

The explicit bijection between fully commutative elements and non-crossing diagrams

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Abstract

Let $W^c(A_n)$ be the set of fully commutative elements in the Coxeter group of type A_n . We determine explicitly the unique bijection between $W^c(A_n)$ and the set of non-crossing diagrams of $n+1$ strands that is compatible with the product of monomials in the Temperley-Lieb algebra of type A_n on one side, and the concatenation of diagrams on the other, and maps Coxeter generators to primitive diagrams in a fixed way. We describe algorithms implementing this bijection and its inverse. We draw consequences, among which the determination of the left and right descent sets or the identification of saturated elements directly from the corresponding diagram.

Keywords: Fully commutative element; Diagram algebra; Catalan number; Coxeter group.

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1 Introduction

We focus in this work on two Catalan phenomena, i.e., counted by the Catalan number C_{n+1} , for $n \geq 0$:

- Fully commutative elements in the Coxeter group of type A_n ;
- Non-crossing diagrams with $n+1$ strands.

Precisely, our aim is to produce an explicit description of the only bijection (and its inverse) between these two objects for all n , that is compatible with the product of monomials in the Temperley-Lieb algebra on the one hand and the product of diagrams by concatenation on the other hand, and maps the Coxeter generators σ_i , $1 \leq i \leq n$, to the primitive diagrams E_i , $1 \leq i \leq n$, of Figure 4, in this order. We

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denote it by $\mathcal{D}$.

Let us first recall the framework. Non-crossing diagrams are long-studied combinatorial objects which play an important part both in algebra and in low-dimensional topology, knot theory in particular. We refer to subsection 2.3 for a definition, and to Example 1 below for an immediate visual help. These classic diagrams form a basis for the well-known Temperley-Lieb algebra of type A_n when viewed as a diagram algebra, this idea dates back to Kauffman (1987).

On the other hand, let $W^c(A_n)$ be the set of *fully commutative* elements, shortened to FC elements, in $W(A_n)$, the Coxeter group of type A_n (see subsection 2.1). The Temperley-Lieb algebra of type A_n has –among many– a *monomial* basis, indexed by FC elements: $\{e_w; w \in W^c(A_n)\}$ (see subsection 2.2). The FC monomial e_w is *well defined* by $e_w = e_{s_1} \cdots e_{s_k}$ if $w = s_1 \cdots s_k$, where $s_1, \dots, s_k$ belong to the set of generators $\{\sigma_1, \dots, \sigma_n\}$ of $W(A_n)$ and $k = \ell(w)$ is the Coxeter length of w , i.e. e_w is *independent* of the choice of the reduced expression for w . Then: $e_w = e_{s_1} \cdots e_{s_k}$ corresponds to a unique diagram D_w coming from the concatenation of $\ell(w)$ primitive diagrams (see Figure 4): D_{s_1} on the top of D_{s_2} ... on the top of D_{s_k} . Implicit here is the condition that the generators $e_{\sigma_1}, \dots, e_{\sigma_n}$ are mapped respectively to the primitive diagrams $D_{\sigma_1}, \dots, D_{\sigma_n}$, where $D_{\sigma_i} = E_i$ is drawn in Figure 4.

We give in this work the explicit description of this bijection $\mathcal{D}: w \mapsto D_w$ between $W^c(A_n)$ and non-crossing diagrams with $n+1$ strands, compatible by construction with the multiplication of monomials and with diagram concatenation, and mapping σ_i to E_i (as detailed in subsection 2.4 and (2)), this is Theorem 2. We emphasize the fact that the theoretical existence of such a bijection does not make it any easier to compute, whereas an explicit description as given below may have numerous uses.

We pause for some comments related to the Catalan number and the many objects that it counts. Our focus here is on $W^c(A_n)$ and non-crossing diagrams on $n+1$ strands, yet we mention briefly two other phenomena, in general better known and considered two of the five fundamental interpretations of C_{n+1} in Stanley (2015):

- Dyck paths of length $2(n+1)$;
- Ballot sequences of length $2(n+1)$.

We start with the well-known canonical reduced expression of an FC element, say w , in $W^c(A_n)$, described in Theorem 1. For now it is sufficient to know that it is given by p pairs of positive integers (i_t, j_t) for $1 \leq t \leq p$, called *standard p -pairs*, such that:

$$i_t \leq j_t, \text{ and } n \geq j_1 > \cdots > j_p \geq 1, \text{ and } n \geq i_1 > \cdots > i_p \geq 1.$$

1. Introduction

In Jones (1983) the late V. Jones has determined a bijection between Dyck paths and standard p -pairs (mentioned as an observation of H. Wilf), call it DF , the bijection sending a path with p peaks having (j_t, i_t) as coordinates, to the p -pairs (i_t, j_t) , thus to the canonical form in (1), since those pairs are standard. Hence, he showed that the Catalan numbers count FC elements via DF . (Actually, in Al Harbat (2024) we show that Catalan numbers count those elements "by definition", i.e. without the help of Dyck paths, and we study the consequences of such interpretation.) We proceed, omitting n for it is fixed.

Now ballot sequences are in bijection with Dyck paths, call it BD , given by sending every $(+)$ to a vertical step and every $(-)$ to a horizontal step. Furthermore, there is a well-known bijection between non-crossing diagrams and ballot sequences, call it NB , defined by orienting the strands into arrows and sending tails to $(-)$ and heads to $(+)$. So by composing $NB.BD.DF$ we get a bijection NF from diagrams to FC elements. This bijection *is not compatible* with monomial multiplication and diagram concatenation: see Al Harbat (2024) for a counter example showing that neither NF nor its variations admit the wanted compatibility.

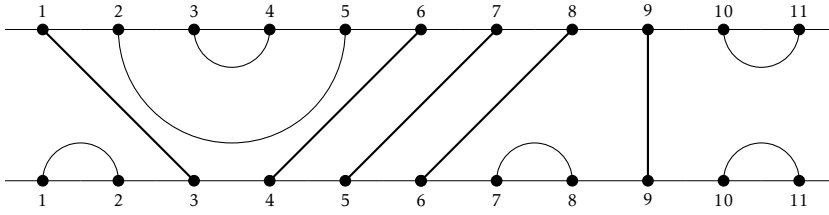
Furthermore, this bijection NF does not respect the natural injections of C_{n+1} -counted into C_{n+2} -counted objects — for diagrams, it consists in adding an $(n+2)$ -th strand, vertical, on the right; for FC elements it is in fact the parabolic inclusion $W^c(A_n) \rightarrow W^c(A_{n+1})$ (a strong advantage of the setting of FC elements over most other settings for the Catalan number) — whereas "the bijection" $\mathcal{D}$ does.

The existence and uniqueness of $\mathcal{D}$ (which yields from an algebra isomorphism defined implicitly in Kauffman (1987)) are explained in section 2.4 below, where we give the main definitions and fix some terminology. In section 3 we define the *main components* of a diagram and state the essential Proposition 1 describing them for $\mathcal{D}(w)$, for $w \in W^c(A_n)$. In section 4 we prove this Proposition with a series of Lemmas. In subsection 5.1 we state Theorem 2 describing the bijection $\mathcal{D}$. We show in particular that *a diagram D is uniquely determined by the set $\mathcal{I}(D)$ of starting points on the top line and the set $\mathcal{J}(D)$ of end points minus one on the bottom line of strictly right-bound strands, and that these two sets make up p -pairs as defined above* (Proposition 3).

Example 1 – In this example, we have $\mathcal{I}(D_0) = \{1, 2, 3, 10\}$ and $\mathcal{J}(D_0) = \{1, 2, 7, 10\}$. The associated FC element by $\mathcal{D}^{-1}$ is the one with p -pairs $\{(10, 10), (3, 7), (2, 2), (1, 1)\}$, namely

$$\mathcal{D}^{-1}(D_0) = \sigma_{10} \sigma_3 \sigma_4 \sigma_5 \sigma_6 \sigma_7 \sigma_2 \sigma_1.$$

This bijection yields two algorithms. One in subsection 5.2 receives any diagram as an input and gives directly the FC element in its canonical form, as easily as in our example above. The other in subsection 5.3 permits to "draw" into a diagram any FC element directly, without any concatenation or multiplication, after putting


 Figure 1 – Example: diagram D_0

it in its canonical expression. Concerning the first, as far as we know the fastest algorithm previously known is one that engages at least standard tableaux and walks as intermediate objects, see for details Lobos, Plaza, and Ryom-Hansen (2021). While concerning the second algorithm, we do not know of another one, other than concatenating primitive diagrams, i.e. concatenating after putting each generating diagram on the top of the next.

We finish in section 6 by proposing some applications and questions. We call attention to Proposition 4 that *expresses the left and right descent sets of a FC element w in terms of the corresponding diagram*, in a very simple way: those sets correspond to minimal cups, left to upper cups and right to bottom ones. In the example above we read on the diagram $\mathcal{L}(\mathcal{D}^{-1}(D_0)) = \{\sigma_3, \sigma_{10}\}$ and $\mathcal{R}(\mathcal{D}^{-1}(D_0)) = \{\sigma_1, \sigma_7, \sigma_{10}\}$. Some consequences on cellularity properties are sure to be expected.

We finish this introduction by pointing out that this work is followed by a second one with C. Blondel in which we deal with the type B FC elements, hence giving the suitable bijection between the "positive" FC elements of type B and blobbed diagrams, and some generalization with the symplectic blob algebra (see Al Harbat, González, and Plaza (2021) for the notion of positivity). Obviously the latter bijection is forced to be an extension of $\mathcal{D}$.

The author believes the first algorithm (read a diagram into a FC monomial) is basic, in the sense that it is an answer to a natural question that emerges immediately when looking at the isomorphism between the first Temperley-Lieb algebra and the first diagram algebra, hence a substantial result that he avoided for a long time, thus he would like to thank L. Poulain d'Andecy for forcing him back into this track by asking this natural question in the author's seminar talk in Reims. The author would like to thank C. Blondel for her careful reading, important comments and suggestions. Last, he would like to thank the anonymous referee for their helpful remarks, in particular concerning the reformulation of the main result.

2 Definitions

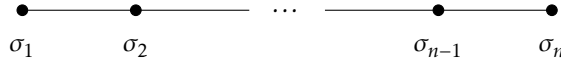
2.1 A canonical form for FC elements

Let (W, S) be a Coxeter system with associated Coxeter graph Γ . For s, t in S we let m_{st} be the order of st in W . Let $w \in W$. We denote by $\ell(w)$ the length of w with respect to S . We call *support of w* and denote by $\text{Supp}(w)$ the subset of S consisting of all generators appearing in a (any) reduced expression of w . We define $\mathcal{L}(w)$ to be the set of $s \in S$ such that $\ell(sw) < \ell(w)$, this is equivalent to saying that s appears at the left edge of some reduced expression of w . We define $\mathcal{R}(w)$ similarly on the right.

We know for a fact that every two reduced expressions of w can be connected via a sequence of braid relations². Among these relations there are *commutation relations*: those that correspond to generators t and s with $m_{st} = 2$.

Definition 1 – Elements for which we can pass from any reduced expression to any other only by applying commutation relations are called *fully commutative elements* (FC elements as mentioned before). We denote by W^c the set of FC elements in W .

Consider the A -type Coxeter group with n generators $W(A_n)$ (n being a positive integer, and say $W(A_0) = 1$), with the following Coxeter-Dynkin graph



We let:

$$[i, j] = \sigma_i \sigma_{i+1} \dots \sigma_j \text{ for } 1 \leq i \leq j \leq n.$$

Considering right classes of $W(A_{n-1})$ in $W(A_n)$ leads to a well-known canonical reduced expression for elements of $W(A_n)$, from which Stembridge has obtained a canonical reduced expression for elements of $W^c(A_n) = A_n^c$:

Theorem 1 – Stembridge (1997, Corollary 5.8) Let n be a positive integer, then A_n^c is the set of elements of the form (reduced expressions):

$$[i_1, j_1][i_2, j_2] \dots [i_p, j_p], \text{ with } 0 \leq p \leq n \text{ and } \begin{cases} n \geq j_1 > \dots > j_p \geq 1, \\ n \geq i_1 > \dots > i_p \geq 1, \\ j_t \geq i_t \text{ for } 1 \leq t \leq p. \end{cases} \quad (1)$$

We call *size of w* the integer p (with 0 as the size of the identity). We call such pairs *standard p -pairs*. We denote by $A_n^{c,p}$ the set of fully commutative elements of size p .

²Bourbaki, 1981, *Groupes et algèbres de Lie*, chapitres 4, 5, 6, §1.5 Proposition 5.

This very canonical form was established by Jones in Jones (1983) using some monomial generators of the Temperley-Lieb algebra which are scalar deformations of monomials in this work (defined in section 2), even before the concept of full commutativity in Coxeter groups emerged.

From now on we mean by FC element, a fully commutative element in $W(A_n)$.

2.2 The monomial basis

We recall briefly the definitions and results, by now classical, about the Temperley-Lieb algebra viewed as a diagram algebra, as written in Westbury (1995), Graham and Lehrer (1996) and Kauffman (1990) to which we refer for proofs and details. We let R be a commutative ring with 1 and we let δ be a fixed element in R .

The set A_n^c indexes a basis, called *monomial basis*, of the Temperley-Lieb algebra $TL(A_n) = TL_{R,\delta}(A_n)$ of type A_n and parameter δ over R . Actually this algebra has a presentation by generators $1, e_1, \dots, e_n$ and relations given, for $1 \leq i, j \leq n$, by:

$$e_i^2 = \delta e_i, \quad e_i e_{i+1} e_i = e_i \text{ if } i < n, \quad e_i e_{i-1} e_i = e_i \text{ if } i > 1, \quad e_i e_j = e_j e_i \text{ if } |i - j| > 1.$$

For $w \in A_n^c$ written in its canonical form (1): $w = [i_1, j_1][i_2, j_2] \dots [i_p, j_p]$, we denote by e_w the monomial $e_{i_1} \dots e_{j_1} \dots e_{i_p} \dots e_{j_p}$; in particular our notation identifies e_{σ_i} with e_i . Those monomials constitute the monomial R -basis of $TL(A_n)$. Recall that e_w doesn't depend on the reduced expression of w used to define it.

2.3 Non-crossing diagrams

A *non-crossing diagram* D on $n+1$ strands consists of two rows of $n+1$ dots in which

- each dot is joined to just one other dot, by a strand, and
- none of the strands intersect when drawn in the rectangle defined by the two rows of $n+1$ dots.

We draw the two rows on horizontal lines and we let $A = \{1, 2, \dots, n+1\}$ be the set of dots on the "top line", numbered from left to right, and $B = \{1', 2', \dots, (n+1)'\}$ be the set of dots on the "bottom line", numbered from left to right. We consider the total order on $A \cup B$ given by

$$1 < 2 < \dots < n+1 < 1' < 2' < \dots < (n+1)'$$

With this order we can consider each strand as a pair (x, y) of dots such that $x < y$, and consider the diagram D as a set of such pairs, that we call *arrows*. The vertices x and y are called the endpoints of the arrow, x is the *tail* of the arrow and y the *head* of the arrow. By convention the two rows of dots are drawn one below the other so that we call an arrow (x, y) with $x \in A$ and $y \in B$ *vertical* if $y = x'$, *positive* if $y > x'$ and *negative* if $y < x'$.

We often shorten *non-crossing diagram* to *diagram*.

2. Definitions

2.4 Multiplication of non-crossing diagrams

We can concatenate two diagrams from top to bottom. We reproduce here (Figures 2 and 3 below) the definition and the very example of Graham and Lehrer (1996), with 5 strands: if D_1 and D_2 are two diagrams, their concatenation $D_1 * D_2$ is the diagram resulting from placing D_2 below D_1 and deleting the circles produced if any; then their product is $D_1 D_2 = \delta^{m(D_1, D_2)} D_1 * D_2$ where $m(D_1, D_2)$ is the number of deleted circles from the concatenation.

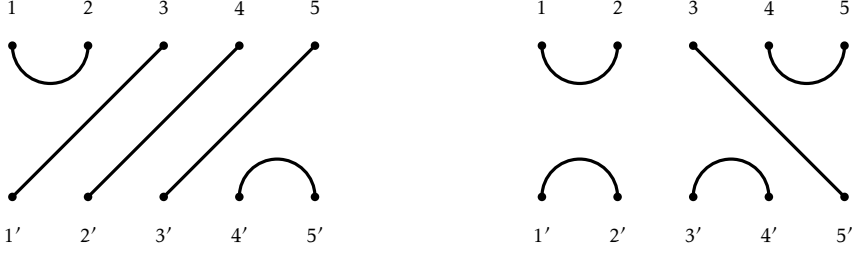


Figure 2 – D_1 (left), D_2 (right)

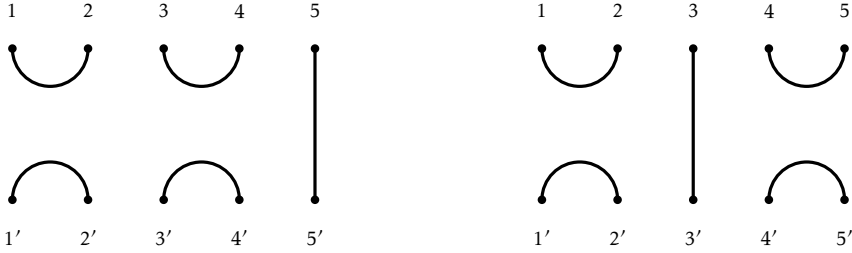
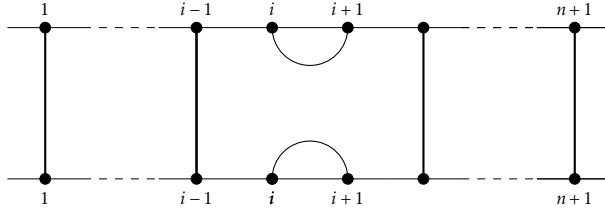


Figure 3 – $D_1 * D_2$ (left), $D_2 * D_1$ (right)

This multiplication makes the free R -module with basis the set of non-crossing diagrams on $n + 1$ strands into an R -algebra $\mathcal{D}_{R, \delta}(n + 1)$, and the map sending 1 to the unit diagram made of $n + 1$ vertical strands, and e_i , $1 \leq i \leq n$, to the diagram E_i drawn in Figure 4:

$$E_i = \{(i, i + 1), (i', (i + 1)')\} \cup \{(j, j') \mid 1 \leq j \leq i - 1 \text{ or } i + 2 \leq j \leq n + 1\}$$

determines an isomorphism of R -algebras Π_n from $TL_{R, \delta}(A_n)$ to $\mathcal{D}_{R, \delta}(n + 1)$. Π_n sends basis elements to basis elements, hence produces the unique bijection $\mathcal{D}$ between A_n^c , viewed as indexing the monomial basis, and the non-crossing diagrams on $n + 1$ strands, that maps e_i to E_i and respects the products.


 Figure 4 – Diagram E_i

To be completely clear, we mean by this the following. Let w and w' in A_n^c and let $w * w'$ be the FC element such that $e_w e_{w'} = \delta^{t(w, w')} e_{w * w'}$, for a well defined integer $t(w, w')$. Then

$$\mathcal{D}(w * w') = \mathcal{D}(w) * \mathcal{D}(w') \text{ and } \mathcal{D}(w)\mathcal{D}(w') = \delta^{t(w, w')} \mathcal{D}(w * w'). \quad (2)$$

We propose below a direct, combinatorial description of this bijection $\mathcal{D}$. And as a consequence we give explicitly Π_n and Π_n^{-1} on the full R-basis rather than on generators only.

3 The main components of a diagram

3.1 Definitions

Let D be a diagram, we write D as a disjoint union of four subsets:

- $AD = \{(x, y) \in D \mid x, y \in A\}$ (arrows in the top row),
- $BD = \{(x, y) \in D \mid x, y \in B\}$ (arrows in the bottom row),
- $D^+ = \{(x, y) \in D \mid x \in A, y \in B, x' < y\}$ (positive arrows from the top row to the bottom row),
- $D^- = \{(x, y) \in D \mid x \in A, y \in B, y \leq x'\}$ (negative or vertical arrows from the top row to the bottom row).

We define:

- $\mathcal{T}(AD)$ the set of tails of the arrows in AD ,
- $\mathcal{T}(D^+)$ the set of tails of the arrows in D^+ ,
- $\mathcal{H}(BD)$ the set of heads of the arrows in BD ,
- $\mathcal{H}(D^+)$ the set of heads of the arrows in D^+ ,

3. The main components of a diagram

- $\mathcal{I}(D) = \{i \in A \mid \exists y \in A \cup B (i, y) \in AD \cup D^+\} = T(AD) \cup T(D^+),$
- $\mathcal{J}(D) = \{j \in A \mid \exists x \in A \cup B (x, (j+1)') \in BD \cup D^+\}$
 $= \{j \in A \mid (j+1)' \in \mathcal{H}(BD) \cup \mathcal{H}(D^+)\},$
- $P_D = \sharp(AD \cup D^+)$, called the *size* of D .

Remark 1 –

1. AD and BD have the same cardinality, hence $\mathcal{I}(D)$ and $\mathcal{J}(D)$ also have the same cardinality.
2. D is fully determined by AD and BD . Indeed, due to the non-crossing condition, after drawing AD and BD , we proceed from left to right and join at each step the leftmost free dot in A to the leftmost free dot in B .
3. For arrows (x, y) in AD or BD , x and y have different parities. For arrows (x, y) in D^+ or D^- , x and y have the same parity.
4. It is straightforward to check that for any two diagrams D and D' :

$$AD \subset A(D * D') \text{ and } BD' \subset B(D * D').$$

5. We will see later (Proposition 3) that $\mathcal{I}(D)$ and $\mathcal{J}(D)$ determine D entirely.

3.2 The components of $\mathcal{D}(w)$

We now state our main Proposition.

Proposition 1 – *Let $w \in W^c(A_n)$ be a fully commutative element, given in its canonical form (1):*

$$w = [i_1, j_1][i_2, j_2] \dots [i_p, j_p], \text{ with } 0 \leq p \leq n \text{ and } \begin{cases} n \geq j_1 > \dots > j_p \geq 1, \\ n \geq i_1 > \dots > i_p \geq 1, \\ j_t \geq i_t \text{ for } 1 \leq t \leq p. \end{cases}$$

and let $\mathcal{D}(w)$ be the corresponding non-crossing diagram. Then

1. $\mathcal{I}(\mathcal{D}(w)) = \{i_k \mid 1 \leq k \leq p\}.$
2. $\mathcal{J}(\mathcal{D}(w)) = \{j_k \mid 1 \leq k \leq p\}.$
3. $\mathcal{D}(w)^+$ is the set of arrows $(i_s, (j_t + 1)')$ where $1 \leq t < s \leq p$ and

$$\begin{aligned} j_t &= i_s + 2(s - t) - 1, \\ i_s + 1 &= i_{s-1}, \\ t < r \leq s - 1 &\implies i_r < i_s + 2(s - r) \leq j_r. \end{aligned}$$

4. We let $A(1) = \{i_1 + 1, i_1 + 2, \dots, j_1 + 1\}$, $f_1 = \min A(1) = i_1 + 1$ and we define inductively on r , $1 < r \leq p$, the subsets:

- if $i_r \in \mathcal{T}(\mathcal{D}(w)^+)$: $A(r) = \emptyset$;
- otherwise:

$$A(r) = \{m \in \mathbb{N} \mid i_r + 1 \leq m \leq j_1 + 1\} - \{i_k \mid 1 \leq k \leq r - 1\} \\ - \{f_i \mid 1 \leq i \leq r - 1, A(i) \neq \emptyset\}.$$

If $A(r)$ is not empty, we let $f_r = \min A(r)$.

Then $AD(w)$ is the set of arrows $\{(i_r, f_r) \mid 1 \leq r \leq p, A(r) \neq \emptyset\}$.

5. We let $B(p) = \{i_p, i_p + 1, \dots, j_p\}$, $g_p = \max B(p) = j_p$ and define inductively on r , $p > r \geq 1$, the subsets:

- if $(j_r + 1)' \in \mathcal{H}(\mathcal{D}(w)^+)$: $B(r) = \emptyset$;
- otherwise:

$$B(r) = \{m \in \mathbb{N} \mid i_p \leq m \leq j_r\} - \{j_k + 1 \mid p \geq k \geq r + 1\} \\ - \{g_i \mid p \geq i \geq r + 1, B(i) \neq \emptyset\}.$$

If $B(r)$ is not empty, we let $g_r = \max B(r)$.

Then $BD(w)$ is the set of arrows $\{(g'_r, (j_r + 1)') \mid 1 \leq r \leq p, B(r) \neq \emptyset\}$.

The following section is devoted to the (long) proof of this (long) Proposition.

4 Proof of Proposition 1

Let w be as in Proposition 1. We keep the notation in the Proposition.

4.1 Implementation of the induction on p

Certainly $\mathcal{D}(1)$ is the identity diagram with arrows joining x to x' for $1 \leq x \leq n + 1$. For $p = 1$ we have $j_1 \geq i_1$ and $w = \sigma_{i_1} \dots \sigma_{j_1}$. It is straightforward to see that the diagram in Figure 5 is indeed the concatenation, from top to bottom, of the diagrams $E_{i_1}, E_{i_1+1}, \dots, E_{j_1}$ (Figure 4) representing $\sigma_{i_1}, \sigma_{i_1+1}, \dots, \sigma_{j_1}$. It satisfies $\mathcal{I}(\mathcal{D}([i_1, j_1])) = \{i_1\}$, $\mathcal{J}(\mathcal{D}([i_1, j_1])) = \{j_1\}$, $\mathcal{D}([i_1, j_1])^+ = \emptyset$, as prescribed in Proposition 1.

We let $\mathcal{D} = \mathcal{D}_p = \mathcal{D}(w)$ and, for $1 \leq s \leq p$, we let $w_s = [i_1, j_1] \dots [i_s, j_s]$ and $\mathcal{D}_s = \mathcal{D}(w_s)$. For $1 < s \leq p$ we have $w_s = w_{s-1}[i_s, j_s]$ hence the diagram $\mathcal{D}_s$ is the concatenation $\mathcal{D}_{s-1} * \mathcal{D}([i_s, j_s])$ as illustrated below in Figure 6.

From now on in this section we assume that **the Proposition holds for w_{s-1}** .

4. Proof of Proposition 1

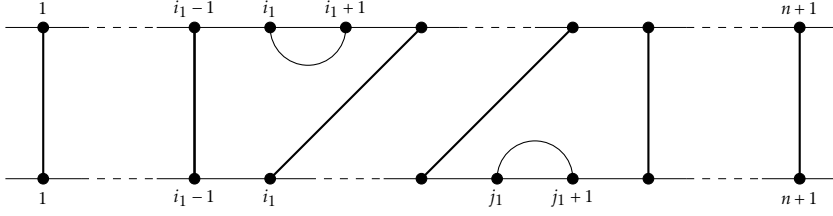


Figure 5 – $\mathcal{D}([i_1, j_1])$

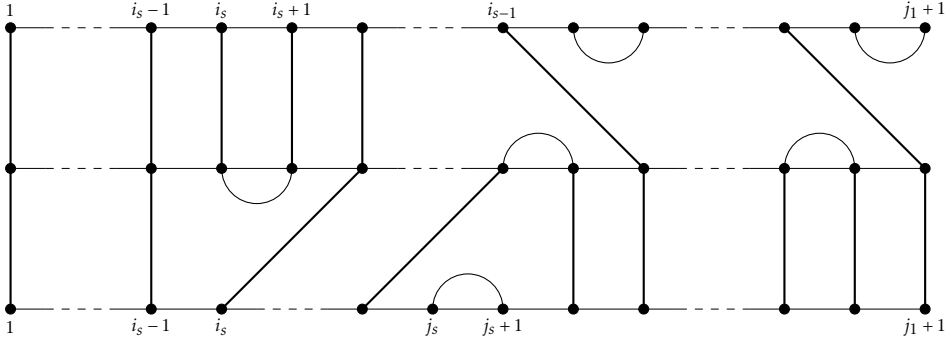


Figure 6 – Concatenation

4.2 A series of Lemmas

As remarked earlier in Remark 1 (4), we have:

Lemma 1 – $AD_{s-1} \subseteq AD_s$ and $(j'_s, (j_s + 1)')$ belongs to BD_s .

Next we examine $\mathcal{D}_s^+$.

Lemma 2 – $\mathcal{D}_{s-1}^-$ contains $\{(z, z') \mid z < i_{s-1}\}$ and $\mathcal{D}_{s-1}^+ \subseteq \mathcal{D}_s^+$.

Proof. The support of w_{s-1} is contained in the subset $\{\sigma_k \mid k \geq i_{s-1}\}$, hence $\mathcal{D}_{s-1}$ is a concatenation of elementary diagrams E_k for $k \geq i_{s-1}$ and $\mathcal{D}_{s-1}^-$ contains the arrows (z, z') for $z < i_{s-1}$. In particular, if $(x, y') \in \mathcal{D}_{s-1}^+$, so $x, y \in A$ and $x < y$, then x satisfies $x \geq i_{s-1} > i_s$. Furthermore $y - 1 \in \mathcal{J}(\mathcal{D}_{s-1})$, hence y is a $j_t + 1$ for some $t \leq s - 1$ so $y > j_s + 1$, hence, when concatenating with $\mathcal{D}([i_s, j_s])$, the arrow is followed downwards by a vertical arrow and indeed we keep $(x, y') \in \mathcal{D}_s^+$. $\square$

Lemma 3 – Let $(x, y') \in \mathcal{D}_{s-1}^-$, so $x, y \in A$, $x \geq y$. Then:

1. If $y < i_s$ or $y > j_s + 1$, we have $(x, y') \in \mathcal{D}_s^-$.

2. If $i_s + 1 < y \leq j_s + 1$, we have $(x, (y - 2)') \in \mathcal{D}_s^-$.
3. If $y = i_s + 1$ we obtain $(i_s, x) \in AD_s$.
4. If $y = i_s$, then $x = i_s$ and we obtain:
 - (a) if $i_s + 1 < i_{s-1}$: $(i_s, i_s + 1) \in AD_s$;
 - (b) if $i_s + 1 = i_{s-1} = j_{s-1}$: $(i_s, (i_s + 2)') = (i_s, (j_{s-1} + 1)')$ belongs to $\mathcal{D}_s^+$;
 - (c) if $i_s + 1 = i_{s-1} < j_{s-1}$:
 - i. either there is a $t < s$ such that $(i'_{s-1}, (j_t + 1)')$ belongs to BD_{s-1} , then $(i_s, (j_t + 1)')$ belongs to $\mathcal{D}_s^+$;
 - ii. or there is an $a > i_{s-1}$ such that (a, i'_{s-1}) belongs to $\mathcal{D}_{s-1}^-$, then (i_s, a) belongs to AD_s .

Proof. The first three statements are immediate from the figure describing the concatenation, we consider the fourth. From Lemma 2 we have $x = i_s$. We compose with the arrow $(i_s, i_s + 1)$ in $\mathcal{D}([i_s, j_s])$ followed by the arrow $\mathcal{A}$ in $\mathcal{D}_{s-1}$ containing $(i_s + 1)'$.

- If $i_s + 1 < i_{s-1}$, $\mathcal{A} = (i_s + 1, (i_s + 1)')$ producing $(i_s, i_s + 1)$ in AD_s .
- If $i_s + 1 = i_{s-1} = j_{s-1}$, then $\mathcal{A} = (i'_{s-1}, (i_{s-1} + 1)')$ belongs to BD_{s-1} by Lemma 1 and we obtain $(i_s, (i_s + 2)')$.
- If $i_s + 1 = i_{s-1} < j_{s-1}$, by induction hypothesis i'_{s-1} is strictly on the left of the smaller head of arrows in $BD_{s-1} \cup \mathcal{D}_{s-1}^+$ which is $(j_{s-1} + 1)'$, hence i'_{s-1} is either the tail of $\mathcal{A} = (i'_{s-1}, (j_t + 1)')$ in BD_{s-1} or the head of $\mathcal{A} = (a, i'_{s-1})$ in $\mathcal{D}_{s-1}^-$. Again by induction hypothesis i_{s-1} belongs to $\mathcal{I}(\mathcal{D}_{s-1})$ so $a > i_{s-1}$.

We remark that case (4b) is similar to (4c)(i): indeed, if $i_{s-1} = j_{s-1}$, the arrow $(i'_{s-1}, (j_{s-1} + 1)')$ belongs to BD_{s-1} . $\square$

Lemma 4 – Let $(x', y') \in BD_{s-1}$, so $x, y \in A$, $x < y$. Then $y = j_t + 1 > j_s + 1$ for some $t \leq s - 1$ and $x \geq i_{s-1} > i_s$, and :

1. if $x > j_s + 1$ then $(x', y') \in BD_s$;
2. if $i_s + 1 < x \leq j_s + 1$ then $((x - 2)', y') \in BD_s$;
3. if $x = i_s + 1$, hence $i_{s-1} = i_s + 1$, then $(i_s, y') = (i_s, (j_t + 1)')$ belongs to $\mathcal{D}_s^+$.

In particular, the arrows in BD_s are either $(j'_s, (j_s + 1)')$, or obtained by concatenation from an arrow in BD_{s-1} with the same head.

Proof. The first statement holds by induction on s , then, as before, we visualize with Figure 6 the concatenation $\mathcal{D}_{s-1} * \mathcal{D}([i_s, j_s])$. We see that y' is always the tail of a vertical strand in $\mathcal{D}([i_s, j_s])$. This is also the case for x' in (1), while in (2) x' is the tail of $(x, (x - 2)')$ in $\mathcal{D}([i_s, j_s])$, and in (3) x' is the head of $(i_s, i_s + 1)$ in $\mathcal{D}([i_s, j_s])$. $\square$

4. Proof of Proposition 1

4.3 Proof of Proposition 1, assertions (1) and (2)

The four Lemmas above give a full account of what becomes of arrows in $\mathcal{D}_{s-1}$ in the concatenation $\mathcal{D}_s = \mathcal{D}_{s-1} * \mathcal{D}([i_s, j_s])$. From the first two:

$$A\mathcal{D}_{s-1} \subseteq A\mathcal{D}_s \text{ and } \mathcal{D}_{s-1}^+ \subseteq \mathcal{D}_s^+, \text{ in particular } \mathcal{I}(\mathcal{D}_{s-1}) \subseteq \mathcal{I}(\mathcal{D}_s).$$

From the third and fourth ones, the only remaining arrow in $A\mathcal{D}_s$ may be some (i_s, a) (with conditions), whereas the only remaining arrow in $\mathcal{D}_s^+$ may be, again, some (i_s, a') . Such an arrow is obtained under the following circumstances:

- $i_{s-1} > i_s + 1$, then $(i_s, i_s + 1)$ always belongs to $A\mathcal{D}_s$;
- $i_{s-1} = i_s + 1$ and there is a $t < s$ such that $(i'_{s-1}, (j_t + 1)')$ belongs to $B\mathcal{D}_{s-1}$,
- $i_{s-1} = i_s + 1$ and there is an $a > i_{s-1}$ such that (a, i'_{s-1}) belongs to $\mathcal{D}_{s-1}^-$.

From Lemma 2, i'_{s-1} always belongs to an arrow of one of the two last cases, so this is true in particular if $i_{s-1} = i_s + 1$. From this we deduce $\mathcal{I}(\mathcal{D}_s) = \mathcal{I}(\mathcal{D}_{s-1}) \cup \{i_s\}$ hence, by induction, assertion (1) in Proposition 1.

From Lemma 4, the heads of arrows in $B\mathcal{D}_{s-1}$ remain heads of arrows in $B\mathcal{D}_s$ or possibly $\mathcal{D}_s^+$, hence, with $\mathcal{D}_{s-1}^+ \subseteq \mathcal{D}_s^+$, we get $\mathcal{J}(\mathcal{D}_{s-1}) \subseteq \mathcal{J}(\mathcal{D}_s)$. Since $\mathcal{I}(\mathcal{D}_s)$ and $\mathcal{J}(\mathcal{D}_s)$ have the same cardinality (Remarks 1) and $(j'_s, (j_s + 1)')$ belongs to $B\mathcal{D}_s$ (Lemma 1), we obtain $\mathcal{J}(\mathcal{D}_s) = \mathcal{J}(\mathcal{D}_{s-1}) \cup \{j_s\}$ and by induction assertion (2) in Proposition 1.

4.4 Proof of Proposition 1, assertion (3)

An arrow in $\mathcal{D}_s^+$ that does not belong to $\mathcal{D}_{s-1}^+$, if any, is unique and has the form $(i_s, (j_t + 1)')$ for some $t < s$. The last two Lemmas produce the same necessary and sufficient condition:

$$i_s + 1 = i_{s-1} \text{ and } (i'_{s-1}, (j_t + 1)') \in B\mathcal{D}_{s-1} \text{ for some } t < s. \quad (3)$$

We have to understand the formation of $B\mathcal{D}_{s-1}$ to find the algebraic meaning of this condition. We distinguish two cases.

- Case 1: $t = s - 1$, i.e. $(i'_{s-1}, (j_{s-1} + 1)') \in B\mathcal{D}_{s-1}$. This condition is equivalent to $j_{s-1} = i_{s-1}$. (Indeed $(j'_{s-1}, (j_{s-1} + 1)')$ always belongs to $B\mathcal{D}_{s-1}$.)
- Case 2: for some $t < s - 1$, $(i'_{s-1}, (j_t + 1)')$ belongs to $B\mathcal{D}_{s-1}$. Then the arrow $(i'_{s-1}, (j_t + 1)')$ must enclose the $s - 1 - t$ arrows in $B\mathcal{D}_{s-1}$ with heads $(j_r + 1)'$ for $t < r \leq s - 1$ so we must have $j_t + 1 = i_{s-1} + 2(s - 1 - t) + 1$, i.e.

$$j_t = i_s + 2(s - t) - 1. \quad (4)$$

Furthermore, the arrow $(i'_{s-1}, (j_t + 1)')$ must come from $(j'_t, (j_t + 1)') \in B\mathcal{D}_t$ through a series of $s - 1 - t$ concatenations. According to Lemma 4 each

concatenation may subtract 2 from the tail, starting with j_t . Comparing with equality (4) we see that the subtraction must occur each time, which means that each iteration falls into condition (2) in Lemma 4, so that for each r , $t \leq r < s-1$, we have, after $r-t$ iterations:

$$i_{r+1} + 1 < j_t - 2(r-t) \leq j_{r+1} + 1,$$

and the next iteration provides the arrow $((j_t - 2(r+1-t))', (j_t + 1)')$. Note that the double inequality above is equivalent, with (4), to:

$$i_{r+1} < i_s + 2(s-1-r) \leq j_{r+1} \text{ for } t \leq r < s-1. \quad (5)$$

This condition implies $i_r < j_r$ for each r , $t < r \leq s-1$. Furthermore, if (5) is satisfied, then from the same iterations as above we arrive at $(i_{s-1}, (j_t + 1)') \in BD_{s-1}$, and adding $i_s + 1 = i_{s-1}$ we obtain $(i_s, (j_t + 1)') \in \mathcal{D}(w)^+$ (again from Lemma 4).

Assertion (3) is proved.

4.5 Description of $\mathcal{D}(w)^+$ in terms of saturated elements

This subsection gives an alternate description of $\mathcal{D}(w)^+$, it may be skipped. As a matter of fact, the description was done in its first version³ in the form of (1) of the next Proposition, i.e., using the notion of "saturated" elements soon to be defined. It was a surprise (say happenstance) that this notion, defined as a tool to explain $\mathcal{D}(w)^+$, turned out to be of remarkable importance: the author has studied saturated elements for their own sake [manuscript in preparation, see Remark 3], not as any subset of FC elements, they are even counted by Catalan numbers "by definition", which means that they can be partitioned in such a way to satisfy the defining recurrence relation for the Catalan number.

Definition 2 – Let (W, S) be a Coxeter system and let w be an element of W . We say that w is *saturated* if $\text{Supp}(w) = S$.

Proposition 2 – For t, s such that $1 \leq t < s \leq p$, the following three sets of conditions are equivalent:

1. The conditions in terms of minimality and saturation:

- $j_t = i_s + 2(s-t) - 1$;
- for $t < r < s$ we have $j_r \neq i_s + 2(s-r) - 1$ and $j_t \neq i_r + 2(r-t) - 1$;
- if $t < s-1$: $\prod_{k=t+1}^{k=s-1} [i_k, j_k]$ is saturated in $\langle \sigma_h \mid i_s + 1 \leq h \leq j_t - 1 \rangle$.

³Al Harbat, 2024, "Catalan numbers: from FC elements to classical diagram algebras".

4. Proof of Proposition 1

2. The conditions:

- $j_t = i_s + 2(s - t) - 1$;
- for $t < r < s$ we have $j_r \neq i_s + 2(s - r) - 1$ and $j_t \neq i_r + 2(r - t) - 1$;
- if $t < s - 1$: $i_{s-1} = i_s + 1$, $j_{t+1} = j_t - 1$ and for $t < u < s - 1$, $i_u \leq j_{u+1} + 1$.

3. The conditions in assertion (3) of Proposition 1.

Proof. For the equivalence between (1) and (2) we need only check the equivalence of the third assertions in both. Indeed for $t < u < s - 1$ no generator can be missing between $[i_u, j_u]$ and $[i_{u+1}, j_{u+1}]$, i.e. between i_u and j_{u+1} , hence the generic inequality, the two others coming from endpoints. We note that for $t = s - 1$ the first assertion reads $j_{s-1} = i_s + 1$ which certainly implies $i_{s-1} = i_s + 1$ and $j_s = j_{s-1} - 1$.

Now we show that assertion (3) in Proposition 1 is equivalent to (1). We assume first that conditions (3) hold. If $t = s - 1$, we do have the only condition in assertion (1), namely $j_{s-1} = i_s + 1$. If $t < s - 1$ we have, for $t < r < s$, $i_s + 2(s - r) \leq j_r$ hence certainly $i_s + 2(s - r) - 1 \neq j_r$. If we rewrite the condition with j_t instead of i_s we find $i_r < j_t - 2(r - t) + 1$ hence certainly $j_t \neq i_r + 2(r - t) - 1$. We check the saturation, which amounts to:

- $j_{t+1} = j_t - 1$: indeed in the inequality $i_s + 2(s - t - 1) \leq j_{t+1}$ the left term is $j_t - 1$, whereas we always have $j_{t+1} < j_t$.
- for $t + 1 < r \leq s - 1$, $j_r \geq i_{r-1} - 1$: assume for a contradiction that there is such an r with $j_r < i_{r-1} - 1$, then we get $i_s + 2(s - r) \leq j_r < i_{r-1} - 1$. Since $r - 1 > t$ we have $i_{r-1} - 1 < i_s + 2(s - r) + 1$ so we find a difference of at least 2 between $i_s + 2(s - r) + 1$ and $i_s + 2(s - r)$, absurd.
- $i_{s-1} = i_s + 1$: assumed in (3).

So indeed conditions (3) imply the conditions in assertion (1).

Now we take a pair (s, t) , $t < s$, satisfying the conditions in assertion (1) of the Proposition and we prove that it satisfies conditions (3).

The first equality $j_t = i_s + 2(s - t) - 1$ is an assumption. If $t = s - 1$ it implies $j_{s-1} = i_s + 1$, hence $i_{s-1} = i_s + 1$ is mandatory and (3) is proven.

If $t < s - 1$, the equality $i_{s-1} = i_s + 1$ is implied by the saturation condition. It remains to establish the inequalities $i_r < i_s + 2(s - r) \leq j_r$ for $t < r \leq s - 1$.

For $r = t + 1$ they read $i_{t+1} < i_s + 2(s - t) - 2 \leq j_{t+1}$ whereas $i_s + 2(s - t) - 2 = j_t - 1$ and $j_t - 1 = j_{t+1}$ from the saturation condition. We just need to rule out $i_{t+1} = j_{t+1}$. But if this was true, we would have $j_t = i_{t+1} + 1$ hence, by minimality, $s = t + 1$, which has been excluded. So (3) holds.

We prove the leftmost inequality by ascending induction, assuming that it holds from $t + 1$ to $r - 1$. We prove it for $r \leq s - 1$. We assume for a contradiction that it doesn't hold. Then $i_r \geq i_s + 2(s - r)$. The inequality for $r - 1$ gives $i_{r-1} < i_s + 2(s - r) + 2$, implying $i_r < i_s + 2(s - r) + 1$, so the only possibility is $i_r = i_s + 2(s - r) = j_t - 2(r - t) + 1$, hence $j_t = i_r + 2(r - t) - 1$, contradicting the minimality conditions.

For $r = s - 1$ the rightmost inequality reads $i_s + 2 \leq j_{s-1}$. The saturation condition implies $i_{s-1} = i_s + 1$ hence $j_{s-1} \geq i_s + 1$. Now $j_{s-1} = i_s + 1$ would contradict the minimality conditions, so indeed $i_s + 2 \leq j_{s-1}$ holds. We proceed by descending induction, assuming that the rightmost inequality holds from $s - 1$ to $r + 1$, and we prove it for r , $t < r < s - 1$.

Assume for a contradiction that it does not hold, then $j_r < i_s + 2(s - r)$. We compare this with the inequality holding for $r + 1$: $i_s + 2(s - r) - 2 \leq j_{r+1}$ and get $i_s + 2(s - r) - 2 \leq j_{r+1} < j_r < i_s + 2(s - r)$, which implies $j_r = i_s + 2(s - r) - 1$, forbidden by the minimality conditions.

Finally (3) holds as we claimed. $\square$

4.6 Proof of Proposition 1, assertion (4): description of $AD(w)$

Certainly $AD_1 = \{(i_1, i_1 + 1)\}$ by Figure 5. Let s such that $1 < s \leq p$, we assume that (4) holds for w_{s-1} and we prove it for w_s . Thanks to Lemmas 1 and 2 we only have to prove:

If i_s is not the tail of an arrow in $\mathcal{D}_s^+$ (or $\mathcal{D}(w)^+$), then $A(s)$ is not empty and (i_s, f_s) belongs to AD_s .

The condition on i_s is the negation of condition (3) in subsection 4.4, so

- either $i_s + 1 < i_{s-1}$, in which case indeed $A(s)$ is not empty and $f_s = i_s + 1$, and by Lemma 3 (4)(a), $(i_s, i_s + 1)$ belongs to AD_s .
- or $i_s + 1 = i_{s-1}$ and i'_{s-1} is not the tail of an arrow in BD_{s-1} , i.e. i'_{s-1} is the head of an arrow in $\mathcal{D}_{s-1}^+$ or $\mathcal{D}_{s-1}^-$. The first is impossible (Lemma 2), so we have an arrow (x, i'_{s-1}) with $x > i_{s-1}$ (equality is impossible because i_{s-1} is the tail of an arrow in $AD_{s-1} \cup \mathcal{D}_{s-1}^+$).

So x is neither an i_k nor an f_k if any for $k \leq s - 1$, and $x \geq i_s + 1$, so x belongs to $A(s)$ which is non empty. The concatenation produces the arrow $(i_s, x) \in AD_s$ so we want to prove that $x = \min A(s)$. Suppose otherwise, then $i_s + 1 = i_{s-1} < f_s < x$. Since (x, i'_{s-1}) is an arrow in $\mathcal{D}_{s-1}^-$, and any dot left of i'_{s-1} is the head of a vertical arrow in $\mathcal{D}_{s-1}^-$, all dots between i_{s-1} and $x - 1$ must belong to arrows in AD_{s-1} , which are all of the form (i_k, f_k) with $k \leq s - 1$ by induction hypothesis. So f_s , that cannot be an i_k with $k \leq s - 1$, has to be an f_k with $k \leq s - 1$, impossible by the definition of $A(s)$.

5. The bijection and the associated algorithms

4.7 Proof of Proposition 1, assertion (5): description of $BD(w)$

With what we know at this point, (5) is equivalent to the following assertion:

Let $1 \leq r \leq p$ and assume $(j_r + 1)'$ is not the head of an arrow in $\mathcal{D}(w)^+$.

Then $B(r)$ is not empty and $(g'_r, (j_r + 1)')$ belongs to $BD(w)$.

Indeed $(j'_p, (j_p + 1)')$ belongs to $BD(w)$, which is the case $r = p$. We now work by descending induction on r : let r with $1 \leq r < p$, we assume our assertion holds for $r + 1$ and prove it for r .

By definition, $BD(w)$ is the set of arrows in $\mathcal{D}(w)$ with head some $(j_r + 1)'$ and not belonging to $\mathcal{D}(w)^+$, so certainly if $(x', (j_r + 1)')$ belongs to $BD(w)$, then $x \leq j_r$, x is not a $j_k + 1$ nor a g_k for $k \geq r + 1$, hence x belongs to $B(r)$ which is not empty. We assume for a contradiction that $x < g_r$. Then the arrow $(x', (j_r + 1)')$ strictly encloses an arrow in $BD(w)$ with head or tail g'_r , actually with tail g'_r since g_r is not a $j_k + 1$. The head of this arrow must be a $(j_k + 1)'$ strictly less than $(j_r + 1)'$, so $k > r$ and by induction we have $g_r = g_k$ which is impossible by construction of $B(r)$. Hence $x = g_r$.

This finishes the proof of assertion (5) and finally of Proposition 1.

5 The bijection and the associated algorithms

5.1 The main Theorem

Proposition 1 ascribes a non-crossing diagram on $n+1$ strands to a fully commutative element w in $W(A_n)$ in a unique way: all constitutive elements of the diagram $\mathcal{D}(w)$, where $\mathcal{D}$ is the bijection defined at the end of section 2, are described directly in terms of w , or rather, in terms of the canonical form of w . So we can describe the inverse of the bijection $\mathcal{D} : w \mapsto \mathcal{D}(w)$, that we denote by $\mathcal{D}^{-1} : \mathcal{D} \mapsto w_D$, as follows:

Proposition 3 – *Let D be a non-crossing diagram on $n + 1$ strands, of size p and write $\mathcal{I}(D)$ and $\mathcal{J}(D)$ in decreasing order, that is:*

$$\mathcal{I}(D) = \{i_1, \dots, i_p\} \quad \text{with } i_1 > i_2 > \dots > i_p,$$

$$\mathcal{J}(D) = \{j_1, \dots, j_p\} \quad \text{with } j_1 > j_2 > \dots > j_p.$$

Then for all k , $1 \leq k \leq p$, we have $i_k \leq j_k$, and D is the image under $\mathcal{D}$ of the fully commutative element

$$w_D = [i_1, j_1][i_2, j_2] \dots [i_p, j_p].$$

In particular $\mathcal{I}(D)$ and $\mathcal{J}(D)$ determine D .

Proof. Indeed $\mathcal{D}$ is surjective so D is a $\mathcal{D}(w)$ given by Proposition 1 for some w in $W^c(A_n)$, hence the elements of $\mathcal{I}(D)$ and $\mathcal{J}(D)$ satisfy the properties of standard p -pairs in Theorem 1. $\square$

We recall that $(i_1, i_1 + 1)$ is necessarily an arrow as well as $(j'_p, (j_p + 1)')$ (Proposition 1). We conclude with our essential statement.

Theorem 2 – Let w be a FC element of $W(A_n)$ given in its normal form (1):

$$w = [i_1, j_1][i_2, j_2] \dots [i_p, j_p], \text{ with } 0 \leq p \leq n \text{ and } \begin{cases} n \geq j_1 > \dots > j_p \geq 1, \\ n \geq i_1 > \dots > i_p \geq 1, \\ j_t \geq i_t \text{ for } 1 \leq t \leq p. \end{cases}$$

1. There is a unique non-crossing diagram on $n+1$ strands $\mathcal{D}(w)$ satisfying $\mathcal{I}(\mathcal{D}(w)) = \{i_1, \dots, i_p\}$ and $\mathcal{J}(\mathcal{D}(w)) = \{j_1, \dots, j_p\}$.
2. The map $w \mapsto \mathcal{D}(w)$ is the unique bijection between the set of FC elements of $W(A_n)$ and the set of non-crossing diagrams on $n+1$ strands that maps σ_i to E_i , $1 \leq i \leq n$, and is compatible with the monomial multiplication and the multiplication of non-crossing diagrams.

Remark 2 –

1. A complete description of $\mathcal{D}(w)$ is given in Proposition 1, and an algorithm to draw $\mathcal{D}(w)$ is given in subsection 5.3 below.
2. Let D be a non-crossing diagram on $n+1$ strands, its image under $\mathcal{D}^{-1}$ is given in Proposition 3, or with an algorithm in subsection 5.2.

5.2 First algorithm: From diagrams to FC elements

It is very easy to write the FC element corresponding to a given diagram D , since it is given by $\mathcal{I}(D)$ and $\mathcal{J}(D)$ (see the example in the Introduction):

- Single out on the top line the tails of arrows going strictly right and list their numbering in decreasing order: $\{i_1, \dots, i_p\}$.
- Single out on the bottom line the heads of arrows coming strictly from the left and list their numbering **minus 1** in decreasing order: $\{j_1, \dots, j_p\}$.
- The FC element that corresponds to D is $[i_1, j_1][i_2, j_2] \dots [i_p, j_p]$.

5.3 Second algorithm: From FC elements to diagrams

We follow Proposition 1 and its proof, starting with a FC element w .

- Write w in its canonical form $w = [i_1, j_1][i_2, j_2] \dots [i_p, j_p]$ and mark the dots i_k (upper line) and $j_k + 1$ (lower line).
- For any u such that $1 \leq u < i_p$ or $j_1 + 1 < u \leq n + 1$, draw the arrow (u, u') .

6. Applications and questions

- Draw the arrow $(i_1, i_1 + 1)$ in $AD(w)$, then working from right to left, i.e. for increasing s , do the following for each s , $2 \leq s \leq p$:
 - If the three following conditions are satisfied, draw the arrow $(i_s, (j_t + 1)')$.
 1. $i_s + 1 = i_{s-1}$,
 2. there exists a highest index $t \leq s - 1$ such that $j_t = i_{s-1} + 2(s - 1 - t)$ and $(j_t + 1)'$ is a free dot (i.e. not already the head of an arrow),
 3. if $t < s - 1$, the element $[i_{t+1}, j_{t+1}] \dots [i_{s-1}, j_{s-1}]$ is saturated in the group generated by the σ_k , $i_{s-1} \leq k \leq j_t - 1$, which amounts to:

$$j_{t+1} = j_t - 1 \text{ and } i_u \leq j_{u+1} + 1 \text{ for } t < u < s - 1.$$
 - If $s < p$ proceed to $s + 1$.
- Proceeding from right to left, i.e. on increasing order on k , $1 \leq k \leq p$, for each k such that i_k is not already the tail of an arrow, find f_k the lowest free dot in A on the right of i_k and draw (i_k, f_k) in $AD(w)$.
- Draw the arrow $(j'_p, (j_p + 1)')$ in $BD(w)$, then, proceeding from left to right, i.e. on decreasing order on k , $p \geq k \geq 1$, for each k such that $(j_k + 1)'$ is not already the head of an arrow, find g'_k the highest free dot in B on the left of $(j_k + 1)'$ and draw $(g'_k, (j_k + 1)')$ in $BD(w)$.
- Proceed from left to right to draw the remaining arrows, all going from top to bottom, and necessarily all vertical or negative. Each joins the lowest free dot on the top line to the lowest free dot on the bottom line.

6 Applications and questions

6.1 Monomial multiplication

Multiplication in the monomial basis may be quite intricate to deal with. We have an alternative here: we can use the algorithm to transform two monomials (of course after putting them in their canonical forms, which is fairly easy) into diagrams, then we multiply the diagrams, an easy step, then we use the algorithm backwards to get the resulting monomial. We can imagine the efficiency of this method when the number of generators increases. Moreover the resulting monomial comes in its canonical form as an output.

6.2 Left and right multiplication

It is easy to identify $\mathcal{L}(w)$ and $\mathcal{R}(w)$ from the diagram $\mathcal{D}(w)$ attached to a FC element w . We only need the following Lemma:

Lemma 5 – Let $w = [i_1, j_1][i_2, j_2] \dots [i_p, j_p]$ be an element of A_n^c given in its normal form (1). Then

$$\mathcal{L}(w) = \{\sigma_{i_1}\} \cup \{\sigma_{i_k} \mid 2 \leq k \leq p \text{ and } i_k < i_{k-1} - 1\},$$

$$\mathcal{R}(w) = \{\sigma_{j_p}\} \cup \{\sigma_{j_k} \mid 1 \leq k \leq p-1 \text{ and } j_k > j_{k+1} + 1\}.$$

Proof. We sketch the proof for $\mathcal{L}(w)$, the other case is similar. The set $\mathcal{L}(w)$ is included in $\{\sigma_{i_k} \mid 1 \leq k \leq p\}$, as we prove now by induction on p . If $p = 1$ the element is rigid, left-reduced only by σ_{i_1} . For $p \geq 2$, assuming the assumption holds up to $p-1$, we multiply w on the left by σ_s for $1 \leq s \leq n$ and assume

$$(H) \quad \ell(\sigma_s w) < \ell(w).$$

If σ_s commutes with $[i_1, j_1]$, that is if $i_1 - 1 > s$ or $s > j_1 + 1$, then $\ell(\sigma_s[i_1, j_1]) > \ell([i_1, j_1])$. We apply the exchange condition⁴ to $\sigma_s[i_1, j_1] \dots [i_p, j_p]$. The index of the element of $[i_1, j_1] \dots [i_p, j_p]$ giving rise to the exchange condition is at least $j_1 - i_1 + 2$, so we can cancel out $[i_1, j_1]$ on both sides and get $\ell(\sigma_s[i_2, j_2] \dots [i_p, j_p]) < \ell([i_2, j_2] \dots [i_p, j_p])$. Then by induction $s = i_k$ for some $k > 1$.

Otherwise we have $i_1 - 1 \leq s \leq j_1 + 1$. If $s = j_1 + 1$ we obtain a new canonical form of size $p+1$ and first block σ_{j_1+1} so we can't have (H). If $s = i_1 - 1$ we obtain a canonical reduced form of size p for an element that may not be FC – we recall that any element in $W(A_n)$ has a canonical form similar to (1), except that the condition $i_1 > i_2 > \dots > i_p$ is removed – so we can't have (H). If $i_1 + 1 \leq s \leq j_1$ we have $\sigma_s[i_1, j_1] = [i_1, j_1]\sigma_{s-1}$, reduced, and based on the exchange condition as before, if σ_s belongs to $\mathcal{L}(w)$, then σ_{s-1} must belong to $\mathcal{L}([i_2, j_2] \dots [i_p, j_p])$ hence $s-1$ is equal to i_k for some $k \geq 2$, impossible since $s-1 \geq i_1$. The only possibility left is $s = i_1$ as claimed.

It remains to prove that if $i_k = i_{k-1} - 1$, then $\ell(\sigma_{i_k} w) > \ell(w)$. But certainly σ_{i_k} commutes with the blocks 1 to $k-2$, so we can write

$$\sigma_{i_k} w = [i_1, j_1] \dots [i_{k-2}, j_{k-2}][i_{k-1} - 1, j_{k-1}][i_{k-1} - 1, j_k] \dots [i_p, j_p]$$

which is a canonical reduced form for a non-FC element, hence reduced. $\square$

From the algorithm linking an FC element to a diagram, we see that $(i, i+1)$ belongs to AD if and only if $i = i_1$ or i is an i_k and $i+1$ is not, and similarly for BD , so:

Proposition 4 – In the diagram $\mathcal{D}(w)$ attached to a FC element w , one can read $\mathcal{L}(w)$ and $\mathcal{R}(w)$ as follows:

$$\mathcal{L}(w) = \{\sigma_i \mid (i, i+1) \in AD(w)\}, \quad \mathcal{R}(w) = \{\sigma_j \mid (j', (j+1)') \in BD(w)\}.$$

In other words, the elements of $\mathcal{L}(w)$ (resp. $\mathcal{R}(w)$) correspond to the tails of the minimal top (resp. bottom) cups. This Proposition is to be considered in a graphic way and in relation to the monomial basis.

⁴Bourbaki, 1981, *Groupes et algèbres de Lie, chapitres 4, 5, 6*, p. IV.1.5.

6. Applications and questions

6.3 Equivalence of diagrams

We introduce the following equivalence on diagrams, hence on A_n^c via the bijection of Theorem 2: we say that two diagrams D_1 and D_2 are equivalent, and we write $D_1 \approx D_2$, if

$$D_1^+ \cup D_1^- = D_2^+ \cup D_2^- \quad \text{that is, } D_1^+ = D_2^+ \text{ and } D_1^- = D_2^-.$$

This clear equivalence gives a mysterious equivalence on A_n^c , and actually on every $A_n^{c,p}$, since obviously the two corresponding FC elements w_{D_1} and w_{D_2} have the same size. Note that we have given in 4.5 various combinatorial descriptions of the elements of D^+ .

6.4 Non-saturated elements

We can see that FC elements corresponding to diagrams containing the arrow (i, i') , $1 < i < n$, are a multiplication of two elements with disjoint commuting supports, for such elements do not contain σ_i in their supports. Actually we can characterize saturated elements by their corresponding diagrams as follows:

Proposition 5 – *Let $\mathcal{D}(w)$ be the diagram attached to a FC element w . The following are equivalent:*

- (i) σ_i does not belong to the support of w .
- (ii) The vertical line at $i + \frac{1}{2}$ separates $\mathcal{D}(w)$ into two disjoint diagrams.

The choice of $\frac{1}{2}$ is of course arbitrary, we mean a parallel to (i, i') cutting the upper line strictly between i and $i + 1$.

Proof. Write w in its normal form $w = [i_1, j_1][i_2, j_2] \dots [i_p, j_p]$. The first condition means: $i > j_1$, or $i < i_p$, or there is a k , $1 < k \leq p$, such that $j_k < i < i_{k-1}$. Considering the diagram E_i drawn in Figure 4 we see that if E_1 , resp. E_n , is missing in a given concatenation, the strand $(1, 1')$, resp. $(n + 1, (n + 1)')$, appears in the resulting diagram, so indeed the vertical line at $1 + \frac{1}{2}$, resp. $n + \frac{1}{2}$, separates it into two disjoint diagrams. This extends directly to cases $i > j_1$, or $i < i_p$. For the third case, we write

$$w = ([i_1, j_1] \dots [i_{k-1}, j_{k-1}]) \cdot ([i_k, j_k] \dots [i_p, j_p]).$$

We obtain $\mathcal{D}(w)$ by concatenating a diagram having vertical strands on the left of i_{k-1} (from $i_{k-1} - 1$ downwards) with one having vertical strands on the right of $j_k + 1$ (from $j_k + 2$ upwards), so any vertical line at x real satisfying $j_k + 1 < x < i_{k-1}$ separates $\mathcal{D}(w)$ into two disjoint diagrams, in particular for $x = i + \frac{1}{2}$.

Conversely, assuming (ii), we see $\mathcal{D}(w)$ as the concatenation of a diagram with vertical strands on the left of $i + \frac{1}{2}$ with one with vertical strands on the right of $i + \frac{1}{2}$, which corresponds to the product of an element with support contained in $\{\sigma_{i+1}, \dots, \sigma_n\}$ with one with support contained in $\{\sigma_1, \dots, \sigma_{i-1}\}$, hence (i). $\square$

Corollary 1 – *The diagrams that correspond to saturated elements are the diagrams that cannot be separated into two disjoint diagrams by a vertical line.*

Remark 3 – In our manuscript in preparation “Counting Catalan objects by definition”, this property is essential to obtain a Catalan-defining partition of the set of non-crossing diagrams.

6.5 Questions

The two algorithms raise a lot of questions, mainly about the interpretation on one side of properties known on the other side. As of now, we invite the interested reader to think about the following questions and remarks.

1. Making use of Proposition 4 which shows that $\mathcal{L}(w)$ and $\mathcal{R}(w)$ are way easier to be determined visually, could it be possible to determine also “cancellable elements”? More: can we determine the a -function on some w visually using $\mathcal{D}(w)$?
2. Can we compute the number of equivalence classes, and the cardinality of each? For this would lead to very interesting partitions of the Catalan number, as follows. Observe first that once $D^+ \cup D^-$ is fixed, what varies in the corresponding equivalence class is nothing but small non-crossing diagrams with exactly the presentation given in ((61)) Stanley (2015), hence the cardinality of an equivalence class is a product of “small” Catalan numbers.

Now let us write $A_n^{c,p}$ as a disjoint union of equivalence classes for $\approx$, say $A_n^{c,p} = \sqcup_{\phi \in \Phi_n^p} \widetilde{D}_\phi$, where we have chosen a system of representatives $D_\phi, \phi \in \Phi_n^p$, for the equivalence classes in $A_n^{c,p}$, with parameter set Φ_n^p , and we denote by $\widetilde{D}$ the equivalence class of D . Accordingly, letting $C_n^p = \#A_n^{c,p}$ and $C_{n+1} = \#A_n^c$ (a Catalan number), we find

$$C_n^p = \sum_{\phi \in \Phi_n^p} \# \widetilde{D}_\phi \quad \text{and} \quad C_{n+1} = \sum_{p=0}^n \sum_{\phi \in \Phi_n^p} \# \widetilde{D}_\phi,$$

which are respectively a one-parameter partition of the Narayana number and a two-parameters partition of the Catalan number C_{n+1} , in which each term is a product of smaller Catalan numbers. More details on these partitions and others can be found in our work⁵.

3. Rethink the latter questions for the rougher equivalence given by: D_1 equivalent to D_2 if $D_1^+ = D_2^+$. For this equivalence the number of classes is easier to compute, but the cardinality of each class is more complicated than for the previous one.

⁵Al Harbat, 2024, “Catalan numbers: from FC elements to classical diagram algebras”.

The last proposed partitions take their importance before any applications. Actually any partition of a number that happens to be C_{n+1} is already important, even before being computed. This was the case of the very C_{n+1} , that took its importance a long time before E. C. Catalan obtained its standard formula:

$$C_{n+1} = \frac{1}{n+2} \binom{2(n+1)}{n+1}.$$

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