma \longrightarrow \mathbb{C}^*$ such that

$$m(\gamma_1\gamma_2) = s(\gamma_1, \gamma_2)m(\gamma_1)m(\gamma_2).$$

An easy remark is that, once a multiplier system m is chosen, the product $A(\gamma, \Omega) = m(\gamma)j(\gamma, \Omega)$ verifies the cocycle condition

$$A(\gamma_1\gamma_2, \Omega) = A(\gamma_1, \gamma_2\Omega)A(\gamma_2, \Omega),$$

for $\gamma_i \in \Gamma$. Then another formulation of the dependence of ζ_γ on γ is to say that it is the multiplier system defined on $\Gamma(1, 2)$. Remark that using the congruence subgroup property due to Mennicke (Mennicke 1965a,b) and Bass, Milnor and Serre (Bass, Milnor, and Serre 1967) any two multiplier systems defined on a subgroup of the theta group are identical on some congruence subgroup.

When $\gamma = \begin{pmatrix} 1_g & B \\ 0 & 1_g \end{pmatrix}$ then the multiplier system is trivial, $\zeta_\gamma = 1$, and eventually for $\gamma = \begin{pmatrix} 0 & -1_g \\ 1_g & 0 \end{pmatrix}$ we have $\zeta_\gamma = \exp(\pi\sqrt{-1}g/4)$. Actually this data determines completely ζ_γ.

Denote $\det^{\frac{1}{2}}(C\Omega + D) = j(\gamma, \Omega)$. Then there exists a map

$$s : Sp(2g, \mathbb{R}) \times Sp(2g, \mathbb{R}) \longrightarrow \{-1, 1\}$$

satisfying

$$j(\gamma_1\gamma_2, \Omega) = s(\gamma_1, \gamma_2) j(\gamma_1, \gamma_2\Omega) j(\gamma_2, \Omega).$$

Consider now the level k theta functions. For $m \in (\mathbb{Z}/k\mathbb{Z})^g$ these are defined by

$$\theta_m(z, \Omega) = \sum_{\ell \in m + k\mathbb{Z}^g} \exp\left(\frac{\pi\sqrt{-1}}{k} (\langle \ell, \Omega\ell \rangle + 2\langle \ell, z \rangle) \right) \tag{40}$$

or, equivalently, by

$$\theta_m(z, \Omega) = \theta(m/k, 0)(kz, k\Omega).$$

where $\theta(*,*)$ are the theta functions with rational characteristics (Mumford 1984) given by

$$\theta(a, b)(z, \Omega) = \sum_{\ell \in \mathbb{Z}^g} \exp\left(\frac{\pi\sqrt{-1}}{k} (\langle \ell + a, \Omega(\ell + a) \rangle + 2\langle \ell + a, z + b \rangle) \right) \tag{41}$$

for $a, b \in \mathbb{Q}^g$. Obviously $\theta(0, 0)$ is the usual theta function.

Let us denote by $R_8 \subset \mathbb{C}^*$ the group of 8^{th} roots of unity. Then R_8 becomes also a subgroup of the unitary group $U(n)$ acting by scalar multiplication. Consider also the theta vector of level k:

$$\Theta_k(z, \Omega) = (\theta_m(z, \Omega))_{m \in (\mathbb{Z}/k\mathbb{Z})^g}.$$

Proposition 15 (Funar 1993, 2000; Gocho 1992) – *The theta vector satisfies the following functional equation:*

$$\Theta_k(\gamma \cdot (z, \Omega)) = \zeta_\gamma \det(C\Omega + D)^{1/2} \exp(k\pi\sqrt{-1}\langle z, (C\Omega + D)^{-1} Cz \rangle) \rho_g(\gamma)(\Theta_k(z, \Omega)) \tag{42}$$

where

1. *γ belongs to the theta group $\Gamma(1, 2)$ if k is odd and to $\mathrm{Sp}(2g, \mathbb{Z})$ elsewhere.*
2. *$\zeta_\gamma \in R_8$ is the (fixed) multiplier system described above.*
3. *$\rho_g : \Gamma(1, 2) \longrightarrow U(\mathbb{C}^{(\mathbb{Z}/k\mathbb{Z})^g})$ is a group homomorphism. For even k the corresponding map $\rho_g : \mathrm{Sp}(2g, \mathbb{Z}) \longrightarrow U(\mathbb{C}^{(\mathbb{Z}/k\mathbb{Z})^g}))$ becomes a group homomorphism (denoted also by ρ_g when no confusion arises) when passing to the quotient $U(\mathbb{C}^{(\mathbb{Z}/k\mathbb{Z})^g})/R_8$.*
4. *ρ_g is determined by the points (1-3) above.*

Remark 16 – This result is stated also in Igusa 1972 for some modified theta functions but in less explicit form.

A.2 Linearizability of Weil representations for odd level k

The proof for the linearizability of the Weil representation associated to $\mathbb{Z}/k\mathbb{Z}$ for odd k was first given by A. Andler (see Andler and Ramanan 1996, Appendix A III) and then extended to other local rings in Cliff, McNeilly, and Szechtman 2000. Let $\eta : Sp(2g,\mathbb{Z}) \times Sp(2g,\mathbb{Z}) \to R_8 \subset U(1)$ be the cocycle determined by the Weil representation associated to $\mathbb{Z}/k\mathbb{Z}$. The image S^2 of $\begin{pmatrix} -1_g & 0 \\ 0 & -1_g \end{pmatrix}$ is the involution

$$S^2\theta_m = \theta_{-m},\ m \in (\mathbb{Z}/k\mathbb{Z})^g.$$

Thus $S^4 = 1$ and S^2 has eigenvalues $+1$ and -1. Moreover S^2 is central and hence the Weil representation splits according to the eigenspaces decomposition. Further, the determinant of each factor representation is a homogeneous function whose degree is the respective dimension of the factor. Therefore we could express, for each one of the two factors, η to the power the dimension of the respective factor as a determinant cocycle. The difference between the two factors' dimensions is the trace of S^2, namely 1 for odd k and 2^g for even k. This implies that η, for odd k, and η^{2^g}, for even k is a boundary cocycle. However $\eta^8 = 1$ and hence for even k and $g \geq 3$ this method could not give any non-trivial information about η.

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References

Andler, A. and S. Ramanan (1996). *Moduli of abelian varieties*. **1644**. Lect. Notes. Math. (cit. on pp. 122, 160).

Barge, J. and J. Lannes (2008). *Suites de Sturm, indice de Maslov et périodicité de Bott [Sturm sequences, Maslov index and Bott periodicity]*. Progress in Mathematics, Birkhäuser Verlag, Basel, (cit. on pp. 113, 115–117).

Bass, H., J. Milnor, and J.-P. Serre (1967). "Solution of the congruence subgroup problem for $\mathrm{SL}_n (n \geq 3)$ and $\mathrm{Sp}_{2n} (n \geq 2)$". *Inst. Hautes Études Sci. Publ. Math.* **33**, pp. 59–137 (cit. on pp. 111, 132–134, 136, 137, 158).

Benson, D. (2019). "Theta functions and a presentation of $2^{1+(2g+1)}Sp(2g,2)$". *J. Algebra* **527**, pp. 204–240 (cit. on p. 110).

Benson, D. et al. (2018). "Cohomology of symplectic groups and Meyer's signature theorem". *Algebr. Geom. Topol.* **18**, pp. 4069–4091 (cit. on pp. 110, 111, 123).

Beyl, R. (1986). "The Schur multiplicator of $SL(2,\mathbb{Z}/m\mathbb{Z})$ and the congruence subgroup property". *Math. Z.* **91**, pp. 23–42 (cit. on p. 110).

Borel, A. (1972). "Cohomologie réelle stable de groupes S-arithmétiques classiques". *C. R. Acad. Sci. Paris Sér. A-B* **274**, A1700–A1702 (cit. on p. 135).

Borel, A. (1974). "Stable real cohomology of arithmetic groups". *Ann. Sci. École Norm. Sup. (4)* **7**, pp. 235–272 (cit. on p. 135).

Borel, A. (1981). "Stable real cohomology of arithmetic groups II". In: *Manifolds and Lie groups, Papers in honor of Yozô Matsushima*. Vol. 14. Progress Math., Springer-Verlag, pp. 21–56 (cit. on p. 135).

Borel, A. and J.-P. Serre (1971). "Cohomologie à supports compacts des immeubles de Bruhat-Tits: applications à la cohomologie des groupes S-arithmétiques". *C. R. Acad. Sci. Paris* **272**, pp. 110–113 (cit. on pp. 111, 135).

Brownstein, A. and R. Lee (1992). "Cohomology of the symplectic group $\mathrm{Sp}_4(\mathbb{Z})$. I. The odd torsion case". *Trans. Amer. Math. Soc.* **334**, pp. 575–596 (cit. on p. 138).

Brownstein, A. and R. Lee (1994). "Cohomology of the symplectic group $\mathrm{Sp}(4,\mathbb{Z})$. II. Computations at the prime 2". *Michigan Math. J.* **41**, pp. 181–208 (cit. on p. 138).

Cliff, G., D. McNeilly, and F. Szechtman (2000). "Weil representations of symplectic groups over rings". *J. London Math. Soc. (2)* **62**, pp. 423–436 (cit. on p. 160).

Deligne, P. (1978). "Extensions centrales non résiduellement finies de groupes arithmétiques". *C. R. Acad. Sci. Paris Sér. A-B* **287**, A203–A208 (cit. on pp. 110–112, 133, 134).

Dennis, R. K. and M. R. Stein (1973). "The functor K: a survey of computations and problems". In: *Algebraic K-Theory II*. Vol. 342. Lecture Notes in Math. 342, Springer-Verlag, pp. 243–380 (cit. on p. 110).

Dennis, R. K. and M. R. Stein (1975). "K_2 of discrete valuation rings". *Advances Math.* **18**, pp. 182–238 (cit. on pp. 117, 119, 138).

Freitag, J. (1987). *Theta functions*. **1487**. Lect. Notes in Math. (cit. on p. 158).

Funar, L. (1993). "Représentations du groupe symplectique et variétés de dimension 3". *C. R. Acad. Sci. Paris Sér. I Math.* **316**, pp. 1067–1072 (cit. on pp. 120, 159).

Funar, L. (2000). "Some abelian invariants of 3-manifolds". *Rev. Roumaine Math. Pures Appl.* **45**, pp. 825–861 (cit. on pp. 120, 121, 159).

Funar, L. (2011). "Two questions on mapping class groups". *Proc. Amer. Math. Soc.* **139**, pp. 375–382 (cit. on pp. 110, 112, 138, 141, 144).

Funar, L. (2013). "Zariski density and finite quotients of mapping class groups". *Int. Math. Res. Not.*, pp. 2078–2096 (cit. on pp. 112, 142).

Funar, L. and R. Kashaev (2014). "Centrally extended mapping class groups from quantum Teichmüller theory". *Advances Math.* **252**, pp. 260–291 (cit. on p. 123).

Funar, L. and W. Pitsch (2020). "The Schur multiplier of the finite symplectic groups". *Bull. Soc. Math. France* **148**, pp. 515–527 (cit. on pp. 110, 111, 115, 117, 132, 134, 138, 140, 149).

Gervais, S. (1996). "Presentation and central extensions of mapping class groups". *Trans. Amer. Math. Soc.* **348**, pp. 3097–3132 (cit. on pp. 114, 123–125, 139).

Gilmer, G. and G. Masbaum (2007). "Integral lattices in TQFT". *Ann. Sci. École Norm. Sup. (4)* **40**, pp. 815–844 (cit. on p. 141).

Gocho, T. (1992). "The topological invariant of three-manifolds based on the U(1) gauge theory". *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **39**, pp. 169–184 (cit. on pp. 120, 121, 159).

Grossman, E. (1974–1975). "On the residual finiteness of certain mapping class groups". *J. London Math. Soc. (2)* **9**, pp. 160–164 (cit. on p. 143).

Grunewald, F., A. Jaikin-Zapirain, and P. Zalesskii (2008). "Cohomological goodness and the profinite completion of Bianchi groups". *Duke Math. J.* **144**, pp. 53–72 (cit. on pp. 112, 147).

Gurevich, S. and R. Hadani (2012). "The Weil representation in characteristic two". *Advances Math.* **230**, pp. 894–926 (cit. on pp. 118, 120).

Gurevich, S., R. Hadani, and R. Howe (2010). "Quadratic reciprocity and the sign of the Gauss sum via the finite Weil representation". *Int. Math. Res. Not.*, pp. 3729–3745 (cit. on p. 120).

Hahn, A. J. and O. T. O'Meara (1989). *The classical groups and K-theory*. **291**. Grund. math. Wissenschaften, Springer Verlag (cit. on pp. 116, 117, 133).

Hannay, J. and M. Berry (1980). "Quantization of linear maps on a torus-Fresnel diffraction by a periodic grating". *Phys. D* **1**, pp. 267–290 (cit. on p. 120).

Hano, J.-I. (1969). "On theta functions and Weil's generalized Poisson summation formula". *Trans. Amer. Math. Soc.* **141**, pp. 195–210 (cit. on p. 120).

Hardy, G. and E. Wright (1979). *An introduction to the theory of numbers*. Oxford at the Claredon Press (cit. on p. 158).

Harer, J. (1983). "The second homology group of the mapping class group of an orientable surface". *Invent. Math.* **72**, pp. 221–239 (cit. on p. 122).

Humphries, S. P. (1979). "Generators for the mapping class group". In: *Topology of low-dimensional manifolds, Proc. 2nd Sussex Conf. 1977*. Vol. 722. Lect. Notes Math., pp. 44–47 (cit. on p. 156).

Igusa, J.-I. (1964). "On the Graded Ring of Theta-Constants". *Amer. J. Math.* **86**, pp. 219–246 (cit. on pp. 120, 121).

Igusa, J.-I. (1972). *Theta functions*. Springer Verlag (cit. on pp. 121, 159).

Kaiblinger, N. and M. Neuhauser (2009). "Metaplectic operators for finite abelian groups and $\mathbb{R}^d$". *Indag. Math. (N.S.)* **20**, pp. 233–246 (cit. on p. 120).

Kloosterman, H. D. (1946). "The behaviour of general theta functions under the modular group and the characters of binary modular congruence groups. I." *Ann. of Math. (2)* **47**, pp. 317–375 (cit. on p. 120).

Korkmaz, M. and A. I. Stipsicz (2003). "The second homology groups of mapping class groups of oriented surfaces". *Math. Proc. Cambridge Philos. Soc.* **134**, pp. 479–489 (cit. on pp. 122, 139).

Lam, T. (2004). *Introduction to quadratic forms over fields*. **67**. Graduate Stud. Math., Amer. Math. Soc. (cit. on p. 120).

Lang, S. (1970). *Algebraic number theory*. Addison-Wesley Comp. (cit. on p. 129).

Malcev, A. I. (1940). "On the faithful representation of infinite groups by matrices". *Mat. Sbornik* **8 (50)**, pp. 405–422 (cit. on p. 140).

Masbaum, G. (2008). "On representations of mapping class groups in integral TQFT". In: *Oberwolfach Reports*. Vol. 5, pp. 1157–1232 (cit. on p. 142).

Masbaum, G. and A. Reid (2012). "All finite groups are involved in the Mapping Class Group". *Geom. Topol.* **16**, pp. 1393–1411 (cit. on pp. 112, 142).

Masbaum, G. and J. D. Roberts (1995). "On central extensions of mapping class groups". *Math. Ann.* **302**, pp. 131–150 (cit. on pp. 112, 139).

Matsumoto, H. (1969). "Sur les groupes arithmétiques des groupes semi-simples déployés". *Ann. Sci. École Norm. Sup. (4)* **2**, pp. 1–62 (cit. on p. 116).

Mennicke, J. L. (1965a). "Finite factor groups of the unimodular group". *Ann. of Math. (2)* **81**, pp. 31–37 (cit. on pp. 137, 158).

Mennicke, J. L. (1965b). "Zur Theorie der Siegelschen Modulgruppe". *Math. Ann.* **159**, pp. 115–129 (cit. on pp. 137, 158).

Milnor, J. (1971). *Introduction to algebraic K-theory*. **72**. Annals Math. Studies, Princeton Univ. Press (cit. on pp. 116, 117).

Milnor, J. and D. Husemoller (1973). *Symmetric bilinear forms*. **73**. Ergeb. Math. Grenz., Springer Verlag (cit. on pp. 117–119).

Moore, C. (1968). "Group extensions of p-adic and adelic linear groups". *Inst. Hautes Etudes Sci. Publ. Math.* **35**, pp. 157–222 (cit. on pp. 111, 133, 135).

Mumford, D. (1984). *Tata lectures on theta I, II*. Progress in Math. (cit. on pp. 157, 159).

Newman, M. and J. R. Smart (1964). "Symplectic modulary groups". *Acta Arith.* **9**, pp. 83–89 (cit. on p. 114).

Pitsch, W. (1999). "Un calcul élémentaire de $H_2(\mathcal{M}_{g,1}, Z)$ pour $g \geq 4$". *C. R. Acad. Sci. Paris Sér. I Math.* **329**, pp. 667–670 (cit. on pp. 122, 139).

Putman, A. (2012). "The Picard group of the moduli space of curves with level structures". *Duke Math. J.* **161**, pp. 623–674 (cit. on pp. 110, 113–115, 140, 149).

Raghunathan, M. S. (1984). "Torsion in cocompact lattices in coverings of Spin(2, n)". *Math. Ann.* **266**, pp. 403–419 (cit. on p. 112).

Ranga Rao, R. (1993). "On some explicit formulas in the theory of Weil representation". *Pacific J. Math.* **157**, pp. 335–371 (cit. on p. 120).

Sakasai, T. (2012). "Lagrangian mapping class groups from group homological point of view". *Algebr. Geom. Topol.* **12**, pp. 267–291 (cit. on p. 122).

Sato, M. (2010). "The abelianization of the level d mapping class group". *J. Topology* **3**, pp. 847–882 (cit. on p. 148).

Serre, J.-P. (1994). *Cohomologie galoisienne, 5-th ed.* Lect. Notes Math., Springer-Verlag (cit. on pp. 112, 145).

Shimura, G. (1966). "Moduli and fibre systems of abelian varieties". *Ann. of Math.* **83**, pp. 294–338 (cit. on p. 120).

Stein, M. R. (1973). "Surjective stability in dimension 0 for K_2 and related functors". *Trans. Amer. Math. Soc.* **178**, pp. 165–191 (cit. on pp. 110, 114, 115, 117, 138).

Stein, M. R. (1975). "The Schur multipliers of $Sp_6(\mathbb{Z})$, $Spin_8(\mathbb{Z})$, $Spin_7(\mathbb{Z})$ and $F_4(\mathbb{Z})$". *Math. Ann.* **215**, pp. 165–172 (cit. on pp. 111, 122, 138).

Stein, M. R. (1978). "Stability theorems for K_1, K_2 and related functors modeled on Chevalley groups". *Japan. J. Math. (N.S.)* **4**, pp. 77–108 (cit. on pp. 113, 114, 136, 138).

Suslin, A. (1987). "Torsion in K_2 of fields". *K-theory* **1**, pp. 5–29 (cit. on p. 120).

Turaev, V. G. (1994). *Quantum invariants of knots and 3-manifolds*. **18**. de Gruyter Studies in Mathematics, Walter de Gruyter & Co. (cit. on p. 120).

Weil, A. (1964). "Sur certains groupes d'opérateurs unitaires". *Acta Math.* **111**, pp. 143–211 (cit. on pp. 120, 122).

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