



A modified Water Wave model with a nonlocal viscous dispersive term

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Abstract

In this paper, we consider the modified water wave model with a nonlocal viscous term

$$u_t + u_x - u_{xx} + \frac{1}{\sqrt{\pi}} \frac{\partial}{\partial t} \int_0^t \frac{u(s)}{\sqrt{t-s}} ds = -u^2 u_x,$$

where $\frac{1}{\sqrt{\pi}} \frac{\partial}{\partial t} \int_0^t \frac{u(s)}{\sqrt{t-s}} ds$ is the Riemann-Liouville half derivative. We discuss theoretically and numerically the well-posedness of the model and the asymptotical behavior of the solutions.

Keywords: water waves, decay rate, diffusive representation, nonlocal term.

MSC: 35B40, 35Q35.

1 Introduction

Modeling the viscosity effects in water wave propagation is a challenging topic. Independently, Liu and Orfila (2004) and Dutykh and Dias (2007) have developed mathematical models for transient long-wave propagation on viscous shallow water in 2D and in 3D where the viscosity is modeled with nonlocal terms in the form of convolution integrals. A nonlinear model in one dimension is presented in Dutykh (2009).

In Chen et al. (2010), the authors investigated the decay rate of the solutions to the nonlocal water wave model

$$u_t + u_x - \alpha u_{xx} + \beta u_{xxx} + \frac{\sqrt{\nu}}{\sqrt{\pi}} \int_0^t \frac{u_t(s)}{\sqrt{t-s}} ds = -\gamma u u_x, \quad (1)$$

where $\frac{\sqrt{\nu}}{\sqrt{\pi}} \int_0^t \frac{u_t(s)}{\sqrt{t-s}} ds$ is the Caputo half derivative in time and it stands for the nonlocal diffusive-dispersive term, u is the horizontal velocity of the fluid, $-\alpha u_{xx}$ is the

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usual diffusion, βu_{xxx} is the geometric dispersion. The parameters ν , α , β and γ are devoted to examine the impact of viscosity, dispersion and nonlinearity on the solution.

Moreover, in Dumont, Goubet, and Manoubi (2020), Dumont and Manoubi (2018), Goubet and Manoubi (2017), and Manoubi (2014), the authors considered a derived model of (1) where the nonlocal term is modeled by the Riemann-Liouville half derivative in time, namely

$$u_t + u_x - \alpha u_{xx} + \beta u_{xxx} + \frac{\sqrt{\nu}}{\sqrt{\pi}} \frac{\partial}{\partial t} \int_0^t \frac{u(s)}{\sqrt{t-s}} ds = -\gamma u u_x. \quad (2)$$

In particular, in Manoubi (2014), the author proved the well-posedness of the initial value problem (2) when $\beta = 0$ for small initial data $u_0 \in L_x^1(\mathbb{R}) \cap L_x^2(\mathbb{R})$ using a fixed point approach. She established that if $\|u_0\|_{L_x^1(\mathbb{R}) \cap L_x^2(\mathbb{R})}$ is small enough, then there exists a unique global solution $u \in C(\mathbb{R}_+; L_x^2(\mathbb{R})) \cap C^{1/2}(\mathbb{R}_+; H_x^{-2}(\mathbb{R}))$. Besides, she proved that the solution u satisfies

$$\max(t^{1/4}, t^{3/4}) \|u(t, \cdot)\|_{L_x^2(\mathbb{R})} + \max(t^{1/2}, t) \|u(t, \cdot)\|_{L_x^\infty(\mathbb{R})} \leq C(u_0).$$

Besides, in Dumont, Goubet, and Manoubi (2020), the authors relaxed the assumption on the initial data. They established that the initial value problem is well-posed for all initial data $u_0 \in L_x^2(\mathbb{R})$ using the diffusive approach. They established that for all initial $u_0 \in L_x^1(\mathbb{R}) \cap L_x^2(\mathbb{R})$,

$$\max(1, t^{3/4}) \|u(t, \cdot)\|_{L_x^2(\mathbb{R})} + \max(t^{1/2}, t) \|u(t, \cdot)\|_{L_x^\infty(\mathbb{R})} \leq C(u_0).$$

Moreover, a great interest was focused on the numerical study of the decay rate of the solutions in the last decades. Several numerical schemes were constructed in order to approximate the nonlocal term³. Over the last years, some authors proposed the so-called diffusive realization of fractional operators (in time). This approach provide a local in time representation of the fractional operators by introducing a new linear differential equation. The resulting diffusive model is local in time, which provide numerous advantages from the mathematical and numerical standpoints. Different applications of this approach can be found in Audounet, Giovangigli, and Roquejoffre (1998), Dumont and Manoubi (2018), H  lie and Matignon (2006), and Montseny, Audounet, and Matignon (2000).

³Chen et al., 2010, "Decay of solutions to a water wave model with a nonlocal viscous dispersive term";

Galucio, De  , and Dubois, 2008, "The G^α -scheme for approximation of fractional derivatives: Application to the dynamics of dissipative systems";

Manoubi, 2014, "Theoretical and numerical analysis of the decay rate of solutions to a water wave model with a nonlocal viscous dispersive term with Riemann-Liouville half derivative".

2. Preliminaries and main results

In order to investigate the effects of the nonlinearity, we consider in this article a modification of the nonlinear term as follows

$$u_t + u_x - u_{xx} + \frac{1}{\sqrt{\pi}} \frac{\partial}{\partial t} \int_0^t \frac{u(s)}{\sqrt{t-s}} ds = -u^2 u_x, \quad (3)$$

where $u(t, x)$ is a real valued function of the variables $t \geq 0$ and $x \in \mathbb{R}$. We study the local and the global well-posedness of the initial value problem for (3). Moreover, we examine the asymptotical behavior of the solutions.

The remaining of this article is as follows. Section 2 concerns some preliminaries and the theoretical results. In section 3, we prove the first main results concerning the well-posedness for the initial value problem under a smallness condition on the initial data using a fixed point argument. Section 4 deals with the second main results related to global well-posedness of the problem using a diffusive representation of the half-derivative in time. Finally, section 5 is concerned with numerical results and discussion.

2 Preliminaries and main results

We begin with introducing some notations. We denote by $L_x^p(\mathbb{R})$ the usual Lebesgue space and by $H_x^m(\mathbb{R})$ with $m \in \mathbb{R}$ the usual Sobolev space on the variable x . The Banach space $L_x^1(\mathbb{R}) \cap H_x^m(\mathbb{R})$ is endowed with norm $\|\cdot\|_{L_x^1(\mathbb{R}) \cap H_x^m(\mathbb{R})} = \|\cdot\|_{L_x^1(\mathbb{R})} + \|\cdot\|_{H_x^m(\mathbb{R})}$. Also we stand by $C^{k+\alpha}(\mathbb{R})$ the usual Hölder space with $k \in \mathbb{N}$ and $\alpha \in (0, 1)$. We set $\langle t \rangle = \sqrt{1+t^2}$.

2.1 First main results using the kernel representation

We introduce the convolution kernels

$$K_{RL}(t, x) = \frac{1}{2\sqrt{\pi t}} e^{-\frac{x^2}{4t}} e^{-x^-} \left(1 - \frac{1}{2} \int_0^{+\infty} e^{-\frac{\mu^2}{4t} - \frac{\mu|x|}{2t} - \frac{\mu}{2}} d\mu \right), \quad (4)$$

$$N(t, x) = \frac{1}{6\sqrt{\pi t}} \partial_x \left(e^{-\frac{x^2}{4t}} e^{-x^-} \left(1 - \frac{1}{2} \int_0^{+\infty} e^{-\frac{\mu^2}{4t} - \frac{\mu|x|}{2t} - \frac{\mu}{2}} d\mu \right) \right), \quad (5)$$

where $x^- = \frac{|x|-x}{2} = \max(-x, 0)$ presents the negative part of $x \in \mathbb{R}$.

We have the following estimations on the convolution kernels from Manoubi (2014).

Lemma 1 – *There exists a constant $C > 0$ such that for all $t > 0$*

$$\max(1, \sqrt{t}) \|K_{RL}(t, \cdot)\|_{L_x^1(\mathbb{R})} + \max(t^{1/4}, t^{3/4}) \|K_{RL}(t, \cdot)\|_{L_x^2(\mathbb{R})} \leq C,$$

and

$$\sqrt{t} \|N(t, \cdot)\|_{L_x^1(\mathbb{R})} + t^{3/4} \|N(t, \cdot)\|_{L_x^2(\mathbb{R})} \leq C.$$

At first, we consider the linear problem associated to (3). The following result is an improvement of that given in Manoubi (2014) when considering more regular initial data.

Theorem 1 – Consider the following Cauchy problem

$$\begin{cases} u_t + u_x + \frac{1}{\sqrt{\pi}} \frac{\partial}{\partial t} \int_0^t \frac{u(s)}{\sqrt{t-s}} ds = u_{xx}, \\ u(0) = u_0, \end{cases} \quad (6)$$

where the initial data $u_0 \in H_x^1(\mathbb{R})$. There exists a unique solution

$$u \in C(\mathbb{R}_+; H_x^1(\mathbb{R})) \cap C^{1/2}(\mathbb{R}_+; H_x^{-1}(\mathbb{R}))$$

for the problem (6) given by

$$u(t, x) = [K_{RL}(t, \cdot) \star u_0](x), \quad (7)$$

where $\star$ represents the usual convolution product and K_{RL} is given by (4). Besides, we have $u \in C^1([0, +\infty[; H_x^{-1}(\mathbb{R}))$.

Now, we state a first global well-posedness result for (3) supplemented with a small initial data u_0 .

Theorem 2 – For $u_0 \in H_x^1(\mathbb{R})$, there exists a unique local solution $u \in C([0, T]; H_x^1(\mathbb{R}))$ of (3). Moreover for $u_0 \in L_x^1(\mathbb{R}) \cap H_x^1(\mathbb{R})$, there exists a positive constant $C > 0$ that depends on u_0 such that if $\|u_0\|_{L_x^1(\mathbb{R}) \cap H_x^1(\mathbb{R})}$ is small enough, there exists a unique global solution $u \in C(\mathbb{R}_+; L_x^2(\mathbb{R}) \cap L_x^\infty(\mathbb{R})) \cap C^{1/2}(\mathbb{R}_+; H_x^{-1}(\mathbb{R}))$ of (3) given by

$$u(t, x) = [K_{RL}(t, \cdot) \star u_0](x) - \int_0^t N(t-s, \cdot) \star u^3(s, x) ds,$$

where K_{RL} and N are given by (4) and (5), respectively. In addition, the solution u satisfies

$$t^{1/4} \|u(t, \cdot)\|_{L_x^2(\mathbb{R})} + t >^{1/2} \|u(t, \cdot)\|_{L_x^\infty(\mathbb{R})} \leq C(\|u_0\|_{L_x^1(\mathbb{R}) \cap H_x^1(\mathbb{R})}).$$

We have the following decay estimate for the solution.

Proposition 1 – Let u_0 such that $\|u_0\|_{L_x^1(\mathbb{R}) \cap H_x^1(\mathbb{R})} < \varepsilon$ where ε is small enough. Then the global solution u defined in Theorem 2 satisfies the following decay estimate

$$\max(t^{1/4}, t^{3/4}) \|u(t, \cdot)\|_{L_x^2(\mathbb{R})} + \max(t^{1/2}, t) \|u(t, \cdot)\|_{L_x^\infty(\mathbb{R})} \leq C(\|u_0\|_{L_x^1(\mathbb{R}) \cap H_x^1(\mathbb{R})}).$$

2. Preliminaries and main results

Remark 1 – We emphasize that the results stated above for the cubic case may be extended to the general case $p > 2$ due to the Sobolev embedding $H_x^1(\mathbb{R}) \hookrightarrow L_x^\infty(\mathbb{R})$. More precisely, the following well-posedness result holds for the nonlinear equation

$$u_t + u_x - u_{xx} + \frac{1}{\sqrt{\pi}} \frac{\partial}{\partial t} \int_0^t \frac{u(s)}{\sqrt{t-s}} ds = -u^p u_x, \quad (8)$$

supplemented with a small initial data u_0 .

Theorem 3 – Let $p > 2$ and $u_0 \in L_x^1(\mathbb{R}) \cap H_x^1(\mathbb{R})$. There exists a positive constant $C > 0$ that depends on u_0 such that if $\|u_0\|_{L_x^1(\mathbb{R}) \cap H_x^1(\mathbb{R})}$ is small enough, there exists a unique global solution $u \in C(\mathbb{R}_+; L_x^2(\mathbb{R}) \cap L_x^\infty(\mathbb{R})) \cap C^{1/2}(\mathbb{R}_+; H_x^{-1}(\mathbb{R}))$ of (3) given by

$$u(t, x) = [K_{RL}(t, \cdot) \star u_0](x) - N_p \otimes u^{p+1}(t, x),$$

where K_{RL} is given by (4) and

$$N_p(t, x) = \frac{1}{2(p+1)\sqrt{\pi}t} \partial_x \left(e^{-\frac{x^2}{4t}} e^{-x^-} \left(1 - \frac{1}{2} \int_0^{+\infty} e^{-\frac{\mu^2}{4t} - \frac{\mu|x|}{2} - \frac{\mu}{2}} d\mu \right) \right).$$

In addition, the solution u satisfies the same decay estimates as those given in Proposition 1.

2.2 Second main results using the diffusive representation

In this subsection, we state a second global well-posedness result for the modified problem (3) without a smallness condition on the initial data u_0 using the so-called diffusive representation. We introduce here a new variable $y \in \mathbb{R}$ that is not physically relevant. Let u be a function that depends on t and on x . We define the vector space

$$X = \left\{ v \in L^2(\mathbb{R}^2) \mid (x, y) \mapsto \frac{\partial v}{\partial y} \in L^2(\mathbb{R}^2) \right\}.$$

We note that X is a Hilbert space for the scalar product

$$\forall v, w \in X, \quad (v, w)_X = (v, w)_{L^2(\mathbb{R}^2)} + \left(\frac{\partial v}{\partial y}, \frac{\partial w}{\partial y} \right)_{L^2(\mathbb{R}^2)}.$$

Hence, the following application

$$\begin{aligned} \Gamma : X &\rightarrow L^2(\mathbb{R}) \\ \phi(t) &\mapsto \Gamma(\phi)(t) = \phi(t, y = 0), \end{aligned}$$

is a linear and continuous operator. Its dual map Γ^* is defined as

$$\begin{aligned}\Gamma^* : L_x^2(\mathbb{R}) &\longrightarrow X' \\ u(t) &\longmapsto \Gamma^*(u)(t) = u(t) \otimes \delta_{y=0},\end{aligned}$$

where X' is the dual space of X , $\delta_{y=0}$ is the Dirac delta function at $y = 0$ and $u(t) \otimes \delta_{y=0}$ is the tensorial product in the distributions sense of the applications $t \mapsto u(t)$ and $y \mapsto \delta_{y=0}$.

A diffusive representation of the half-order integral $I^{1/2}u(t)$ is given by

$$I^{1/2}u(t) = 2\Gamma(\phi)(t),$$

where Φ is a solution of the diffusive problem

$$\begin{cases} \partial_t \Phi(t, y) = \Phi_{yy}(t, y) + \Gamma^*(u)(t), \\ \Phi(0, y) = 0, \quad y \in \mathbb{R}. \end{cases}$$

Hence, the modified model (3) reads as follows

$$\begin{cases} u_t + 2\partial_t(\Gamma(\Phi)) + u_x - u_{xx} + u^2u_x = 0, \\ \Phi_t - \Phi_{yy} - \Gamma^*(u) = 0, \end{cases} \quad (9)$$

supplemented with the initial data $u(0) = u_0$ and $\Phi(0) = 0$.

Now, we state the second global well-posedness result.

Theorem 4 – *For all initial data $u_0 \in H_x^1(\mathbb{R})$, the problem (9) has a unique global weak solution (u, Φ) . Moreover,*

$$\begin{aligned}u &\in L^\infty(\mathbb{R}_+, H_x^1(\mathbb{R})) \cap C_w(\mathbb{R}_+, L_x^2(\mathbb{R})), \\ u_x &\in L^2(\mathbb{R}_+, H_x^1(\mathbb{R})), \\ \forall T > 0, \quad \Phi &\in L^\infty([0, T], X) \cap C(\mathbb{R}_+, \mathbb{L}_{x,y}^2(\mathbb{R}^2)).\end{aligned}$$

Remark 2 – The particular role of the cubic nonlinearity ($p = 2$) in theorem 4 is worth noting. For the generalized KdV equation, a different behavior is observed in the L^2 -critical case ($p = 4$), where finite-time blow-up solutions are known (see Martel and Merle (2002) and Merle (2001) and references therein for more details). This provides a motivation for investigating the role of the nonlinear term in the global behavior of solutions for the general model (8). In addition, the sharpness of the cubic case for the present model remains an interesting open question.

3 Proof of First main results

3.1 Proof of Theorem 1

Following Manoubi (2014), a solution of (6) is formally given by (7). Let $u_0 \in H_x^1(\mathbb{R})$. The well-posedness of the linear problem (7) may be justified following the proof of

3. Proof of First main results

Theorem 2.1 in Manoubi (2014). Now, we study the regularity of the solution for initial data in $H_x^1(\mathbb{R})$. Let $t_0 > 0$ and t such that $t \rightarrow t_0$. We have

$$\begin{aligned} \|u(t, \cdot) - u(t_0, \cdot)\|_{H_x^1(\mathbb{R})} &= \|K_{RL}(t, \cdot) \star u_0 - K_{RL}(t_0, \cdot) \star u_0\|_{H_x^1(\mathbb{R})} \\ &\leq \|K_{RL}(t, \cdot) - K_{RL}(t_0, \cdot)\|_{L_x^1(\mathbb{R})} \|u_0\|_{H_x^1(\mathbb{R})}. \end{aligned}$$

Here, the function $t \mapsto K_{RL}(t, x)$ is continuous for all $t > 0$. Then using the Lebesgue dominated convergence theorem, we get

$$\lim_{t \rightarrow t_0} \|u(t, x) - u(t_0, \cdot)\|_{H_x^1(\mathbb{R})} = 0.$$

On $t_0 = 0$, since $K_{RL}(t, \cdot)$ is an approximate of the identity, then

$$\lim_{t \rightarrow 0^+} \|K_{RL}(t, \cdot) \star u_0 - u_0\|_{H_x^1(\mathbb{R})} = 0,$$

which implies that $u \in C(\mathbb{R}_+; H_x^1(\mathbb{R}))$.

To prove that $u \in C^{1/2}(\mathbb{R}_+; H_x^{-1}(\mathbb{R}))$, we apply the Fourier transform in space to (6). Following Manoubi (2014), we establish that, for all $t \geq 0$,

$$\hat{u}(t, \xi) \in C^{1/2}(\mathbb{R}_+, L^2((1 + |\xi|^2)^{-1} d\xi)(\mathbb{R})).$$

Using Plancherel formula, we conclude that the solution u is in $C^{1/2}(\mathbb{R}_+, H_x^{-1}(\mathbb{R}))$. Moreover, we observe that $\hat{u}(t, \xi) \in C^1([0, +\infty[, L_\xi^2((1 + |\xi|^2)^{-1} d\xi)(\mathbb{R}))$. Hence we deduce that $u \in C^1([0, +\infty[, H_x^{-1}(\mathbb{R}))$.

3.2 Proof of Theorem 2

The proof is divided into two steps. The first one, we establish the well-posedness of the linear inhomogeneous problem. Then, we establish the well-posedness of the nonlinear problem using a fixed point method.

The linear inhomogeneous problem

We start by given the results associated to the linear inhomogeneous problem. Consider

$$u_t + u_x - u_{xx} + \frac{1}{\sqrt{\pi}} \frac{\partial}{\partial t} \int_0^t \frac{u(s)}{\sqrt{t-s}} ds = \frac{1}{3} (f^3)_x, \quad (10)$$

supplemented with the initial condition $u(0, x) = 0$ for all $x \in \mathbb{R}$. We establish the following result.

Proposition 2 – Consider a function $f \in C(\mathbb{R}_+; H_x^1(\mathbb{R}))$, then there exists a unique solution u to (10) given by

$$u(t, x) = \int_0^t N(t-s, \cdot) \star f^3(s, x) ds. \quad (11)$$

Moreover, we have $u \in C(\mathbb{R}_+; H_x^1(\mathbb{R})) \cap C^1(\mathbb{R}_+; H_x^{-1}(\mathbb{R}))$.

Proof. From Manoubi (2014), a solution of (10) is formally given by (11). It is classical now to prove the existence and the uniqueness of the solution. In the sequel, we study the regularity of the solution. Let $t_0 > 0$ and t such that $t \rightarrow t_0$. We have

$$\begin{aligned} & \|u(t, \cdot) - u(t_0, \cdot)\|_{H_x^1(\mathbb{R})} \\ &= \left\| \int_0^t N(s, \cdot) \star f^3(t-s, \cdot) ds - \int_0^{t_0} N(s, \cdot) \star f^3(t_0-s, \cdot) ds \right\|_{H_x^1(\mathbb{R})} \\ &\leq \int_0^{t_0} \|N(s, \cdot) \star (f^3(t-s, \cdot) - f^3(t_0-s, \cdot))\|_{H_x^1(\mathbb{R})} ds \\ &\quad + \int_{t_0}^t \|N(s, \cdot) \star f^3(t-s, \cdot)\|_{H_x^1(\mathbb{R})} ds. \end{aligned}$$

On the one hand, using Sobolev inequalities and Lemma 1, it results

$$\begin{aligned} & \int_0^{t_0} \|N(s, \cdot) \star (f^3(t-s, \cdot) - f^3(t_0-s, \cdot))\|_{H_x^1(\mathbb{R})} ds \\ &\leq \int_0^{t_0} \|N(s, \cdot)\|_{L_x^1(\mathbb{R})} \|f^3(t-s, \cdot) - f^3(t_0-s, \cdot)\|_{H_x^1(\mathbb{R})} ds \\ &\leq C \sup_{v \in [0, t_0]} \|f(v, \cdot)\|_{H_x^1(\mathbb{R})}^2 \int_0^{t_0} \frac{1}{s^{1/2}} \|f(t-s, \cdot) - f(t_0-s, \cdot)\|_{H_x^1(\mathbb{R})} ds. \end{aligned}$$

We observe that $s \mapsto \frac{1}{s^{1/2}}$ is integrable on $[0, t_0]$. Hence, using the fact that f lies in $C(\mathbb{R}_+, H_x^1(\mathbb{R}))$ and the dominated convergence theorem, we deduce that

$$\lim_{t \rightarrow t_0} \int_0^{t_0} \|N(s, \cdot) \star (f^3(t-s, \cdot) - f^3(t_0-s, \cdot))\|_{H_x^1(\mathbb{R})} ds = 0. \quad (12)$$

On the other hand,

$$\begin{aligned} & \int_{t_0}^t \|N(s, \cdot) \star f^3(t-s, \cdot)\|_{H_x^1(\mathbb{R})} ds \leq \int_{t_0}^t \|N(s, \cdot)\|_{L_x^1(\mathbb{R})} \|f(t-s, \cdot)\|_{H_x^1(\mathbb{R})}^3 ds \\ &\leq C \sup_{v \in [0, t_0]} \|f(v, \cdot)\|_{H_x^1(\mathbb{R})}^3 \int_{t_0}^t \frac{ds}{s^{1/2}}. \end{aligned}$$

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Passing to the limit when t tends to t_0 , we deduce that

$$\lim_{t \rightarrow t_0} \int_{t_0}^t \|N(s, \cdot) \star f^3(t-s, \cdot)\|_{H_x^1(\mathbb{R})} ds = 0. \quad (13)$$

It follows from (12) and (13) that $u \in C(\mathbb{R}_+; H_x^1(\mathbb{R}))$.

To prove that $u \in C^1(\mathbb{R}_+; H_x^{-1}(\mathbb{R}))$, we apply the Fourier transform in space to (6). Since f belongs to $C(\mathbb{R}_+, H_x^1(\mathbb{R}))$ then f^3 belongs also to $C(\mathbb{R}_+, H_x^1(\mathbb{R}))$. Using the fact that $u \in C(\mathbb{R}_+, H_x^1(\mathbb{R}))$, we establish that, for all $t \geq 0$, $\hat{u}_t(t, \xi)$ belongs to $C(\mathbb{R}_+, L^2((1 + |\xi|^2)^{-1} d\xi)(\mathbb{R}))$. This ends the proof of the Proposition. $\square$

The formal solution

Formally, we have that a solution of (3) supplemented with an initial condition $u_0 \in H_x^1(\mathbb{R})$ or $u_0 \in H_x^1(\mathbb{R}) \cap L_x^1(\mathbb{R})$ is a solution of the following (mild) Duhamel formulation

$$u(t, x) = [K_{RL}(t, \cdot) \star u_0](x) - \int_0^t N(t-s, \cdot) \star u^3(s, x) ds. \quad (14)$$

We solve (14) by applying a contraction principle using a Picard fixed point theorem to a functional Φ defined by

$$\Phi(u)(t, x) = [K_{RL}(t, \cdot) \star u_0](x) - \int_0^t N(t-s, \cdot) \star u^3(s, x) ds,$$

in an appropriate metric space that we will define in the sequel.

Local existence

In this paragraph, we justify the local existence of a solution to (3) with an initial condition in $H_x^1(\mathbb{R})$. Let $T > 0$ and set

$$X_T = C(0, T; H_x^1(\mathbb{R})) = \{u : [0, T] \rightarrow H_x^1(\mathbb{R}) \text{ continuous} \}.$$

X_T is a Banach space when endowed with the norm $\|u\|_{X_T} := \sup_{t \in [0, T]} \|u(t)\|_{H_x^1(\mathbb{R})}$. In the following we give some properties satisfied by the functional Φ on X_T with $u_0 \in H_x^1(\mathbb{R})$. Let $u \in X_T$, we have

$$\|K_{RL}(t, \cdot) \star u_0\|_{H_x^1(\mathbb{R})} \leq \|K_{RL}(t, \cdot)\|_{L_x^1(\mathbb{R})} \|u_0\|_{H_x^1(\mathbb{R})}.$$

Using Lemma 1, we get

$$\|K_{RL}(t, \cdot) \star u_0\|_{H_x^1(\mathbb{R})} \leq C \min(1, \frac{1}{\sqrt{t}}) \|u_0\|_{H_x^1(\mathbb{R})}.$$

Since we seek for a local solution, we choose that

$$\forall t \in [0, T], \quad \|K_{RL}(t, \cdot) \star u_0\|_{H_x^1(\mathbb{R})} \leq C_1 \|u_0\|_{H_x^1(\mathbb{R})}. \quad (15)$$

Moreover, using Lemma 1, we have

$$\begin{aligned} \left\| \int_0^t N(t-s, \cdot) \star u^3(s, x) ds \right\|_{H_x^1(\mathbb{R})} &\leq \int_0^t \|N(t-s, \cdot) \star u^3(s)\|_{H_x^1(\mathbb{R})} ds \\ &\leq \int_0^t \|N(t-s)\|_{L_x^1(\mathbb{R})} \|u^3(s)\|_{H_x^1(\mathbb{R})} ds \\ &\leq C \int_0^t \frac{1}{\sqrt{t-s}} \|u(s)\|_{H_x^1(\mathbb{R})}^3 ds \\ &\leq C \sqrt{t} \|u\|_{X_T}^3 ds. \end{aligned}$$

This implies that

$$\left\| \int_0^t N(t-s, \cdot) \star u^3(s, x) ds \right\|_{X_T} \leq C_2 \sqrt{T} \|u\|_{X_T}^3. \quad (16)$$

By (15) and (16) we deduce that, for all $u \in X_T$,

$$\|\Phi(u)\|_{X_T} \leq C_1 \|u_0\|_{H_x^1(\mathbb{R})} + C_2 \sqrt{T} \|u\|_{X_T}^3. \quad (17)$$

Let u and v in X_T then for all $t > 0$ we have

$$\begin{aligned} \|\Phi(u)(t, \cdot) - \Phi(v)(t, \cdot)\|_{H_x^1(\mathbb{R})} &\leq \left\| \int_0^t N(t-s) \star (u^3(s) - v^3(s)) ds \right\|_{H_x^1(\mathbb{R})} \\ &\leq \int_0^t \|N(t-s)\|_{L_x^1(\mathbb{R})} \|u^3(s) - v^3(s)\|_{H_x^1(\mathbb{R})} ds \\ &\leq C \int_0^t \frac{1}{\sqrt{t-s}} \|u^3(s) - v^3(s)\|_{H_x^1(\mathbb{R})} ds. \end{aligned}$$

Using Sobolev inequalities, we get

$$\begin{aligned} \|u^3(s) - v^3(s)\|_{H_x^1(\mathbb{R})} &\leq \|u(s) - v(s)\|_{H_x^1(\mathbb{R})} \left(\|u^2(s)\|_{H_x^1(\mathbb{R})} + \|v^2(s)\|_{H_x^1(\mathbb{R})} \right) \\ &\leq \|u(s) - v(s)\|_{H_x^1(\mathbb{R})} \left(\|u(s)\|_{H_x^1(\mathbb{R})}^2 + \|v(s)\|_{H_x^1(\mathbb{R})}^2 \right). \end{aligned}$$

This implies that

$$\|\Phi(u) - \Phi(v)\|_{X_T} \leq C_3 \sqrt{T} \|u - v\|_{X_T} \left(\|u\|_{X_T}^2 + \|v\|_{X_T}^2 \right). \quad (18)$$

3. Proof of First main results

Now we define a set B_0 invariant under the action of Φ . Let $u_0 \in H_x^1(\mathbb{R})$ and take $R_0 = 2C_1\|u_0\|_{H^1}$. Let B_0 be the closed ball in X_T of radius R_0 centered at the origin. Thanks to (17) and (18) we get

$$\forall u \in B_0 \quad \|\Phi(u)\|_{X_T} \leq C_1\|u_0\|_{H_x^1(\mathbb{R})} + C_2\sqrt{T}R_0^3,$$

and

$$\forall u, v \in B_0 \quad \|\Phi(u) - \Phi(v)\|_{X_T} \leq C_3R_0^2\sqrt{T}\|u - v\|_{X_T}.$$

In conclusion, we take $T > 0$ small such that $C_{R_0} = \max(C_2, C_3)R_0^2\sqrt{T} \leq \frac{1}{2}$. It follows that $\Phi(B_0) \subset B_0$ and thus the map Φ is a contraction on B_0 . Using the fixed point theorem we deduce that there exists a unique fixed point u of the functional Φ in the ball B_0 .

In the sequel we prove the global existence of the solution.

Global existence

In the following, we assume that $u_0 \in L_x^1(\mathbb{R}) \cap H_x^1(\mathbb{R})$. We define the Banach space

$$X := \{v \in C(\mathbb{R}_+; L_x^2(\mathbb{R})) / \sup_{t>0} t^{1/4} \|v(t, \cdot)\|_{L_x^2(\mathbb{R})} < \infty \text{ and } \sup_{t>0} \langle t \rangle^{1/2} \|v(t, \cdot)\|_{L_x^\infty(\mathbb{R})} < \infty\},$$

endowed with the norm $\|v\|_X = \sup_{t>0} t^{1/4} \|v(t, \cdot)\|_{L_x^2(\mathbb{R})} + \sup_{t>0} \langle t \rangle^{1/2} \|v(t, \cdot)\|_{L_x^\infty(\mathbb{R})}$.

Now we give some properties satisfied by Φ on X when $u_0 \in L_x^1(\mathbb{R}) \cap H_x^1(\mathbb{R})$. First we have for all $t \geq 0$

$$\|K_{RL}(t, \cdot) \star u_0\|_{L_x^\infty(\mathbb{R})} \leq \|K_{RL}(t, \cdot)\|_{L_x^1(\mathbb{R})} \|u_0\|_{L_x^\infty(\mathbb{R})}.$$

Using Lemma 1, we have

$$\|K_{RL}(t, \cdot) \star u_0\|_{L_x^\infty(\mathbb{R})} \leq C \min(1, \frac{1}{\sqrt{t}}) \|u_0\|_{L_x^\infty(\mathbb{R})}.$$

Then, if $t < 1$, we get

$$\|K_{RL}(t, \cdot) \star u_0\|_{L_x^\infty(\mathbb{R})} \leq C \|u_0\|_{L_x^\infty(\mathbb{R})}. \quad (19)$$

However, if $t > 1$, we get

$$\|K_{RL}(t, \cdot) \star u_0\|_{L_x^\infty(\mathbb{R})} \leq \frac{C}{\sqrt{t}} \|u_0\|_{L_x^\infty(\mathbb{R})}. \quad (20)$$

We recall that for $t \geq 1$ and $\alpha > 0$

$$t^\alpha \leq \langle t \rangle^\alpha \leq 2^{\alpha/2} t^\alpha. \quad (21)$$

Hence, by applying the previous estimate with $\alpha = \frac{1}{2}$ in (20) and taking into account (19), we deduce

$$\forall t > 0, \quad \|K_{RL}(t, \cdot) \star u_0\|_{L_x^\infty(\mathbb{R})} \leq \frac{C_1}{\langle t \rangle^{1/2}} \|u_0\|_{L_x^\infty(\mathbb{R})}.$$

Due to the continuous embedding $H_x^1(\mathbb{R}) \hookrightarrow L_x^\infty(\mathbb{R})$, we have

$$\forall t > 0, \quad \|K_{RL}(t, \cdot) \star u_0\|_{L_x^\infty(\mathbb{R})} \leq \frac{C_1}{\langle t \rangle^{1/2}} \|u_0\|_{H_x^1(\mathbb{R})}. \quad (22)$$

In addition, let $u \in X$ then for all $t \geq 0$ we have

$$\left\| \int_0^t N(t-s, \cdot) \star u^3(s, x) ds \right\|_{L_x^\infty(\mathbb{R})} \leq \int_0^t \|N(t-s, \cdot) \star u^3(s)\|_{L_x^\infty(\mathbb{R})} ds.$$

Let $t < 1$, then using Lemma 1 and Hölder inequality, we get

$$\begin{aligned} \left\| \int_0^t N(t-s, \cdot) \star u^3(s, x) ds \right\|_{L_x^\infty(\mathbb{R})} &\leq \int_0^t \|N(t-s)\|_{L_x^2(\mathbb{R})} \|u^3(s)\|_{L_x^2(\mathbb{R})} ds \\ &\leq C \int_0^t \frac{1}{(t-s)^{\frac{3}{4}}} \|u(s)\|_{L_x^2(\mathbb{R})} \|u(s)\|_{L_x^\infty(\mathbb{R})}^2 ds \\ &\leq C \left(\int_0^t \frac{1}{(t-s)^{\frac{3}{4}} s^{\frac{1}{4}} < s >} ds \right) \|u\|_X^3. \end{aligned}$$

Taking into account that $\int_0^t \frac{ds}{(t-s)^{\frac{3}{4}} s^{\frac{1}{4}} < s >} \leq \int_0^t \frac{ds}{(t-s)^{\frac{3}{4}} s^{\frac{1}{4}}} \leq C$, it follows

$$\forall t < 1, \quad \left\| \int_0^t N(t-s, \cdot) \star u^3(s, x) ds \right\|_{L_x^\infty(\mathbb{R})} \leq C \|u\|_X^3. \quad (23)$$

Now, let $t > 1$. Using Lemma 1 and Hölder inequality, we get

$$\begin{aligned} \int_0^t \|N(t-s) \star u^3(s)\|_{L_x^\infty(\mathbb{R})} ds &\leq \int_0^t \|N(t-s)\|_{L_x^2(\mathbb{R})} \|u(s)\|_{L_x^2(\mathbb{R})} \|u(s)\|_{L_x^\infty(\mathbb{R})}^2 ds \\ &\leq C \int_0^t \frac{\sqrt{s} \|u(s)\|_{L_x^\infty(\mathbb{R})}}{(t-s)^{\frac{3}{4}} s^{\frac{1}{4}} s^{\frac{1}{2}} < s >^{\frac{1}{2}}} ds \|u\|_X^2. \end{aligned}$$

Using the fact that

$$\forall 0 \leq s \leq t, \quad \sqrt{s} \leq \sqrt{\langle s \rangle} \leq \sqrt{\langle t \rangle},$$

we get

$$\int_0^t \|N(t-s) \star u^3(s)\|_{L_x^\infty(\mathbb{R})} ds \leq C \int_0^t \frac{ds}{(t-s)^{\frac{3}{4}} s^{\frac{3}{4}}} \|u\|_X^3.$$

3. Proof of First main results

It follows that

$$\forall t > 1, \quad \left\| \int_0^t N(t-s, \cdot) \star u^3(s, x) ds \right\|_{L_x^\infty(\mathbb{R})} \leq \frac{C}{\sqrt{t}} \|u\|_X^3. \quad (24)$$

Hence, from (23), (21) and (24), we deduce

$$\forall t > 0, \quad \left\| \int_0^t N(t-s, \cdot) \star u^3(s, x) ds \right\|_{L_x^\infty(\mathbb{R})} \leq \frac{C_2}{\langle t \rangle^{1/2}} \|u\|_X^3. \quad (25)$$

Now, we seek for similar estimations en L^2 -norm. First, using Lemma 1 and Hölder inequality, we have

$$\|K_{RL}(t, \cdot) \star u_0\|_{L_x^2(\mathbb{R})} \leq C \|K_{RL}(t, \cdot)\|_{L_x^2(\mathbb{R})} \|u_0\|_{L_x^1(\mathbb{R})} \leq C \min\left(\frac{1}{t^{1/4}}, \frac{1}{t^{3/4}}\right) \|u_0\|_{L_x^1(\mathbb{R})}.$$

This implies that

$$\forall t > 0, \quad \|K_{RL}(t, \cdot) \star u_0\|_{L_x^2(\mathbb{R})} \leq \frac{C_3}{t^{1/4}} \|u_0\|_{L_x^1(\mathbb{R})}. \quad (26)$$

In addition, using Lemma 1 and Hölder inequality, we get

$$\begin{aligned} \left\| \int_0^t N(t-s, \cdot) \star u^3(s, x) ds \right\|_{L_x^2(\mathbb{R})} &\leq \int_0^t \|N(t-s, \cdot) \star u^3(s)\|_{L_x^2(\mathbb{R})} ds \\ &\leq \int_0^t \|N(t-s, \cdot)\|_{L_x^2(\mathbb{R})} \|u^3(s)\|_{L_x^1(\mathbb{R})} ds \\ &\leq \int_0^t \|N(t-s, \cdot)\|_{L_x^2(\mathbb{R})} \|u(s)\|_{L_x^2(\mathbb{R})}^2 \|u(s)\|_{L_x^\infty(\mathbb{R})} ds \\ &\leq C \left(\int_0^t \frac{ds}{(t-s)^{\frac{3}{4}} s^{\frac{1}{2}} < s >^{\frac{1}{2}}} \right) \|u\|_X^3. \end{aligned}$$

Taking into account that $\int_0^t \frac{ds}{(t-s)^{\frac{3}{4}} s^{\frac{1}{2}} < s >^{\frac{1}{2}}} \leq \int_0^t \frac{ds}{(t-s)^{\frac{3}{4}} s^{\frac{1}{2}}} \leq \frac{C}{t^{\frac{1}{4}}}$, it follows that

$$\forall t > 0, \quad \left\| \int_0^t N(t-s, \cdot) \star u^3(s, x) ds \right\|_{L_x^2(\mathbb{R})} \leq \frac{C_4}{t^{\frac{1}{4}}} \|u\|_X^3. \quad (27)$$

Hence, recalling (22), (25), (26) and (27), we get for all $u \in X$

$$t^{1/4} \|\Phi(u(s))\|_{L_x^2(\mathbb{R})} + \langle t \rangle^{1/2} \|\Phi(u(s))\|_{L_x^\infty(\mathbb{R})} \leq A_1 \|u_0\|_{L_x^1(\mathbb{R}) \cap H_x^1(\mathbb{R})} + A_2 \|u\|_X^3. \quad (28)$$

Next, we seek estimations to deal with the contraction of the function Φ . Let u and v be in X and $t \geq 0$, we have

$$\|\Phi(u)(t, \cdot) - \Phi(v)(t, \cdot)\|_{L_x^\infty(\mathbb{R})} \leq \int_0^t \|N(t-s) \star (u^3(s) - v^3(s))\|_{L_x^\infty(\mathbb{R})} ds.$$

Let $t < 1$, then using Lemma 1 and Hölder inequality, we get

$$\begin{aligned}
\|\Phi(u)(t, \cdot) - \Phi(v)(t, \cdot)\|_{L_x^\infty(\mathbb{R})} &\leq \int_0^t \|N(t-s)\|_{L_x^2(\mathbb{R})} \|u^3(s) - v^3(s)\|_{L_x^2(\mathbb{R})} ds \\
&\leq C \int_0^t \frac{1}{(t-s)^{\frac{3}{4}}} \|u(s) - v(s)\|_{L_x^2(\mathbb{R})} \left(\|u(s)\|_{L_x^\infty(\mathbb{R})}^2 + \|v(s)\|_{L_x^\infty(\mathbb{R})}^2 \right) ds \\
&\leq C \int_0^t \frac{ds}{(t-s)^{\frac{3}{4}} s^{\frac{1}{4}} < s >} \|u - v\|_X \left(\|u\|_X^2 + \|v\|_X^2 \right).
\end{aligned}$$

It follows that

$$\forall t < 1, \quad \|\Phi(u)(t, \cdot) - \Phi(v)(t, \cdot)\|_{L_x^\infty(\mathbb{R})} \leq C \|u - v\|_X \left(\|u\|_X^2 + \|v\|_X^2 \right). \quad (29)$$

Now, let $t > 1$, then using Lemma 1 and Hölder inequality, we get

$$\begin{aligned}
\|\Phi(u)(t, \cdot) - \Phi(v)(t, \cdot)\|_{L_x^\infty(\mathbb{R})} &\leq \int_0^t \|N(t-s) \star (u^3(s) - v^3(s))\|_{L_x^\infty(\mathbb{R})} ds \\
&\leq \int_0^t \|N(t-s)\|_{L_x^2(\mathbb{R})} \|u(s) - v(s)\|_{L_x^2(\mathbb{R})} \left(\|u(s)\|_{L_x^\infty(\mathbb{R})}^2 + \|v(s)\|_{L_x^\infty(\mathbb{R})}^2 \right) ds \\
&\leq C \int_0^t \frac{1}{(t-s)^{\frac{3}{4}} s^{\frac{3}{4}} < s >^{\frac{1}{2}}} \left(\sqrt{s} \|u(s)\|_{L_x^\infty(\mathbb{R})} + \sqrt{s} \|v(s)\|_{L_x^\infty(\mathbb{R})} \right) ds \\
&\quad \times \|u - v\|_X (\|u\|_X + \|v\|_X).
\end{aligned}$$

Since we have

$$\forall 0 \leq s \leq t, \quad \sqrt{s} \leq \sqrt{< s >} \leq \sqrt{< t >},$$

then we get

$$\|\Phi(u)(t, \cdot) - \Phi(v)(t, \cdot)\|_{L_x^\infty(\mathbb{R})} ds \leq C \int_0^t \frac{ds}{(t-s)^{\frac{3}{4}} s^{\frac{3}{4}}} \|u - v\|_X \left(\|u\|_X^2 + \|v\|_X^2 \right).$$

It follows that

$$\forall t > 1, \quad \|\Phi(u)(t, \cdot) - \Phi(v)(t, \cdot)\|_{L_x^\infty(\mathbb{R})} \leq \frac{C}{\sqrt{t}} \|u - v\|_X \left(\|u\|_X^2 + \|v\|_X^2 \right). \quad (30)$$

Now, recalling (29) and (30), we deduce

$$\forall t > 0, \quad \|\Phi(u)(t, \cdot) - \Phi(v)(t, \cdot)\|_{L_x^\infty(\mathbb{R})} \leq \frac{C_5}{< t >^{1/2}} \left(\|u\|_X^2 + \|v\|_X^2 \right) \|u - v\|_X. \quad (31)$$

3. Proof of First main results

Next, we seek estimations on L^2 -norm. Let u and v be in X and $t \geq 0$, we have

$$\begin{aligned}
& \|\Phi(u)(t, \cdot) - \Phi(v)(t, \cdot)\|_{L_x^2(\mathbb{R})} \\
& \leq \int_0^t \|N(t-s) \star (u^3(s) - v^3(s))\|_{L_x^2(\mathbb{R})} ds \\
& \leq \int_0^t \|N(t-s)\|_{L_x^2(\mathbb{R})} \|u^3(s) - v^3(s)\|_{L_x^1(\mathbb{R})} ds \\
& \leq C \int_0^t \frac{1}{(t-s)^{3/4}} \|u(s) - v(s)\|_{L_x^2(\mathbb{R})} \left(\|u^2(s)\|_{L_x^2(\mathbb{R})} + \|v^2(s)\|_{L_x^2(\mathbb{R})} \right) ds.
\end{aligned}$$

Using the fact that $\|u^2(s)\|_{L_x^2(\mathbb{R})} \leq \|u(s)\|_{L_x^2(\mathbb{R})} \|u(s)\|_{L_x^\infty(\mathbb{R})}$, we get

$$\|\Phi(u)(t, \cdot) - \Phi(v)(t, \cdot)\|_{L_x^2(\mathbb{R})} \leq C \left(\int_0^t \frac{ds}{(t-s)^{3/4} s^{1/2}} \right) \left(\|u\|_X^2 + \|v\|_X^2 \right) \|u - v\|_X.$$

It follows that

$$\forall t > 0, \quad \|\Phi(u)(t, \cdot) - \Phi(v)(t, \cdot)\|_{L_x^2(\mathbb{R})} \leq \frac{C_6}{t^{1/4}} \left(\|u\|_X^2 + \|v\|_X^2 \right) \|u - v\|_X. \quad (32)$$

Using (31) and (32) we get: for all $t > 0$

$$\begin{aligned}
t^{1/4} \|\Phi(u)(t, \cdot) - \Phi(v)(t, \cdot)\|_{L_x^2(\mathbb{R})} &+ < t^{1/2} \|\Phi(u)(t, \cdot) - \Phi(v)(t, \cdot)\|_{L_x^\infty(\mathbb{R})} \\
&\leq A_3 \left(\|u\|_X^2 + \|v\|_X^2 \right) \|u - v\|_X.
\end{aligned} \quad (33)$$

We denote by $B = \{u \in X; \|u\|_X \leq R\}$ the closed ball of X of radius R centered at the origin. Recalling (28) and (33), we get

$$\forall u \in B \quad \|\Phi(u)\|_X \leq A_1 \|u_0\|_{L_x^1(\mathbb{R}) \cap H_x^1(\mathbb{R})} + A_2 R^3,$$

and

$$\forall u, v \in B \quad \|\Phi(u) - \Phi(v)\|_X \leq A_3 R^2 \|u - v\|_X.$$

We choose $R = 2A_1 \|u_0\|_{L_x^1(\mathbb{R}) \cap H_x^1(\mathbb{R})}$. Hence, taking $\|u_0\|_{L_x^1(\mathbb{R}) \cap H_x^1(\mathbb{R})}$ small enough such that $C_R = \max(A_2, A_3) R^2 \leq \frac{1}{2}$. Hence, we get $\Phi(B) \subset B$, and thus the map Φ is a contraction on B . We deduce that Φ admits a unique fixed point in B given by (14).

3.3 Proof of Proposition 1

For all t and $s \in \mathbb{R}_+$, we define

$$M_2(t) = \sup_{s \leq t} \max(s^{1/4}, s^{3/4}) \|u(s, \cdot)\|_{L_x^2(\mathbb{R})} \quad \text{and} \quad M_\infty(t) = \sup_{s \leq t} \max(s^{1/2}, s) \|u(s, \cdot)\|_{L_x^\infty(\mathbb{R})}.$$

We proceed in two steps.

First step. Let $\varepsilon > 0$ be sufficiently small and let $u_0 \in L_x^1(\mathbb{R}) \cap H_x^1(\mathbb{R})$ such that $\|u_0\|_{L^1(\mathbb{R}) \cap H_x^1(\mathbb{R})} \leq \varepsilon$. We have the general result for all $t \in \mathbb{R}_+$,

$$t^{1/4} \|u(t, \cdot)\|_{L_x^2(\mathbb{R})} \leq C\varepsilon.$$

First, we consider the case $0 < t < 3$, we observe that

$$\max(t^{1/4}, t^{3/4}) \|u(t, \cdot)\|_{L_x^2(\mathbb{R})} \leq Ct^{1/4} \|u(t, \cdot)\|_{L_x^2(\mathbb{R})} \leq C\varepsilon.$$

Now, let $t \geq 2$, we have

$$\|u(t, \cdot)\|_{L_x^2(\mathbb{R})} \leq \|K_{RL}(t, \cdot) \star u_0\|_{L_x^2(\mathbb{R})} + \int_0^t \|N(t-s, \cdot) \star u^3(s, \cdot)\|_{L_x^2(\mathbb{R})} ds.$$

On the one hand

$$\|K_{RL}(t, \cdot) \star u_0\|_{L_x^2(\mathbb{R})} \leq \frac{C\varepsilon}{t^{3/4}}. \quad (34)$$

On the other hand,

$$\begin{aligned} \int_0^t \|N(t-s, \cdot) \star u^3(s, \cdot)\|_{L_x^2(\mathbb{R})} ds &\leq \int_0^1 \|N(t-s, \cdot) \star u^3(s, \cdot)\|_{L_x^2(\mathbb{R})} ds \\ &\quad + \int_1^t \|N(t-s, \cdot) \star u^3(s, \cdot)\|_{L_x^2(\mathbb{R})} ds. \end{aligned} \quad (35)$$

Now we estimate each integral on the right hand side of (35). First, since $0 \leq s \leq 1$, we have

$$\begin{aligned} \int_0^1 \|N(t-s, \cdot) \star u^3(s, \cdot)\|_{L_x^2(\mathbb{R})} ds &\leq \int_0^1 \|N(t-s)\|_{L_x^2(\mathbb{R})} \|u^3(s)\|_{L_x^1(\mathbb{R})} ds \\ &\leq \int_0^1 \|N(t-s)\|_{L_x^2(\mathbb{R})} \|u(s)\|_{L_x^2(\mathbb{R})}^2 \|u(s)\|_{L_x^\infty(\mathbb{R})} ds. \end{aligned}$$

We recall that, we have established that

$$\forall t > 0, \quad \|u(t, \cdot)\|_{L_x^\infty(\mathbb{R})} \leq \frac{C\varepsilon}{\sqrt{t}} \leq C\varepsilon. \quad (36)$$

Hence, using Hölder inequality, we get

$$\begin{aligned} \int_0^1 \|N(t-s, \cdot) \star u^3(s, \cdot)\|_{L_x^2(\mathbb{R})} ds &\leq \int_0^1 \frac{1}{(t-s)^{\frac{3}{4}} s^{\frac{1}{2}}} \left(s^{1/4} \|u(s, \cdot)\|_{L_x^2(\mathbb{R})} \right)^2 C\varepsilon ds \\ &\leq C\varepsilon \int_0^1 \frac{ds}{(t-s)^{\frac{3}{4}} s^{\frac{1}{2}}} M_2^2(1). \end{aligned}$$

3. Proof of First main results

Now, using a change of variables on the last integral, we get

$$\int_0^1 \|N(t-s, \cdot) \star u^3(s, \cdot)\|_{L_x^2(\mathbb{R})} ds \leq \frac{C\varepsilon}{t^{\frac{1}{4}}} \int_0^{1/t} \frac{d\sigma}{(1-\sigma)^{3/4} \sigma^{1/2}} M_2^2(t).$$

It follows that

$$\int_0^1 \|N(t-s, \cdot) \star u^3(s, \cdot)\|_{L_x^2(\mathbb{R})} ds \leq \frac{C\varepsilon}{t^{\frac{3}{4}}} M_2^2(t). \quad (37)$$

Moreover,

$$\begin{aligned} \int_1^t \|N(t-s, \cdot) \star u^3(s, \cdot)\|_{L_x^2(\mathbb{R})} ds &\leq \int_1^t \|N(t-s)\|_{L_x^2(\mathbb{R})} \|u^3(s)\|_{L_x^1(\mathbb{R})} ds \\ &\leq \int_1^t \|N(t-s)\|_{L_x^2(\mathbb{R})} \|u(s)\|_{L_x^2(\mathbb{R})}^2 \|u(s)\|_{L_x^\infty(\mathbb{R})} ds. \end{aligned}$$

Since for $1 \leq s \leq t$, we get

$$\begin{aligned} \int_1^t \|N(t-s, \cdot) \star u^3(s, \cdot)\|_{L_x^2(\mathbb{R})} ds &\leq \int_1^t \frac{1}{(t-s)^{\frac{3}{4} s^{\frac{3}{2}}}} \left(s^{3/4} \|u(s, \cdot)\|_{L_x^2(\mathbb{R})} \right)^2 C\varepsilon ds \\ &\leq C\varepsilon \int_1^t \frac{ds}{(t-s)^{\frac{3}{4} s^{\frac{3}{2}}}} M_2^2(t). \end{aligned}$$

Now, using a change of variables on the last integral, we get

$$\int_1^t \|N(t-s, \cdot) \star u^3(s, \cdot)\|_{L_x^2(\mathbb{R})} ds \leq \frac{C\varepsilon}{t^{\frac{5}{4}}} \int_{\frac{1}{t}}^1 \frac{d\sigma}{(1-\sigma)^{3/4} \sigma^{1/2}} M_2^2(t).$$

Since $t \geq 2$, it follows that

$$\int_1^{\frac{t}{2}} \|N(t-s, \cdot) \star u^3(s, \cdot)\|_{L_x^2(\mathbb{R})} ds \leq \frac{C\varepsilon}{t^{\frac{3}{4}}} M_2^2(t). \quad (38)$$

Collecting the estimates (34), (37) and (38), we conclude that

$$M_2(t) = t^{3/4} \|u(t, \cdot)\|_{L_x^2(\mathbb{R})} \leq C_2\varepsilon + C_1\varepsilon M_2^2(t).$$

We define $f(X) = C_1\varepsilon X^2 - X + C_2\varepsilon$. Hence M_2 satisfies the inequality $f(X) \geq 0$. Let $\Delta = 1 - 4C_1C_2\varepsilon^2$. We observe that for ε small enough that there exists two distinct positive roots of f , namely, $X_1 < X_2$. Since $M_2(0) = 0$ and $t \mapsto M_2(t)$ is continuous then $M_2(t) \in [0, X_1]$. Moreover, we can choose $\varepsilon > 0$ small enough such

that $f(2C_2\varepsilon) = C_2\varepsilon(4C_1C_2\varepsilon^2 - 1) < 0$ which implies that $M_2(t) \leq 2C_2\varepsilon$. Hence for $t \geq 2$, we have

$$\max(t^{1/4}, t^{3/4}) \|u(t, \cdot)\|_{L_x^2(\mathbb{R})} \leq C\varepsilon. \quad (39)$$

This completes the proof of the upper bound in L^2 norm.

Second step. Let $\|u_0\|_{L^1(\mathbb{R}) \cap H^1(\mathbb{R})} \leq \varepsilon$ with $\varepsilon > 0$ small. We have the general result for all $t \in \mathbb{R}_+$

$$\langle t \rangle^{1/2} \|u(t, \cdot)\|_{L_x^\infty(\mathbb{R})} \leq C\varepsilon.$$

Since, for all $t \geq 0$, we have $t \leq \langle t \rangle$, we deduce that

$$\forall t \geq 0, \quad t^{1/2} \|u(t, \cdot)\|_{L_x^\infty(\mathbb{R})} \leq C\varepsilon.$$

Now, let $0 < t < 3$, we observe that

$$\max(\sqrt{t}, t) \|u(t, \cdot)\|_{L_x^\infty(\mathbb{R})} \leq C\sqrt{t} \|u(t, \cdot)\|_{L_x^\infty(\mathbb{R})} \leq C\varepsilon.$$

Now let $t \geq 2$ then

$$\|u(t, \cdot)\|_{L_x^\infty(\mathbb{R})} \leq \|K_{RL}(t, \cdot) \star u_0\|_{L_x^\infty(\mathbb{R})} + \int_0^t \|N(t-s, \cdot) \star u^3(s, \cdot)\|_{L_x^\infty(\mathbb{R})} ds.$$

Concerning the linear kernel, we have

$$\|K_{RL}(t, \cdot) \star u_0\|_{L_x^\infty(\mathbb{R})} \leq \|K_{RL}(t, \cdot)\|_{L_x^1(\mathbb{R})} \|u_0\|_{L_x^\infty(\mathbb{R})} \leq \frac{C\varepsilon}{t^{1/2}}.$$

Besides

$$\|K_{RL}(t, \cdot) \star u_0\|_{L_x^\infty(\mathbb{R})} \leq \|K_{RL}(t, \cdot)\|_{L_x^\infty(\mathbb{R})} \|u_0\|_{L_x^1(\mathbb{R})} \leq \frac{C\varepsilon}{t}.$$

It results that

$$\|K_{RL}(t, \cdot) \star u_0\|_{L_x^\infty(\mathbb{R})} \leq C \min\left(\frac{1}{t^{1/2}}, \frac{1}{t}\right) \varepsilon \leq \frac{C\varepsilon}{t}. \quad (40)$$

We seek the estimation for the nonlinear kernel. We have

$$\begin{aligned} \int_0^t \|N(t-s, \cdot) \star u^3(s, \cdot)\|_{L_x^\infty(\mathbb{R})} ds &\leq \int_0^1 \|N(t-s, \cdot) \star u^3(s, \cdot)\|_{L_x^\infty(\mathbb{R})} ds \\ &\quad + \int_1^{\frac{t}{2}} \|N(t-s, \cdot) \star u^3(s, \cdot)\|_{L_x^\infty(\mathbb{R})} ds \\ &\quad + \int_{\frac{t}{2}}^t \|N(t-s, \cdot) \star u^3(s, \cdot)\|_{L_x^\infty(\mathbb{R})} ds. \end{aligned} \quad (41)$$

3. Proof of First main results

Now we estimate each integral on the right hand side of (41). First, since $0 \leq s \leq 1$, we have

$$\int_0^1 \|N(t-s, \cdot) \star u^3(s, \cdot)\|_{L_x^\infty(\mathbb{R})} ds \leq \int_0^1 \|N(t-s)\|_{L_x^\infty(\mathbb{R})} \|u(s)\|_{L_x^2(\mathbb{R})}^2 \|u(s)\|_{L_x^\infty(\mathbb{R})} ds.$$

Using (36) and Hölder inequality, we get

$$\int_0^1 \|N(t-s, \cdot) \star u^3(s, \cdot)\|_{L_x^\infty(\mathbb{R})} ds \leq C\varepsilon \int_0^1 \frac{ds}{(t-s)s^{\frac{1}{2}}} M_2^2(1).$$

Now, using a change of variables on the last integral, we get

$$\int_0^1 \|N(t-s, \cdot) \star u^3(s, \cdot)\|_{L_x^\infty(\mathbb{R})} ds \leq \frac{C\varepsilon}{t^{1/2}} \int_0^{1/t} \frac{d\sigma}{(1-\sigma)^{3/4} \sigma^{1/2}} M_2^2(t).$$

It follows that

$$\int_0^1 \|N(t-s, \cdot) \star u^3(s, \cdot)\|_{L_x^\infty(\mathbb{R})} ds \leq \frac{C\varepsilon}{t} M_2^2(t). \quad (42)$$

Moreover,

$$\int_1^{\frac{t}{2}} \|N(t-s, \cdot) \star u^3(s, \cdot)\|_{L_x^\infty(\mathbb{R})} ds \leq \int_1^{\frac{t}{2}} \|N(t-s)\|_{L_x^\infty(\mathbb{R})} \|u(s)\|_{L_x^2(\mathbb{R})}^2 \|u(s)\|_{L_x^\infty(\mathbb{R})} ds.$$

Since for $1 \leq s \leq \frac{t}{2}$, we get

$$\int_1^{\frac{t}{2}} \|N(t-s, \cdot) \star u^3(s, \cdot)\|_{L_x^\infty(\mathbb{R})} ds \leq C\varepsilon \int_1^{\frac{t}{2}} \frac{ds}{(t-s)s^{\frac{3}{2}}} M_2^2(t).$$

Now, using a change of variables on the last integral, we get

$$\int_1^{\frac{t}{2}} \|N(t-s, \cdot) \star u^3(s, \cdot)\|_{L_x^\infty(\mathbb{R})} ds \leq \frac{C\varepsilon}{t^{\frac{3}{2}}} \int_{\frac{1}{t}}^{\frac{1}{2}} \frac{d\sigma}{(1-\sigma)\sigma^{3/2}} M_2^2(t).$$

Since $t \geq 2$, it follows that

$$\int_1^{\frac{t}{2}} \|N(t-s, \cdot) \star u^3(s, \cdot)\|_{L_x^\infty(\mathbb{R})} ds \leq \frac{C\varepsilon}{t} M_2^2(t). \quad (43)$$

Finally,

$$\begin{aligned} \int_{\frac{t}{2}}^t \|N(t-s, \cdot) \star u^3(s, \cdot)\|_{L_x^\infty(\mathbb{R})} ds &\leq \int_{\frac{t}{2}}^t \|N(t-s)\|_{L_x^2(\mathbb{R})} \|u(s)\|_{L_x^2(\mathbb{R})} \|u(s)\|_{L_x^\infty(\mathbb{R})}^2 ds \\ &\leq C\varepsilon \int_{\frac{t}{2}}^t \|N(t-s)\|_{L_x^2(\mathbb{R})} \|u(s)\|_{L_x^2(\mathbb{R})} \|u(s)\|_{L_x^\infty(\mathbb{R})} ds. \end{aligned}$$

Since $\frac{t}{2} \leq s \leq t$, then in particular $s \geq 1$. Hence, we get

$$\int_{\frac{t}{2}}^t \|N(t-s, \cdot) \star u^3(s, \cdot)\|_{L_x^\infty(\mathbb{R})} ds \leq C\varepsilon \int_{\frac{t}{2}}^t \frac{ds}{(t-s)^{\frac{3}{4}} s^{\frac{7}{4}}} M_2(t) M_\infty(t).$$

Now, using a change of variables on the last integral, we get

$$\int_{\frac{t}{2}}^t \|N(t-s, \cdot) \star u^3(s, \cdot)\|_{L_x^\infty(\mathbb{R})} ds \leq \frac{C\varepsilon}{t^{\frac{3}{2}}} \int_{\frac{1}{2}}^1 \frac{d\sigma}{(1-\sigma)^{\frac{3}{4}} \sigma^{\frac{7}{4}}} M_2(t) M_\infty(t).$$

Since $t \geq 2$, it follows that

$$\int_{\frac{t}{2}}^t \|N(t-s, \cdot) \star u^3(s, \cdot)\|_{L_x^\infty(\mathbb{R})} ds \leq \frac{C\varepsilon}{t} M_2(t) M_\infty(t). \quad (44)$$

Collecting the estimates (40), (42), (43) and (44), we conclude that

$$M_\infty(t) = t \|u(t, \cdot)\|_{L_x^\infty(\mathbb{R})} \leq C\varepsilon + C\varepsilon M_2^2(t) + C\varepsilon M_2(t) M_\infty(t).$$

Recalling (39), we conclude that $M_2^2(t) \leq 1$ and $C\varepsilon M_2(t) M_\infty(t) \leq \frac{1}{2} M_\infty(t)$. Hence we get for all $t \geq 2$

$$M_\infty(t) = \max(\sqrt{t}, t) \|u(t, \cdot)\|_{L_x^\infty(\mathbb{R})} \leq C\varepsilon.$$

This completes the proof of the upper bound in L^∞ norm.

4 Proof of the second main results

In this section, we establish Theorem 4. We start by defining a weak solution of (9).

Definition 1 – We denote by weak solution of the regularized problem (9), a couple (u, Φ) such that

- $u \in L^\infty(\mathbb{R}_+, H_x^1(\mathbb{R})) \cap C_w(\mathbb{R}_+, L_x^2(\mathbb{R}))$,
- $u_x \in L^2(\mathbb{R}_+, H_x^1(\mathbb{R}))$,
- $\forall T > 0, \Phi \in L^\infty([0, T], X) \cap C(\mathbb{R}_+, \mathbb{L}_{x,y}^2(\mathbb{R}^2))$,

that verifies for all $\tilde{u} \in H_x^1(\mathbb{R})$ and $\tilde{\Phi} \in X'$

$$\begin{cases} \frac{d}{dt} \left[(u(t), \tilde{u})_{L_x^2(\mathbb{R})} + 2(\Gamma\Phi(t), \tilde{u})_{L_x^2(\mathbb{R})} \right] + (u_x(t), \tilde{u})_{L_x^2(\mathbb{R})} + (u_x(t), \tilde{u}_x)_{L_x^2(\mathbb{R})} \\ \hspace{15em} = -(u^2(t)u_x(t), \tilde{u})_{L_x^2(\mathbb{R})}, \\ \frac{d}{dt} [\Phi(t), \tilde{\Phi}]_{\mathbb{L}_{x,y}^2(\mathbb{R}^2)} + [\Phi_y(t), \tilde{\Phi}_y]_{\mathbb{L}_{x,y}^2(\mathbb{R}^2)} = (u(t), \tilde{\Phi}(x, 0))_{L_x^2(\mathbb{R})}, \\ u(0) = u_0 \in H_x^1(\mathbb{R}), \\ \Phi(0) = 0. \end{cases}$$

4. Proof of the second main results

We recall the following result from Goubet and Manoubi (2017).

Lemma 2 – For all $\Phi \in X$, we have

$$|\Gamma(\Phi)|_{L_x^2(\mathbb{R})} \leq \|\Phi\|_{\mathbb{L}_{x,y}^2(\mathbb{R}^2)} \|\Phi_y\|_{\mathbb{L}_{x,y}^2(\mathbb{R}^2)}.$$

In the sequel, we establish the following a priori estimates.

Lemma 3 – Let $T > 0$. We have that

- $u \in L^\infty(0, T, L_x^2(\mathbb{R}))$,
- $u_x \in L^\infty(0, T, L_x^2(\mathbb{R}))$,
- $u_{xx} \in L^2(0, T, L_x^2(\mathbb{R}))$,
- $\Phi \in L^\infty(0, T, X)$,
- $\partial_t \Phi \in L^2(0, T, \mathbb{L}_{x,y}^2(\mathbb{R}^2))$.

Proof. First, we proceed as in the proof of Theorem 1.1 in Goubet and Manoubi (2017), we get

$$\begin{aligned} |u(t)|_{L_x^2(\mathbb{R})} &\leq |u_0|_{L_x^2(\mathbb{R})}, \\ \|\Phi_y(t)\|_{\mathbb{L}_{x,y}^2(\mathbb{R}^2)} &\leq \frac{1}{\sqrt{2}} |u_0|_{L_x^2(\mathbb{R})}, \\ \int_0^t |u_x(s)|_{L_x^2(\mathbb{R})}^2 ds &\leq \frac{1}{2} |u_0|_{L_x^2(\mathbb{R})}^2, \\ \int_0^t \|\Phi_t(s)\|_{\mathbb{L}_{x,y}^2(\mathbb{R}^2)}^2 ds &\leq \frac{1}{2} |u_0|_{L_x^2(\mathbb{R})}^2, \\ \|\Phi(t)\|_{\mathbb{L}_{x,y}^2(\mathbb{R}^2)} &\leq c(u_0)t. \end{aligned}$$

In order to get more regularity on the solution for initial data $u_0 \in H_x^1(\mathbb{R})$, we take the scalar product of the first equation in (9) with u_{xx} in $L_x^2(\mathbb{R})$ and the scalar product of the second equation in (9) with Φ_{txx} in $\mathbb{L}_{x,y}^2(\mathbb{R}^2)$, we get

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} |u_x(t)|_{L_x^2(\mathbb{R})}^2 + 2(\Gamma(\Phi_{tx}), u_x) + |u_{xx}(t)|_{L_x^2(\mathbb{R})}^2 &= (u^2(t)u_x(t), u_{xx}(t)), \\ \frac{1}{2} \frac{d}{dt} \|\Phi_{xy}(t)\|_{\mathbb{L}_{x,y}^2(\mathbb{R}^2)}^2 + \|\Phi_{tx}(t)\|_{\mathbb{L}_{x,y}^2(\mathbb{R}^2)}^2 &= (\Gamma(\Phi_{tx}), u_x). \end{aligned}$$

It follows that

$$\begin{aligned} \frac{d}{dt} \left(\frac{1}{2} |u_x(t)|_{L_x^2(\mathbb{R})}^2 + \|\Phi_{xy}(t)\|_{\mathbb{L}_{x,y}^2(\mathbb{R}^2)}^2 \right) + |u_{xx}(t)|_{L_x^2(\mathbb{R})}^2 \\ + 2\|\Phi_{tx}(t)\|_{\mathbb{L}_{x,y}^2(\mathbb{R}^2)}^2 = (u^2(t)u_x(t), u_{xx}(t)). \end{aligned} \quad (45)$$

Using first the Cauchy-Schwarz inequality, we get

$$\begin{aligned} (u^2(t)u_x(t), u_{xx}(t)) &\leq |u(t)|_{L_x^\infty(\mathbb{R})}^2 |u_x(t)|_{L_x^2(\mathbb{R})} |u_{xx}(t)|_{L_x^2(\mathbb{R})} \\ &\leq \frac{1}{2} |u(t)|_{L_x^\infty(\mathbb{R})}^4 |u_x(t)|_{L_x^2(\mathbb{R})}^2 + \frac{1}{2} |u_{xx}(t)|_{L_x^2(\mathbb{R})}^2. \end{aligned}$$

Using the Agmon inequality and the a priori estimate on u from Lemma 3, we deduce

$$(u^2(t)u_x(t), u_{xx}(t)) \leq \frac{1}{2} |u_0|_{L_x^2(\mathbb{R})}^2 |u_x(t)|_{L_x^2(\mathbb{R})}^4 + \frac{1}{2} |u_{xx}(t)|_{L_x^2(\mathbb{R})}^2.$$

Substituting the above estimate into (45), we deduce that there exists a constant $c = c(u_0)$ depending on u_0 such that

$$\frac{d}{dt} \left(\frac{1}{2} |u_x(t)|_{L_x^2(\mathbb{R})}^2 + \|\Phi_{xy}(t)\|_{\mathbb{L}_{x,y}^2(\mathbb{R}^2)}^2 \right) + \frac{1}{2} |u_{xx}(t)|_{L_x^2(\mathbb{R})}^2 \leq c(u_0) |u_x(t)|_{L_x^2(\mathbb{R})}^4. \quad (46)$$

Define the functions $\Psi(t) = \frac{1}{2} |u_x(t)|_{L_x^2(\mathbb{R})}^2 + \|\Phi_{xy}(t)\|_{\mathbb{L}_{x,y}^2(\mathbb{R}^2)}^2$ and $g(t) = |u_x(t)|_{L_x^2(\mathbb{R})}^2$. We deduce from (46)

$$\frac{d}{dt} (\Psi(t)) + \frac{1}{2} |u_{xx}(t)|_{L_x^2(\mathbb{R})}^2 \leq c(u_0) g(t) \Psi(t). \quad (47)$$

From Lemma 3, we have that $g \in L^1(\mathbb{R}_+)$. Now, we integrate (47) in time from 0 to t , we get

$$\Psi(t) \leq \Psi(0) + c(u_0) \int_0^t g(s) \Psi(s) ds,$$

where $\Psi(0) = |u_{0x}|_{L_x^2(\mathbb{R})}^2$. We deduce by Gronwall Lemma that, there exists a constant $\tilde{c}(u_0)$ that depends on u_0 and u_{0x} such that

$$\frac{1}{2} |u_x(t)|_{L_x^2(\mathbb{R})}^2 \leq \tilde{c}(u_0). \quad (48)$$

Moreover, integrating (46) in time from 0 to t and using (48), we get

$$\frac{1}{2} \int_0^t |u_{xx}(s)|_{L_x^2(\mathbb{R})}^2 ds \leq |u_{0x}|_{L_x^2(\mathbb{R})}^2 + \tilde{c}(u_0) \int_0^t g(s) ds.$$

This completes the proof of the lemma. $\square$

Proposition 3 – *The problem (9) has a unique maximal solution defined on the interval $[0, T_{\max}[$, where $0 < T_{\max} \leq +\infty$.*

Corollary 1 – *Using Lemma 3 and the blow up alternative, we deduce that $T_{\max} = +\infty$.*

5. Numerical computation for the fractional PDE

Following the work of Goubet and Manoubi (2017), we justify the existence of a weak solution (u, Φ) of (9) by a limiting argument using the Galerkin method. We note that there is a quiet difference only in the estimation of the nonlinear term. Moreover, it is standard to justify the continuity and the uniqueness of the solution (u, Φ) of (9). We omit the details for the sake of conciseness.

5 Numerical computation for the fractional PDE

5.1 Numerical schema using the diffusive representation

This section deals with the numerical solution of the generalized model

$$u_t + u_x + \beta u_{xxx} + \frac{\sqrt{v}}{\sqrt{\pi}} \frac{\partial}{\partial t} \int_0^t \frac{u(s)}{\sqrt{t-s}} ds + \gamma u^p u_x = \alpha u_{xx}, \quad (49)$$

when $p > 1$ supplemented with the initial condition $u(0, x) = u_0(x)$. Here ν and α are non negative parameters.

In order to write a numerical scheme that is local in time, we use the diffusive representation of the Riemann-Liouville half-derivative, following the work of Dumont and Manoubi (2018). We choose here the following representation (see Dumont and Manoubi (2018) and Montseny, Audounet, and Matignon (2000))

$$\begin{cases} \partial_t \phi(t, \sigma) = -\sigma^2 \phi(t, \sigma) + \frac{2}{\pi} u(t), & \phi(0, \sigma) = 0, \quad \sigma \geq 0, \\ D^{1/2} u(t) = \int_0^{+\infty} \left(\frac{2}{\pi} u(t) - \sigma^2 \phi(t, \sigma) \right) d\sigma. \end{cases}$$

Hence, the integro-differential model (49) is equivalent to the system

$$\begin{cases} \partial_t u(t,x) + \partial_x(u + \frac{\gamma}{(p+1)}u^{p+1}) = -\sqrt{\nu} \int_0^\infty (\frac{2}{\pi}u(t,x) - \sigma^2 \psi(t,x,\sigma))d\sigma \\ \quad \quad \quad + \alpha u_{xx}(t,x) - \beta u_{xxx}(t,x), & (50) \\ \partial_t \psi(t,x,\sigma) = -\sigma^2 \psi(t,x,\sigma) + \frac{2}{\pi}u(t,x), \end{cases}$$

supplemented with the initial conditions $\psi(0, x, \sigma) = 0$ and $u(0, x) = u_0(x)$, for all $t > 0, x \in \mathbb{R}$ and $\sigma \geq 0$.

In the sequel, we approximate the integral in (50) using the Gauss-Jacobi quadrature with N nodes. We denote by w_i and σ_i for $i = 1, \dots, N$ the weights and abscissas,

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Then a Strang Splitting method of order 2 (see Holden et al. (2011) and LeVeque (1992)) between t_n and t_{n+1} is used to solve respectively (53) and (54), which provides an approximation of order 2 in time to the problem (52).

On the one hand, to approximate the solution of the propagation equation, we write a semi-discret in time scheme, using the Crank-Nicolson scheme for the linear part and Adams-Bashforth scheme for the nonlinear part. We denote by u^m the approximate value of $u(t^m)$. Then the space approximation of the solutions is performed by standard Fourier methods:

$$\frac{\hat{u}^{m+1} - \hat{u}^m}{\Delta t} = \frac{\hat{u}^{m+1} + \hat{u}^m}{2} (-\alpha \xi^2 - i\xi + i\beta \xi^3) - \frac{i\gamma \xi}{p+1} \left(\frac{3(\widehat{u^m})^{p+1} - (\widehat{u^{m-1}})^{p+1}}{2} \right). \quad (55)$$

On the other hand, the diffusion equation (54) is calculated numerically using the formula

$$\mathbf{H}\left(\frac{\Delta t}{2}\right)U = e^{\mathcal{S}\frac{\Delta t}{2}}U. \quad (56)$$

where $\mathbf{H}$ is the discret operator of the solution of (54).

5.2 Numerical results and discussion

We take an initial datum u_0 which provides a small amplitude and long wave of a generalized-KdV soliton when $\alpha = \nu = 0$, $\beta = 1$ and $\gamma = 6$. We denote by x_0 the middle of the space interval then the initial datum reads

$$u_0(x) = 0.054(p+1)(p+2) \operatorname{sech}^{\frac{2}{p}}(0.4 * p(x - x_0)).$$

All numerical simulations are constructed over the space interval $[0, A] = [0, 500]$ and the time interval $[0, T]$. We consider the time step $\Delta t = 0.1$, the space step $h = 0.1$ and periodic boundary conditions i.e $u|_{x=0} = u|_{x=A}$.

Our goal is to investigate the impact of the parameters in (49) on the asymptotical behavior of the numerical solution and to compare these results with those presented in Dumont and Manoubi (2018) and Manoubi (2014) for the classical case $p = 1$.

The study of the convergence in number of grid points N are already done in Dumont and Manoubi (2018). We observed that the use of $N = 15$ points of the Gauss-Jacobi quadrature method is sufficient to get fulfilling numerical results.

In figure 1, we illustrate the influence of the parameter p on the solution. We consider the parameters $\alpha = 1$, $\beta = 0$, $\gamma = 2$, $\nu = 0$ (the case of the generalized KdV (GKdV) solution) and $\nu = 1$ (with the nonlocal term). We plot the solution at time $T = 200$. We observe that for the two cases (a) and (b), the solutions have the same shape. However, the solutions amplitude increases slightly when p increases. Moreover, when $\nu \neq 0$, the solutions amplitude is very decreasing (at the order of 0.005 when $p = 3$) comparing with the corresponding GKdV-solution (at the order

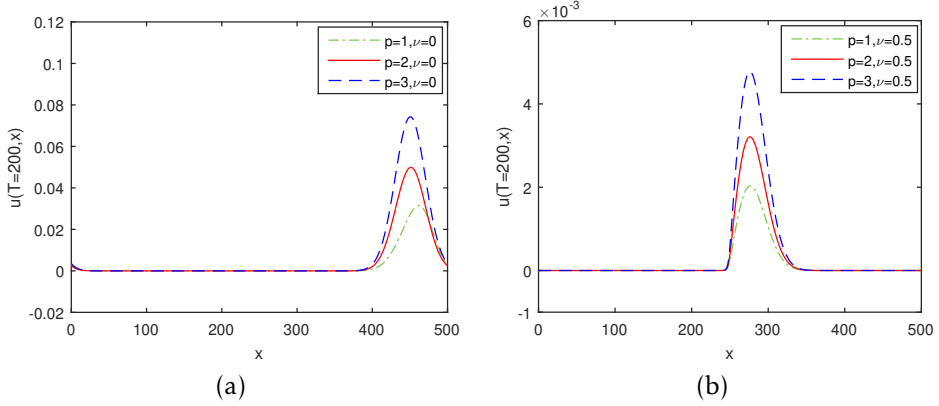


Figure 1 – Solutions at time $T = 200$ for different values of p when $\alpha = 1$, $\beta = 0$ and $\gamma = 2$ (a) $\nu = 0$, (b) $\nu = 0.5$.

of 0.07 when $p = 3$). Moreover, the presence of the nonlocal viscous term slows the wave down (the solutions centered at 290) comparing with the GKdV-case (the solutions centered at 450).

Now, we compare the theoretical and numerical results on the decay rate of the solution. Hence, we take $\alpha = 1$, $\nu = 0.5$, $\gamma = 2$ and $\beta = 0$. We plot the decay rates L^2 and L^∞ norms on the interval $(0, 200)$ for different values of p . In figure 2, we first present the decay rates of the GKdV-solution ($\nu = 0$) in (a)-(b). Then, we plot the decay rates of solutions to the scheme (55)-(56) with $\nu = 0.5$.

On the one hand, we observe that, when there $\nu = 0$, the L^2 -decay rate approaches -0.25 and the L^∞ -decay rate approaches -0.5 . The difference between the decay rates for selected values of p is quite small. We deduce that the influence of the parameter p is very negligible. These results are similar to those given for $p = 1$ in Dumont and Manoubi (2018).

On the other hand, considering the case $\nu \neq 0$, we observe that the L^2 -decay rate approaches -0.75 and the L^∞ -decay rate approaches -1 for selected values of p . We note that similar results are observed for several values of the parameters, when considering the same selected values of p . Moreover, the numerical results are in correspondance with those proven theoretically for $p = 1$ in Manoubi (2014).

Now, we simulate the scheme (55)-(56) when theoretical results are not provided, namely when $\beta \neq 0$. Since the value of p is quite significant, we provide numerical results when $p = 2$. We take $(\alpha, \nu) = ((0.5, 0), (0, 1))$ and $(0.5, 1)$, $\gamma = 1$ and $\beta = 6$. We note that, when $\alpha = \nu = 0$, the exact solution is the soliton $u(t, x) = u_0(x - 1.64t)$.

In figure 3, we plot the solutions when $T = 100$. The geometric dispersion term βu_{xxx} plays a role on the appearance of the oscillations on the solution. When the nonlocal term is present, the local term is quite significant. Moreover, the nonlocal

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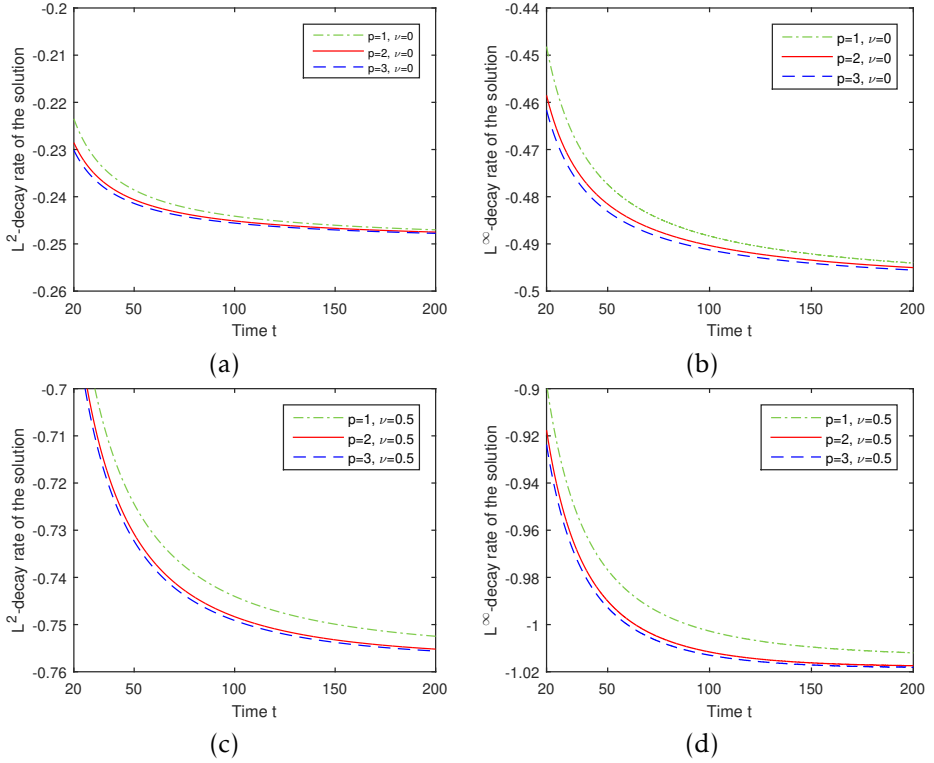


Figure 2 – L^2 and L^∞ -decay rates of the solution for different values of p when $\alpha = 1$, $\gamma = 2$, $\beta = 0$ and $\nu = 0$ or 1 .

viscous term reduces the amplitude and slows down the wave considerably.

In figure 4, we plot the L^2 and the L^∞ decay rates on the interval $(0, 1000)$. Again, when $\nu \neq 0$, the L^2 -decay rate approaches -0.75 and the L^∞ -decay rate approaches -1 . This is in coherence with the theoretical results established in Proposition 1.

Norm	$\alpha = 0, \nu = 0.5$				$\alpha = 0.5, \nu = 0$				$\alpha = \nu = 0.5$			
	$\gamma = 0$	$\gamma = 1$			$\gamma = 0$	$\gamma = 1$			$\gamma = 0$	$\gamma = 1$		
		$p = 1$	$p = 2$	$p = 3$		$p = 1$	$p = 2$	$p = 3$		$p = 1$	$p = 2$	$p = 3$
L^∞	-1.014	-1.	-1.014	-1.016	-0.410	-0.403	-0.41	-0.411	-0.9821	-0.976	-0.982	-0.982
L^2	-0.782	-0.78	-0.782	-0.782	-0.241	-0.234	-0.241	-0.242	-0.8	-0.79	-0.8	-0.802

Table 1 – Decay rates of the solution when $\beta = 6$.

Finally, in table 1, we compute the decay rate of solutions to (55)-(56) when $\beta = 6$ for different values of the parameters. These computations cover the linear

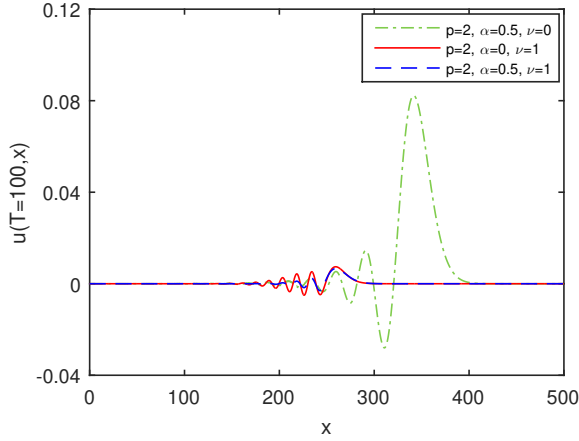


Figure 3 – Solutions at $T = 100$ when $p = 2$, $(\alpha, \nu) = ((0.5, 0), (0, 1), (0.5, 1))$, $\gamma = 1$ and $\beta = 6$.

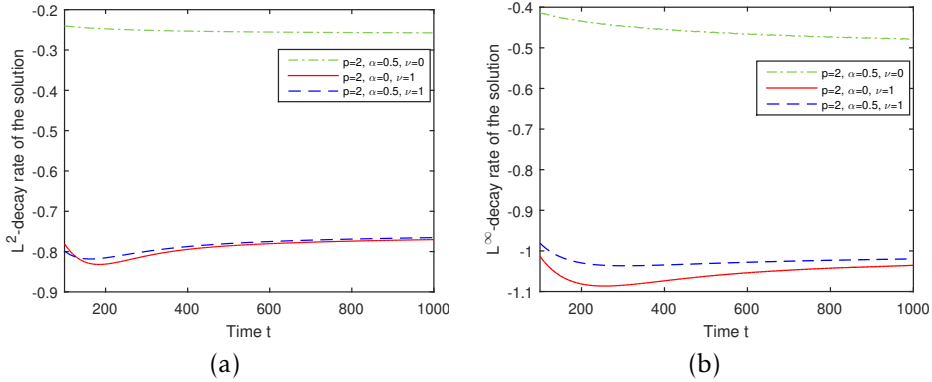


Figure 4 – (a) L^2 -decay rate of the solution, (b) L^∞ -decay rate of the solution when $p = 2$ $(\alpha, \nu) = ((0.5, 0), (0, 1), (0.5, 1))$, $\gamma = 1$ and $\beta = 6$.

and the nonlinear cases for different values of p . The presented results confirm well the theoretical results given in Proposition 1 and the observed numerical results.

Conclusion

In this article, we investigate a modified water wave model with power nonlinearity in the presence of a fractional operator in time. We first establish the existence and the decay estimates of solutions for a general class of nonlinearities, under a

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smallness condition on the initial data. We then adopted a diffusive realization on the fractional term and we proved the global wellposedness of the problem in the cubic case removing the smallness assumption. Finally, numerical investigation is provided to exhibit the behavior of the model for different power nonlinearities. We observe that, for the selected values of p , the nonlinear term does not effect the asymptotical behavior of the solutions. In particular, we get the same decay rate as in the classical case⁴. This was expected since the nonlinearity is asymptotically weak (see the classification in Hayashi et al. (2006)). An interesting open problem is to establish the same decay estimates of solutions by extending the analysis to arbitrary initial data. Moreover, it would be interesting to explore the possible occurrence of blow-up for large nonlinearities.

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