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Banach spaces with the Blum-Hanson Property

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Abstract

We are interested in a sufficient condition given in an article by P. Lefèvre, É. Matheron and A. Primot[2] to obtain the Blum-Hanson property and we then partially answer two questions asked in this same article on other possible conditions to have this property for a separable Banach space.

Keywords: Blum-Hanson, property (m_p), property sub-(m_p), AUS norm, Banach space, property (M^*).

MSC: 46B03, 46B06, 46B10, 46B20.

1 Introduction

These notes are essentially inspired by the article cited in the abstract[3] in which sufficient new conditions to justify that a Banach space has the Blum-Hanson property were obtained.
We recall that, for a (real or complex) Banach space X, and a contraction T on X (T is a bounded operator on X with $\|T\| \leq 1$), we say that T has the *Blum-Hanson property* if, for $x, y \in X$ such that $T^n x$ weakly converges to $y \in X$ when n tends to infinity, the mean

$$\frac{1}{N}\sum_{k=1}^{N} T^{n_k} x$$

tends toward y in norm for any increasing sequence of integers $(n_k)_{k\geq 1}$.
The space X is said to have the *Blum-Hanson property* if every contraction on X has the Blum-Hanson property.
Note, to understand the interest in this property and its historical aspect, that, when X is a Hilbert space and the linear operator T is a contraction, for all $x \in X$ such that $T^n x \xrightarrow{w} 0$, the arithmetic mean

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[2] Lefèvre, Matheron, and Primot, 2016, “Smoothness, asymptotic smoothness and the Blum-Hanson property”.

[3] Ibid.

$$\frac{1}{N}\sum_{k=1}^{N} T^{n_k}x$$

is norm convergent to 0 for any increasing sequence of integers $(n_k)_{k\geq 1}$. This result was first proved by J.R. Blum and D.L. Hanson[4] for isometries induced by measure-conserving transformations, then in two other papers[5] for arbitrary contractions. The most notable spaces having the Blum-Hanson property are the Hilbert spaces and the ℓ_p spaces for $1 \leq p < \infty$.
Note that this property is not preserved under renormings[6]. This raises the following question: "Which Banach spaces can be renormed to have the Blum-Hanson property ?", already asked before[7]. This question motivated the writing of this article.
To understand the main results of this work, we give first the following definition of an asymptotically uniformly smooth norm.

Definition 1 – Consider a Banach space $(X, \|\cdot\|)$. By following the definitions due to V. Milman[8] and the notations of two papers cited below[9], for $t \in [0,\infty)$, $x \in S_X$ and Y a closed vector subspace of X, we define the modulus of asymptotic uniform smoothness, $\overline{\rho}_X(t)$:

$$\overline{\rho}_X(t,x,Y) = \sup_{y \in S_Y}(\|x+ty\| - 1).$$

Then

$$\overline{\rho}_X(t,x) = \inf_{Y \in \mathrm{cof}(X)} \overline{\rho}_X(t,x,Y) \qquad \text{and} \qquad \overline{\rho}_X(t) = \sup_{x \in S_X} \overline{\rho}_X(t,x).$$

The norm $\|\cdot\|$ is said to be *asymptotically uniformly smooth* (in short AUS) if

$$\lim_{t\to 0} \frac{\overline{\rho}_X(t)}{t} = 0.$$

Now, we can give the main property of this paper which partially answers the previous question:

[4] Blum and Hanson, 1960, "On the mean ergodic theorem for subsequences".
[5] Akcoglu, Huneke, and Rost, 1974, "A counterexample to Blum-Hanson theorem in general spaces"; Jones and Kuftinec, 1971, "A note on the Blum-Hanson theorem".
[6] Müller and Tomilov, 2007, "Quasi-similarity of power-bounded operators and Blum-Hanson property".
[7] Lefèvre, Matheron, and Primot, 2016, "Smoothness, asymptotic smoothness and the Blum-Hanson property".
[8] Milman, 1971, "Geometric theory of Banach spaces. II. Geometry of the unit ball (Russian)".
[9] Johnson et al., 2002, "Almost Fréchet differentiability of Lipschitz mappings between infinite-dimensional Banach spaces";
Lancien and Raja, 2018, "Asymptotic and Coarse Lipschitz structures of quasi-reflexive Banach spaces".

Theorem 1 – *Let Y be a separable Banach space whose norm is AUS. Then Y has an equivalent norm with the Blum-Hanson property.*

Remark 1 – A Banach space Y which has an AUS norm is an Asplund space. Consequently, Y is separable if and only if its dual is separable.

2 Banach space with property (m_p)

N. Kalton and D. Werner introduced the property (m_p)[10]:

Definition 2 – A Banach space X has property (m_p), where $1 \leq p \leq \infty$ if, for any $x \in X$ and every weakly null sequence $(x_n) \subset X$, it holds that:

$$\limsup_{n\to\infty} \|x + x_n\| = (\|x\|^p + \limsup \|x_n\|^p)^{\frac{1}{p}}.$$

For $p = \infty$, the right-hand side is of course to be interpreted as $\max(\|x\|, \limsup \|x_n\|)$.

Example 1 – ℓ_p has property (m_p), c_0 has property m_∞.

Remark 2 – We shall say that X has property *sub*-(m_p) if, for any $x \in X$ and every weakly null sequence $(x_n) \subset X$, it holds that:

$$\limsup_{n\to\infty} \|x + x_n\| \leq (\|x\|^p + \limsup \|x_n\|^p)^{\frac{1}{p}}.$$

As before, for $p = \infty$, the right-hand side is of course to be interpreted as $\max(\|x\|, \limsup \|x_n\|)$.

P. Lefèvre, É. Matheron and A. Primot[11] obtained the following property which was a corollary of one of the main theorems of their paper. It is this property which allowed us in particular to obtain Theorem 1.

Proposition 1 – [12] *For any $p \in (1, \infty]$, property sub-(m_p) implies Blum-Hanson property.*

Example 2 – [13]
We recall the definition of the James space J_p. This is the real Banach space of all sequences $x = (x(n))_{n\in\mathbb{N}}$ of real numbers satisfying $\lim_{n\to\infty} x(n) = 0$, endowed with the norm

[10] Kalton and Werner, 1995, “Property (M), M-ideals, and almost isometric structure of Banach spaces”.

[11] Lefèvre, Matheron, and Primot, 2016, “Smoothness, asymptotic smoothness and the Blum-Hanson property”.

[12] Ibid.

$$\|x\|_{J_p} = \sup \left\{ \left(\sum_{i=1}^{n-1} |x(p_{i+1}) - x(p_i)|^p \right)^{\frac{1}{p}} : 1 \leq p_1 < p_2 < \cdots < p_n \right\}.$$

This is a quasi-reflexive Banach space which is isomorphic to its bidual.
Historically, R.C. James has focused exclusively on $J = J_2$[14], and I.S. Edelstein and B.S. Mityagin[15] are apparently the first to have observed that we could generalize the definition to $p \geq 1$ arbitrary and to have observed the quasi-reflexivity of J_p for any $p > 1$.
There exists an equivalent norm $|\cdot|$ on J_p[16] (Corollary 2.4 of the cited paper for the proof) such that, for all $x, y \in J_p$ verifying $\max\{i \in \mathbb{N} : x(i) \neq 0\} < \min\{i \in \mathbb{N} : y(i) \neq 0\}$, it holds that

$$|x + y|^p \leq |x|^p + |y|^p.$$

Thus, $\tilde{J_p} := (J_p, |\cdot|)$ has the sub-(m_p) property, and therefore the Blum-Hanson property.

We now introduce a notion that is essentially dual to sub-(m_p).

Definition 3 – Let X be a separable Banach space and $q \in (1, \infty)$. We say that X^* has property sup-$(m_q)^*$ if, for any $x^* \in X^*$ and any weak*-null sequence (x_n^*) in X^*, we have:

$$\liminf_{n\to\infty} \|x^* + x_n^*\|^q \geq \|x^*\|^q + \liminf_{n\to\infty} \|x_n^*\|^q.$$

The following is an easy adaptation of the proof of Proposition 2.6 from an article by G. Godefroy, N.J. Kalton and G. Lancien[17].

Proposition 2 – *Let X be a separable Banach space. Let $p \in (1, \infty)$ and q be its conjugate exponent. Assume that X^* has property sup-$(m_q)^*$, then X has property sub-(m_p).*

Proof. Let $x \in X$ and (x_n) be a weakly null sequence in X and denote $s = \limsup_n \|x_n\|$. Pick $y_n^* \in X^*$ so that $\|y_n^*\| = 1$ and $y_n^*(x + x_n) = \|x + x_n\|$. After extracting a subsequence, we may assume that (y_n^*) is weak* converging to $x^* \in B_{X^*}$. Denote $x_n^* = y_n^* - x^*$ and assume also, as we may, that $\lim_n \|x_n^*\| = t$. Since X^* has sup-$(m_q)^*$, we have that $\|x^*\|^q + t^q \leq 1$. Therefore

(Cont. next page)

$$\limsup_n \|x + x_n\| = \limsup_n (x^* + x_n^*)(x + x_n) \leq x^*(x) + st$$

[13] García-Lirola and Petitjean, 2021, "On the weak maximizing properties"; Netillard, 2018, "Coarse Lipschitz embeddings of James spaces".
[14] James, 1950, "Bases and reflexivity of Banach spaces".
[15] Edelstein and Mityagin, 1970, "Homotopy type of linear groups of two classes of Banach spaces".
[16] Netillard, 2018, "Coarse Lipschitz embeddings of James spaces".
[17] Godefroy, Kalton, and Lancien, 2001, "Szlenk indices and uniform homeomorphisms".

$$\leq (\|x^*\|^q + t^q)^{1/q}(\|x\|^p + s^p)^{1/p} \leq (\|x\|^p + s^p)^{1/p}. \qquad \square$$

This concludes our proof.

3 Main results

We give some definitions that will be used later.

Definition 4 – Given an FDD (E_n), (x_n) is said to be a block sequence with respect to (H_i) if there exists a sequence of integers $0 = m_1 < m_2 < \cdots$ such that $x_n \in \bigoplus_{j=m_n}^{m_{n+1}-1} E_j$.

Definition 5 – Let $1 \leq q \leq p \leq \infty$ and $C < \infty$. A (finite or infinite) FDD (E_i) for a Banach space Z is said to satisfy $C-(p,q)$ estimates if for all $n \in \mathbb{N}$ and block sequences $(x_i)_{i=1}^n$ with respect to (E_i):

$$C^{-1}\left(\sum_1^n \|x_i\|^p\right)^{\frac{1}{p}} \leq \left\|\sum_1^n x_i\right\| \leq C\left(\sum_1^n \|x_i\|^q\right)^{\frac{1}{q}}.$$

For the central theorem for this work, we now recall the definition of the Szlenk index.

Definition 6 – Let X be a Banach space and K be a weak*-compact subset of X^*. For $\epsilon > 0$, let $\mathcal{V}$ be the set of all weak*-open subsets of K such that the norm diameter (for the norm of X^*) of V is less than ϵ, and

$$s_\epsilon K = K \setminus \bigcup \{V : V \in \mathcal{V}\}.$$

As a remark, $s_\epsilon^\alpha B_{X^*}$ is defined inductively for any ordinal α by

$$s_\epsilon^{\alpha+1} B_{X^*} = s_\epsilon(s_\epsilon^\alpha B_{X^*})$$

and

$$s_\epsilon^\alpha B_{X^*} = \bigcap_{\beta<\alpha} s_\epsilon^\beta B_{X^*} \text{ if } \alpha \text{ is a limit ordinal.}$$

We define $Sz(X,\epsilon)$ to be the least ordinal α so that $s_\epsilon^\alpha B_{X^*} = \emptyset$ if such an ordinal exists. Otherwise we write $Sz(X,\epsilon) = \infty$ by convention.
We will then denote $Sz(X)$ the Szlenk index of X, defined by

$$Sz(X) = \sup_{\epsilon>0} Sz(X,\epsilon).$$

Remark 3 – For a detailed report about the Szlenk index, one can refer to the article by G. Lancien quoted below[18].
Note that the Szlenk index was introduced by W. Szlenk[19] to show that there is no universal reflexive space for the class of separable reflexive spaces.

The main ingredient of our argument is the following result, which is deduced from a work of H. Knaust, E. Odell and T. Schlumprecht[20] (Corollary 5.3) and is already cited in an article by G. Lancien[21] (in the proof of Theorem 4.15). However, in this last paper, we do not find the detailed proof of this property, that we include now.

Proposition 3 – *Let Y be a separable Banach space such that $Sz(Y) \leq \omega$, where ω denote the first infinite ordinal.*
Then Y can be renormed so as to have property sub-(m_q) for some value $q \in (1, \infty)$.

Proof. According to Corollary 5.3 of the article by H. Knaust, E. Odell and T. Schlumprecht cited above[22], $\mathrm{Sz}(Y) \leq \omega$ implies that there exists a Banach space Z with a boundedly complete FDD (E_i) (in particular Z is isometric to a dual space X^*) with the following properties.

1. There exists $p \in (1, \infty)$ such that (E_i) satisfies $1-(p, 1)$ estimates.
2. Y^* is isomorphic (norm and weak*) to a weak*-closed subspace F of $Z = X^*$.

Let us denote $S : Y^* \to F$ this isomorphism. Then, there exists a subspace G of X such that $G^{\perp} = F$ and S is the adjoint of an isomorphism T from X/G onto Y. Let now q be the conjugate exponent of p. It is thus enough to prove that $E = X/G$ has sub-(m_q). Since $X^* = Z$ satisfies $1-(p, 1)$ estimates with respect to the boundedly complete FDD (E_i), it is immediate that X^* has sup-$(m_p)^*$. Now this property passes clearly to its weak*-closed subspace F and E^* has sup-$(m_p)^*$. Finally, we deduce from Proposition 2 that E has sub–(m_q). This finishes our proof. □

Thanks to this Proposition, we obtain the Theorem 1.

Proof. The proof is immediate by applying the Proposition 1. □

We will now talk about property (M^*).
It was studied by N.J. Kalton and D. Werner[23] :

Definition 7 – A Banach space X has property (M^*) if, for $u^*, v^* \in S_{X^*}$ and $(x_n^*) \subseteq X^*$ a weak* null sequence, it holds that

[18] Lancien, 2006, "A survey on the Szlenk index and some of its applications".
[19] Szlenk, 1968, "The non existence of a separable reflexive space universal for all reflexive Banach spaces".
[20] Knaust, Odell, and Schlumprecht, 1999, "On asymptotic structure, the Szlenk index and UKK properties in Banach spaces".
[21] Lancien, 2012, "A short course on non linear geometry of Banach spaces".
[22] Knaust, Odell, and Schlumprecht, 1999, "On asymptotic structure, the Szlenk index and UKK properties in Banach spaces".
[23] Kalton and Werner, 1995, "Property (M), M-ideals, and almost isometric structure of Banach spaces".

$$\limsup_n \|u^* + x_n^*\| = \limsup_n \|v^* + x_n^*\|.$$

Remark 4 – It has been shown[24] that, if X is a separable Banach space having property (M^*), then its dual is separable.

The following Proposition follows from an article by S. Dutta and A.Godard[25] (Proposition 2.2 of this article).

Proposition 4 – *Let X be a separable Banach space with property (M^*). Then X is asymptotically uniformly smooth for a norm $\|\cdot\|_M$.*

Corollary 1 – *Let X be a separable Banach space with property (M^*). Then X has an equivalent norm with the Blum-Hanson property.*

Proof. It follows from the Proposition 4 and from the Theorem 1. □

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