North-Western European Journal of Mathematics

A simple Master Theorem for Discrete Divide and Conquer Recurrences

Olivier Garet[1]

Received: June 4, 2021/Accepted: January 25, 2022/Online: May 3, 2022

Abstract

The aim of this note is to provide a Master Theorem for some discrete divide and conquer recurrences:

$$X_n = a_n + \sum_{j=1}^{m} b_j X_{\lfloor \frac{n}{m_j} \rfloor},$$

where the m_i's are integers with $m_i \geq 2$. The main novelty of this work is there is no assumption of regularity or monotonicity for (a_n). Then, this result can be applied to various sequences of random variables $(a_n)_{n\geq 0}$, for example such that $\sup_{n\geq 1} \mathbb{E}(|a_n|) < +\infty$.

Keywords: Divide-and-conquer recurrence, Dirichlet series, Tauberian theorem.

MSC: 11B37, 68W40, 60F15.

1 Introduction

Divide-and-conquer methods are widely used in Computer Science. The analysis of the cost of the algorithm naturally leads to divide-and-conquer recurrences. The methods to study these recurrences are popularized as "Master theorems" in the literature of Computer Science. See e.g. the reference books by Cormen et al[2] or Goodrich and Tamassia[3].

In the sequel, we consider sequences $(X_n)_{n\geq 0}$ that are defined by $X_0 = a_0$, then

$$X_n = a_n + \sum_{j=1}^{m} b_j X_{\lfloor \frac{n}{m_j} \rfloor}, \tag{1}$$

where the m_i's are integer with $m_i \geq 2$ and $\lfloor x \rfloor$ denotes the only $n \in \mathbb{Z}$ such that $x - n \in [0, 1)$.

---

[1] Université de Lorraine, CNRS, IECL, F-54000 Nancy, France

[2] Cormen et al., 2010, *Introduction to Algorithms*.

[3] Goodrich and Tamassia, 2002, *Algorithm Design: Foundations, Analysis, and Internet Examples*.

Of course, in Computer Science, a_n and X_n represent computation times and are therefore positive. However, the case of negative a_n and X_n can be of theoretical interest.

In the literature of Computer Science, (a_n) is supposed to be deterministic. Nevertheless, in the context of randomized algorithm, eventually involving Monte-Carlo simulation, it is natural to consider the case of a random (a_n) and observe the fluctuations of the computation time.

One of the most general results in the field of Computer Science is due to Akra and Bazzi[4]. They do not seek for an exact asymptotic limit, focusing of the order of the fluctuations. Their methods rely on classical real analysis.

The mathematical literature is more focused on exact methods, that rely on generating functions. The first paper in this spirit is Erdős et al[5], which solved the case $a_n = 0$ with the help of renewal equations. Tauberian theorems lead to simpler proofs of their result, see e.g. Choimet and Queffelec[6]. Recent results by Drmota and Szpankowski[7]) also rely on Tauberian theorems and some other tools in complex analysis. They request some assumptions of monotonicity.

If one wants to cover the case of a random (a_n), the sequence (a_n) obviously can not be supposed to be monotonic. Quite surprisingly, we did not find in the literature any theorem of this kind, computing an exact limit without making some assumption of monotonicity.

Let us clarify the assumptions: we assume that the b_i's are positive numbers with $\sum_{j=1}^{m} b_j > 1$, that the m_i are integers with $m_i \geq 2$ and such that there exists j, ℓ with $\frac{\ln m_j}{\ln m_\ell} \notin \mathbb{Q}$. The rational case, which is not considered here, is also of great interest in Computer Science – see e.g. Roura[8] or Drmota and Szpankowski[9].

It is known that the general growth of (X_n) is governed by the value of the positive root s_0 for the equation

$$\sum_{j=1}^{m} b_j m_j^{-s} = 1.$$

As said before, the originality of the present paper lies in the assumption on the (a_n): under the assumption that

$$\sum_{n=1}^{+\infty} \frac{|a_n|}{n^{s_0}} < +\infty,$$

[4]Akra and Bazzi, 1998, "On the solution of linear recurrence equations".
[5]Erdős et al., 1987, "The asymptotic behavior of a family of sequences".
[6]Choimet and Queffélec, 2015, *Twelve landmarks of twentieth-century analysis*.
[7]Drmota and Szpankowski, 2013, "A master theorem for discrete divide and conquer recurrences".
[8]Roura, 2001, "Improved master theorems for divide-and-conquer recurrences".
[9]Drmota and Szpankowski, 2013, "A master theorem for discrete divide and conquer recurrences".

we prove that the sequence $\frac{X_n}{n^{s_0}}$ admits a limit L when n tends to infinity and give a fairly simple closed expression for it.

As we will see, this allow to apply our Theorem to a large class of random variables. Then, the limit L is a random variable, which appears as the sum of a random series.

If we specialize to the case where the (a_n) are independent, then one can easily control the random fluctuations of L.

2 The deterministic Theorem

Theorem 1 – *Let $m \geq 1$, $(b_1,\dots,b_m)$ be a family of non-negative numbers and $(m_1,\dots,m_m)$ be a family of integers with $m_i \geq 2$ and such that*

- *there exists j,ℓ with $\frac{\ln m_j}{\ln m_\ell} \notin \mathbb{Q}$;*
- $\sum_{j=1}^{m} b_j > 1$.

We denote by s_0 the positive root s_0 for the equation

$$\sum_{j=1}^{m} b_j m_j^{-s} = 1.$$

Then, there exists a sequence $(\ell_j)_{j\geq 0}$ of positive numbers such that for every sequence $(a_n)_{n\geq 0}$ with

$$\sum_{n=1}^{+\infty} \frac{|a_n|}{n^{s_0}} < +\infty,$$

then the sequence $(X_n)_{n\geq 0}$ defined by $X_0 = a_0$ and the recursion (1) *satisfies*

$$\lim_{n\to+\infty} \frac{X_n}{n^{s_0}} = \sum_{j=0}^{+\infty} \ell_j a_j.$$

Note that if the sequence $(a_j)_{j\geq 0}$ is non-negative and not identically zero, the limit $\sum_{j=0}^{+\infty} \ell_j a_j$ is positive, so we have found the correct speed for the growth of $(X_n)_{n\geq 0}$.

Proof. We denote by $L_n(a)$ the value of X_n corresponding to the recursion (1) for some sequence a.

The recursion equation

Let n_0 be a non-negative integer and suppose first that $a_n = 0$ for $n > n_0$.

For $n > n_0$, we have $X(n) = \sum_{j=1}^{m} b_j X(\lfloor \frac{n}{m_j} \rfloor)$.

We can choose C such that $|X_k| \le Ck^{s_0}$ for $0 < k \le n_1 = \max(n_0, m_1, \ldots, m_m)$. Then, it follows by natural induction that $|X_k| \le Ck^{s_0}$ for each $k \in \mathbb{N}^*$. In the sequel, we put $X(t) = X(\lfloor t \rfloor)$ to simplify some notation. Now define

$$\phi(s) = s \int_{n_0+1}^{+\infty} \frac{X(t)}{t^{s+1}}\, dt \tag{2}$$

for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > s_0$. The recursion Equation leads to

$$\begin{aligned}\phi(s) &= s \int_{n_0+1}^{+\infty} \sum_{j=1}^{m} b_j \frac{X(\frac{t}{m_j})}{t^{s+1}}\, dt = s \sum_{j=1}^{m} b_j m_j^{-s} \int_{\frac{n_0+1}{m_j}}^{+\infty} \frac{X(t)}{t^{s+1}}\, dt \\ &= \left(\sum_{j=1}^{m} b_j m_j^{-s} \right) \phi(s) + s \sum_{j=1}^{m} b_j m_j^{-s} \int_{\frac{n_0+1}{m_j}}^{n_0+1} \frac{X(t)}{t^{s+1}}\, dt.\end{aligned}$$

Since

$$|\sum_{j=1}^{m} b_j m_j^{-s}| \le \sum_{j=1}^{m} |b_j m_j^{-s}| = \sum_{j=1}^{m} b_j m_j^{-\mathrm{Re}(s)} < \sum_{j=1}^{m} b_j m_j^{-s_0} = 1,$$

we can write, for $\mathrm{Re}(s) > s_0$:

$$\phi(s) = \frac{P(s)}{1 - \sum_{j=1}^{m} b_j m_j^{-s}}, \text{ with } P(s) = s \sum_{j=1}^{m} b_j m_j^{-s} \int_{\frac{n_0+1}{m_j}}^{n_0+1} \frac{X(t)}{t^{s+1}}\, dt \tag{3}$$

Tauberian magic

Now, fix a non-negative integer n_0 and suppose that the sequence $a = (a_n)_{n \ge 0}$ is $a = I^{n_0}$ with

$$I_i^{n_0} = \mathbb{1}_{i \le n_0} = \begin{cases} 1 & \text{if } i \le n_0 \\ 0 & \text{else} \end{cases}.$$

By natural induction, it is easy to see that $(X_n)_{n \ge 0}$ is non-decreasing.

It is also not difficult to see that $1 - \sum_{j=1}^{m} b_j m_j^{-s}$ does not vanish for $s \in \mathbb{C}$ with $\mathrm{Re}(s) \ge s_0$ and $s \ne s_0$. Proceeding as in Choimet and Queffelec (see[10], section 4), we can note that, for $\mathrm{Re}(s) = s_0$

$$\mathrm{Re}\left(\sum_{j=1}^{m} b_j m_j^{-s} \right) = \sum_{j=1}^{m} b_j m_j^{-s_0} \cos(\ln m_j \,\mathrm{Im}(s)) \le \sum_{j=1}^{m} b_j m_j^{-s_0} = 1.$$

In fact, the inequality in strict when $\mathrm{Im}(s) \neq 0$. Overwise, we would have $\ln m_j \,\mathrm{Im}(s) \in 2\pi\mathbb{Z}$ for each j, whence $\frac{\ln m_j}{\ln m_k} \in \mathbb{Q}$ for each j,k, which has been excluded.

It follows that for

$$c = \mathrm{Res}_{s_0} \phi = \frac{P(s_0)}{\sum_{j=1}^m b_j m_j^{-s_0} \ln(m_j)},$$

the map $s \mapsto \phi(s) - \frac{c}{s-s_0}$ is holomorphic on $\{s \in \mathbb{C}; \mathrm{Re}(s) \geq s_0\}$.

Now note $b(x) = \sum_{n_0<n\leq x}(X_n - X_{n-1})$. The Abel transformation gives

$$\sum_{n=n_0+1}^{+\infty} \frac{X_n - X_{n-1}}{n^s} = s\int_{n_0+1}^{+\infty} \frac{b(t)}{t^{s+1}}\, dt.$$

Since $b(t) = X(t) - X_{n_0}$, we have

$$\sum_{n=n_0+1}^{+\infty} \frac{X_n - X_{n-1}}{n^s} = s\int_{n_0+1}^{+\infty} \frac{X(t)}{t^{s+1}}\, dt - \frac{X_{n_0}}{(n_0+1)^s} = \phi(s) - \frac{X_{n_0}}{(n_0+1)^s}.$$

Now, we will apply the Ikehara–Newman Theorem for series:

Proposition 1 – *Let $(u_n)_{n\geq 1}$ be a sequence of non-negative real numbers, and a, c be positive real numbers. Suppose that the Dirichlet series $\Phi(s) = \sum_{n=1}^{+\infty} u_n n^{-s}$ is defined on the open half-plane $\mathrm{Re}(s) > a$ and that, more precisely, with $A(x) = \sum_{n\leq x} u_n$ for $x \geq 0$, the following properties are verified:*

- *$A(x)x^{-a}$ is bounded on $\mathbb{R}^+$;*
- *$\Phi(s) - \frac{c}{s-a}$ has a holomorphic extension G on the closed half-plane $\mathrm{Re}(s) \geq a$.*

Then we have $A(x) \sim \frac{c}{a} x^a$ as $x \to +\infty$.

Since $(X_n)_{n\geq 0}$ is non-decreasing, the sequence $(X_n - X_{n-1})_{n>n_0}$ is non-negative, so the Wiener-Ikehara Theorem for series applies: since $b(t) = O(t^{s_0})$ when $t \to +\infty$, we get $b(x) \sim \frac{c}{s_0} x^{s_0}$,so

$$\lim_{n\to+\infty} \frac{L_n(I^{n_0})}{n^{s_0}} = \frac{\sum_{j=1}^m b_j m_j^{-s_0} \int_{\frac{n_0+1}{m_j}}^{n_0+1} \frac{L_t(I^{n_0})}{t^{s_0+1}}\, dt}{\sum_{j=1}^m b_j m_j^{-s_0} \ln(m_j)}.$$

For $n_0 = 0$, we have $\ell_0 = \lim_{n\to+\infty} \frac{L_n(\delta^0)}{n^{s_0}} = \lim_{n\to+\infty} \frac{L_n(I^0)}{n^{s_0}}$, so

$$\ell_0 = \frac{1}{\sum_{j=1}^m b_j m_j^{-s_0} \ln(m_j)} \sum_{j=1}^m b_j m_j^{-s_0} \int_{\frac{1}{m_j}}^{1} \frac{1}{t^{s_0+1}}\, dt \quad \text{or}$$

$$\ell_0 = \frac{1}{\sum_{j=1}^m b_j m_j^{-s_0} \ln(m_j)} \sum_{j=1}^m b_j m_j^{-s_0} \frac{m_j^{s_0}-1}{s_0}.$$

Note that this equality and the related convergence form the result by Erdős et al[11].

Let $n_0 \geq 1$. The sequence $(\delta_n^{n_0})_{n\geq 0}$ is defined by

$$\delta_n^{n_0} = \begin{cases} 1 & \text{if } n = n_0 \\ 0 & \text{else} \end{cases}.$$

Since $\delta^{n_0} = I^{n_0} - I^{n_0-1}$, it follows that

$$L_n(\delta^{n_0})n^{-s_0} = L_n(I^{n_0})n^{-s_0} - L_n(I^{n_0-1})n^{-s_0}$$

has a limit when n tends to infinity. Let us denote it by ℓ_{n_0}.

To compute it, take $a = \delta^{n_0}$ and consider again the associated ϕ. From (2), we get $\ell_{n_0} = \lim_{s\to s_0^+} \frac{1}{s_0}(s-s_0)\phi(s)$. On the other side, Equation (3) is still valid, with

$$\begin{aligned} P(s) &= s\sum_{j=1}^m b_j m_j^{-s} \int_{\frac{n_0+1}{m_j}}^{n_0+1} \frac{X(t)}{t^{s+1}}\,dt \\ &= s\sum_{j=1}^m b_j m_j^{-s} \int_{\max(n_0,\frac{n_0+1}{m_j})}^{n_0+1} \frac{1}{t^{s+1}}\,dt, \end{aligned}$$

also

$$\frac{1}{s_0}(s-s_0)\phi(s) = -\frac{s}{s_0}\frac{s_0-s}{1-\sum_{j=1}^m b_j m_j^{-s}} \sum_{j=1}^m b_j m_j^{-s} \int_{\max(n_0,\frac{n_0+1}{m_j})}^{n_0+1} \frac{1}{t^{s+1}}\,dt$$

and, considering that $m_j \geq 2$, we get

$$\ell_{n_0} = \frac{1}{\sum_{j=1}^m b_j m_j^{-s_0} \ln(m_j)} \sum_{j=1}^m b_j m_j^{-s_0} \int_{n_0}^{n_0+1} \frac{1}{t^{s_0+1}}\,dt.$$

Thanks to this expression and the previous one, it is clear that $\ell_j > 0$ holds for each $j \geq 0$.

The general case

For $n, j \geq 0$, we note $K_n^j = L_n(\delta^j)$. It is obvious that $K_n^j = 0$ for $n < j$ and $K_j^j = 1$. It easily follows by natural induction on n that $0 \leq K_n^j \leq \frac{K_n^0}{K_j^0}$. Now, the affine nature of the recursion gives

$$X_n = \sum_{j=0}^{n} K_n^j a_j.$$

For each $j \geq 0$, we have $\lim_{n \to +\infty} \frac{K_n^j}{n^{s_0}} = \ell_j$. Also, the K_j^0's are positive, with $\lim_{j \to +\infty} \frac{K_j^0}{j^{s_0}} = \ell_0 > 0$, so there exists M such that $0 < \frac{1}{K_j^0} \leq \frac{M}{j^{s_0}}$ for each $j \geq 1$ Then, for each $j, n \geq 1$, we have

$$|\frac{K_n^j a_j}{n^{s_0}}| \leq \frac{K_n^0}{n^{s_0}} \frac{|a_j|}{K_j^0} \leq \frac{|a_j|}{K_j^0} \leq M \frac{|a_j|}{j^{s_0}}$$

and by the Weierstrass criterion,

$$\lim_{n \to +\infty} \frac{X_n}{n^{s_0}} = \sum_{j=0}^{+\infty} \ell_j a_j.$$

□

3 Application to sequences of random variables

We give below some applications of Theorem 1 to sequences of random variables.

3.1 Convergence

Theorem 2 – *Assume that the m_i's, the b_i's and s_0 fulfill the assumptions of Theorem 1 and (a_n) is a sequence of random variables. Under each of the following sets of supplementary assumptions, the sequence $(X_n)_{n\geq 0}$ defined by $X_0 = a_0$ and the recursion* (1) *is such that $\frac{X_n}{n^{s_0}}$ almost surely converges to some random variable, given as the sum of the random series:*

$$L = \sum_{j=0}^{+\infty} \ell_j a_j.$$

[10] Choimet and Queffélec, 2015, *Twelve landmarks of twentieth-century analysis.*
[11] Erdős et al., 1987, "The asymptotic behavior of a family of sequences".

(A) $\sum_{j=1}^{m} \frac{b_j}{m_j} > 1$ *and the* (a_n) *are integrable random variables with*

$$C = \sup_{n\geq 1} \mathbb{E}|a_n| < +\infty.$$

(B) $\sum_{j=1}^{m} \frac{b_j}{m_j^2} > 1$ *and there exists* $C > 0$ *such that for each* $n \geq 1$ *and* $t \geq 1$*, we have* $\mathbb{P}(|a_n| > t) \leq \frac{C}{t}$.

Proof. (A) the condition $\sum_{j=1}^{m} \frac{b_j}{m_j} > 1$ implies that $s_0 > 1$.
We have $\mathbb{E}(\sum_{n=1}^{+\infty} \frac{|a_n|}{n^{s_0}}) \leq C\zeta(s_0) < +\infty$, so $\sum_{n=1}^{+\infty} \frac{|a_n|}{n^{s_0}} < +\infty$ almost surely, which gives the almost sure behavior of $\frac{X_n}{n^{s_0}}$.

(B) the condition $\sum_{j=1}^{m} \frac{b_j}{m_j^2} > 1$ implies that $s_0 > 2$. We fix $\eta > 1$ with $s_0 - \eta > 1$. Then $\mathbb{P}(|a_n| > n^{\eta}) = O(n^{-\eta})$ and $\sum_{n=1}^{+\infty} \mathbb{P}(|a_n| > n^{\eta}) < +\infty$, so by the Borel-Cantelli Lemma, for almost every ω, there exists $n_0(\omega)$ with $|a_n(\omega)| \leq n^{\eta}$ for $n \geq n_0(\omega)$, which gives the convergence of $\sum_{n\geq 1} \frac{|a_n|}{n^{s_0}}$ and our Master Theorem still applies. □

3.2 Non-vanishing limit

We have already noticed that the limit does not vanish when the a_j are non-negative. In the case of random independent a_n, it is very unlikely that the limit is null, even for signed variables.

Theorem 3 – *Assume that the* a_i*'s,* m_i*'s, the* b_i*'s and* s_0 *fulfill the assumptions of Theorem 2 and also that* (a_n) *is a sequence of independent random variables, with at least one* $j_0 \geq 0$ *such that* a_j *is non-atomic. Then, the limit* $L = \sum_{j=0}^{+\infty} \ell_j a_j$ *is non-atomic, and particularly* $\mathbb{P}(L = 0) = 0$.

Proof. By independence, the characteristic function of L satisfies

$$\forall t \in \mathbb{R} \quad |\phi_L(t)| = \prod_{j=0}^{+\infty} |\phi_{\ell_j a_j}(t)| \leq |\phi_{\ell_{j_0} a_{j_0}}(t)|.$$

Therefore

$$\lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} |\phi_L(t)|^2 \, dt \leq \lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} |\phi_{\ell_0 a_{j_0}}(t)|^2 \, dt = 0,$$

which implies that L is non-atomic (see e.g. Durrett[12], section 3.3). □

[12] Durrett, 2019, *Probability—theory and examples.*

3.3 Exponential moments

Theorem 4 – *Assume that the m_i's, the b_i's and s_0 fulfill the assumptions of Theorem 1 and (a_n) is a sequence of independent random variables. The sequence $(X_n)_{n\geq 0}$ is defined by $X_0 = a_0$ and the recursion (1).*

- *If there exists a distribution μ with exponential moments such that $|a_n|$ is stochastically dominated by μ^{*n} for each $n \geq 0$, then $|X_n|$ has exponential moments for each n.*
- *If $s_0 > 1$ (or equivalently $\sum_{j=1}^{m} \frac{b_j}{m_j} > 1$) and there exists a distribution μ with exponential moments such that $|a_n|$ is stochastically dominated by μ for each $n \geq 0$, then $\frac{X_n}{n^{s_0}} \to L$ a.s. where $|L|$ has exponential moments.*
- *If $s_0 > 2$ (or equivalently $\sum_{j=1}^{m} \frac{b_j}{m_j^2} > 1$) and there exists a distribution μ with exponential moments such that $|a_n|$ is stochastically dominated by μ^{*n} for each $n \geq 0$, then $\frac{X_n}{n^{s_0}} \to L$ a.s. where $|L|$ has exponential moments.*

Proof. We begin with an easy lemma:

Lemma 1 – *Let X be a random variable with $\mathbb{E}(e^{\alpha X}) < +\infty$ and Y a random variable following the exponential law $\mathcal{E}(\alpha)$ Then, for $a = \frac{1}{\alpha} \ln \mathbb{E}(e^{\alpha X_1})$, we have the stochastic domination $X \prec Y + a$.*

Proof. We just have to prove that for $t \in \mathbb{R}$, $\mathbb{P}(X \geq t) \leq \mathbb{P}(Y + a \geq t)$, or equivalently $\mathbb{P}(X \geq t) \leq \mathbb{P}(Y \geq t - a)$. For $t \leq a$, we have $\mathbb{P}(X \geq t) \leq 1 = \mathbb{P}(Y \geq t - a)$. For $t \geq a$, the Markov inequality gives

$$\mathbb{P}(X \geq t) \leq \frac{\mathbb{E}e^{\alpha X}}{e^{\alpha t}} = \frac{e^{\alpha a}}{e^{\alpha t}} = \exp(-\alpha(t-a)) = P(Y \geq t - a).$$

This completes the proof. □

Now, we have a and α such that for each $n \geq 1$

$$|a_n| \prec \mu^{*n} \prec (\delta^a * \mathcal{E}(\alpha))^{*n} = \delta^{na} * \Gamma(n, \theta).$$

Let $(Z_n)_{n\geq 0}$ be a sequence of independent variables with $Z_n \sim \Gamma(n, \theta)$, where $\Gamma(a, \gamma)$ is the Law with the density

$$x \mapsto \frac{\gamma^a}{\Gamma(a)} x^{a-1} e^{-\gamma x} \, \mathbb{1}_{]0,+\infty[}(x).$$

$\frac{|X_n|}{n^{s_0}}$ is stochastically dominated by $M \sum_{j=0}^{n} \frac{ja + Z_j}{(j+1)^{s_0}}$, so for $t < 1/\alpha$, we have

(Cont. next page)

$$\mathbb{E}(e^{t\frac{|X_n|}{n^{s_0}}}) \le \exp(Ma\sum_{j=1}^{n+1} j^{-s_0})\prod_{j=0}^{n}\mathbb{E}\exp(\frac{tZ_j}{(j+1)^{s_0}})$$
$$\le \exp(Ma\sum_{j=1}^{n+1} j^{-s_0})\prod_{j=0}^{n}(1-\frac{\alpha t}{(j+1)^{s_0}})^{-j}. \qquad \square$$

When j is large enough, $(1-\frac{\alpha t}{(j+1)^{s_0}})^{-j} \le \exp(\frac{\alpha t}{j^{s_0-1}})$, which gives the existence of an exponential moment for $s_0 > 2$.

The proof in the case $|a_n| \prec \mu$ and $s_0 > 1$ is similar.

As an example of domination by μ^{*n}, we can think about the case where a recursive function called with parameter n requires n simulations with an acceptance-rejection method. Then, a_n appears as the sum of n independent variables following a geometric distribution $\mu = \mathcal{G}(p)$.

References

Akra, M. and L. Bazzi (1998). "On the solution of linear recurrence equations". *Comput. Optim. Appl.* **10** (2), pp. 195–210. ISSN: 0926-6003. DOI: 10.1023/A:1018353700639 (cit. on p. 92).

Choimet, D. and H. Queffélec (2015). *Twelve landmarks of twentieth-century analysis*. Illustrated by Michaël Monerau, Translated from the 2009 French original by Danièle Gibbons and Greg Gibbons, With a foreword by Gilles Godefroy. Cambridge University Press, New York, pp. xv+508. DOI: 10.1017/CBO9781107445017 (cit. on pp. 92, 97).

Cormen, T. H. et al. (2010). *Introduction to Algorithms*. PHI Learning (cit. on p. 91).

Drmota, M. and W. Szpankowski (2013). "A master theorem for discrete divide and conquer recurrences". *J. ACM* **60** (3), Art. 16, 49. ISSN: 0004-5411. DOI: 10.1145/2487241.2487242 (cit. on p. 92).

Durrett, R. (2019). *Probability—theory and examples*. **49**. Cambridge Series in Statistical and Probabilistic Mathematics. Fifth edition of [MR1068527]. Cambridge University Press, Cambridge, pp. xii+419. ISBN: 978-1-108-47368-2. DOI: 10.1017/9781108591034 (cit. on p. 98).

Erdős, P. et al. (1987). "The asymptotic behavior of a family of sequences". *Pacific J. Math.* **126** (2), pp. 227–241. ISSN: 0030-8730 (cit. on pp. 92, 97).

Goodrich, M. T. and R. Tamassia (2002). *Algorithm Design: Foundations, Analysis, and Internet Examples*. Wiley (cit. on p. 91).

Roura, S. (2001). "Improved master theorems for divide-and-conquer recurrences". *J. ACM* **48** (2), pp. 170–205. ISSN: 0004-5411. DOI: 10.1145/375827.375837 (cit. on p. 92).

Contents

1	Introduction	91
2	The deterministic Theorem	93
	The recursion equation	94
	Tauberian magic	94
	The general case	97
3	Application to sequences of random variables	97
	3.1 Convergence	97
	3.2 Non-vanishing limit	98
	3.3 Exponential moments	99
References		100
Contents		i